# Normalizations about weighted Pascal's triangles and extended visualizations of weighted skipped generalized Fibonacci or Padovan sequences


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## 1. Introduction

The author described the previous works including RIMS Kôkyûroku No. 2304-13[1-8] about the various modifications of Pascal's triangles and these matrices to display the weighted skipped Gibonacci sequences as the summation of these diagonals. The original Gibonacci sequence [9-12] is constructed with arbitrary positive initial constants $(g_0, g_1)$ instead of original Fibonacci sequence. Lucas sequence is one of them. We would like to define weighted Gibonacci sequences with the weights $(a, b)$ to describe the ideas in this study. Therefore, weighted Gibonacci sequences be also admitted as generalized Fibonacci sequences with flexibly changing initial constants. The author suggested various upward sequences [13] such as generalized Padovan sequence [7,14,15] including Perrin sequence [16,17] using the tendencies of plastic ratio [18], and similar type sequences using super-golden ratio [19], super-silver ratio [20], and so [1].

In this study, firstly, it is dealt with the tendencies about the various diagonal summations based on the horizontal normalization of weighted Pascal's triangles with the weight $(a, b)$. At this time, we can understand that these tendencies convergence the constants as the meaningful fractions with horizontal step number $h$, vertical step number $v$ and the weights $(a, b)$ of weighted Pascal's triangle precisely. Therefore, we would like to explain those numerical results as the first rapid announcement in this study. Secondly, if we think of the horizontal leg extensions of modified Pascal's triangles, we would also like to inform the readers that various skipped weighted Gibonacci sequences and generalized Padovan sequences on that as leg extensional versions are clearly and numerically shown as the extended suggestions of RIMS Kôkyûroku No. 2304-13 in this study [1]. According to the second proportion of this study, we would like to display the 2 or 3 skipped generalized Padovan sequences as the leg extensional versions of the modified Pascal's triangles. In the same manner, we would also like to illustrate the weighted Gibonacci sequences on that.

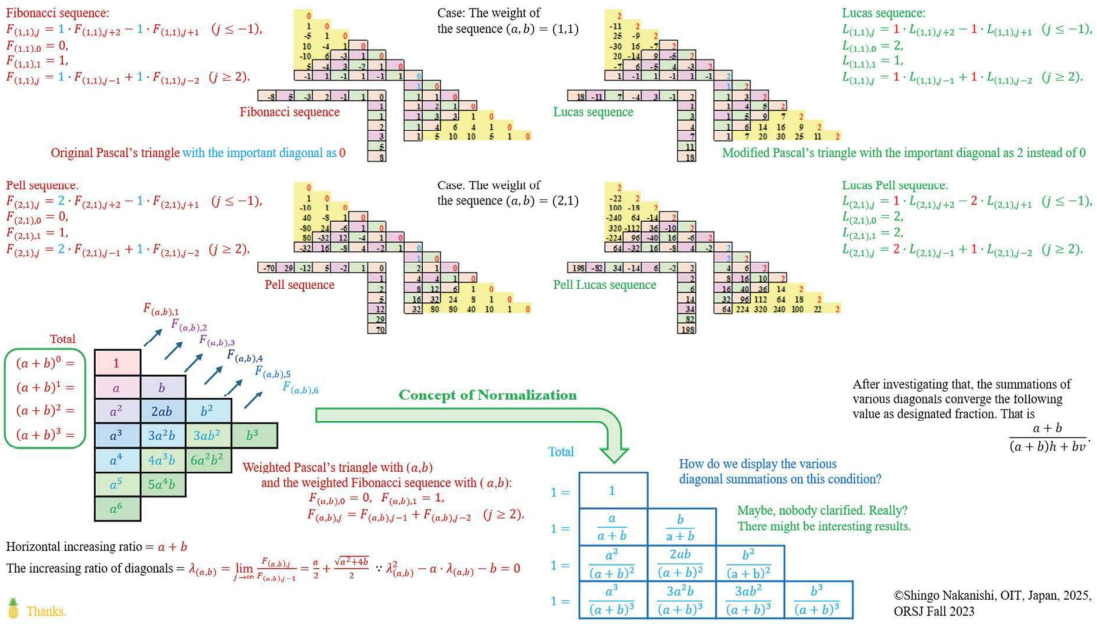


**Figure 1** Visualizations about weighted Gibonacci sequences on the weighted and modified Pascal's triangles as the summations of these diagonals, and horizontal normalizations of that based on the concepts of this study.

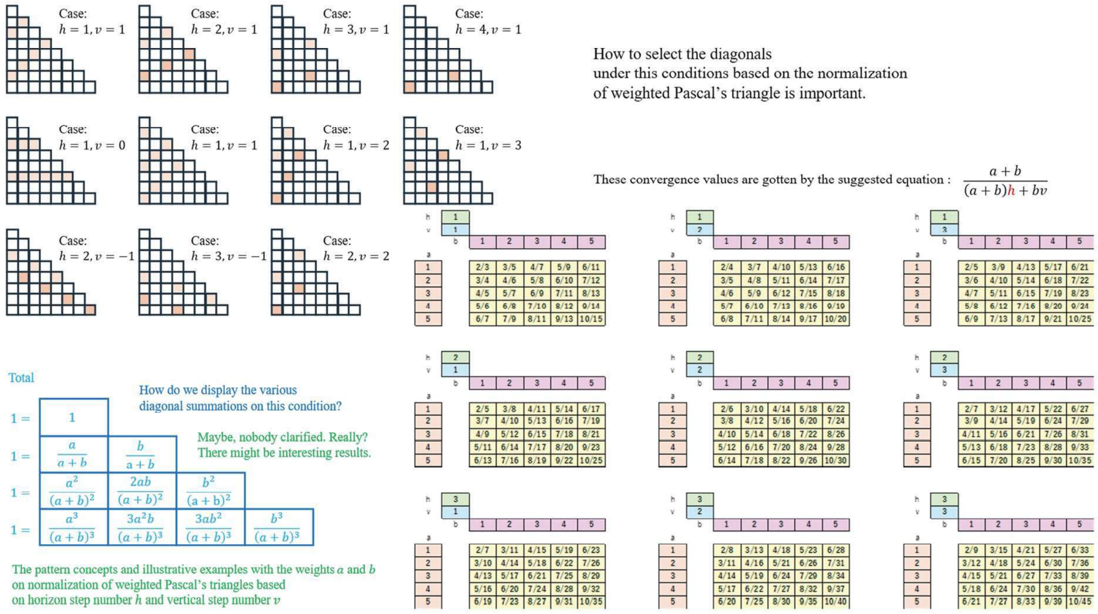


**Figure 2** Illustrative visualizations about various diagonals based on the normalization of weighted Pascal's triangle and these numerical tendencies of various diagonals based on weighed Pascal's triangle with the weights $a$ and $b$, and horizontal and vertical step numbers $h$ and $v$.

## 2. Normalization of weighted Pascal's triangles

If we think that weighted Pascal's triangles are constructed by the weights $a$ and $b$ shown in Figure 1, we can also calculate the diagonal summations as the weighted Fibonacci sequences, (Generalized Fibonacci sequences)

$$F_{(a,b),0} = 0, \qquad F_{(a,b),1} = 1, \qquad F_{(a,b),j} = a \cdot F_{(a,b),j-1} + b \cdot F_{(a,b),j-2} \quad (j \geq 2). \tag{2.1}$$

$$G_{(a,b),0} = g_0, \qquad G_{(a,b),1} = g_1, \qquad G_{(a,b),j} = a \cdot G_{(a,b),j-1} + b \cdot G_{(a,b),j-2} \quad (j \geq 2). \tag{2.2}$$

In Equation (2.1), $a$ and $b$ are supposed to be arbitrary positive integers. For example, we can admit the Equation (2.1) as $n$-Pell sequences [21,22] if we estimate $a = n \in \mathbb{N}$ and $b = 1$. For one of the other examples, we can also consider the Equation (2.1) as $n$-Jacobsthal sequences [23,24] in case that $a = 1$ and $b = n$. When we think that the Equations (2.1) and (2.2) and the two initial constants $G_{(a,b),0} = g_0$, and $G_{(a,b),1} = g_1$, we can also get the weighted Gibonacci sequences [9-12] (Generalized Fibonacci sequences with arbitrary positive initial constants). In Equation (2.2), $g_0$ and $g_1$ are supposed to be arbitrary integers. For example, we can admit the Equation (2.2) as weighted Lucas sequences if we estimate $g_0 = 2$, $g_1 = a$. In case that $a = n$ and $b = 1$, we can show the $n$-Pell Lucas sequences. In case that $a = 1$ and $b = n$, we can also describe the $n$-Jacobsthal Lucas sequences from Equation (2.2). For example, Pell and Pell Lucas sequences are shown in Figure 1 based on these polynomials [18,19]

In Figure 1, since we can calculate the $s$-th horizontal summations are defined as $(a+b)^s$ according to the binomial theorem on the weighted Pascal's triangle, we understand that the increasing ratios are shown $(a+b)$ as the constants. At the same time, we can estimate the increasing ratios about the diagonal summations converge the value

$$\lambda_{(a,b)} = \frac{a}{2} + \frac{\sqrt{a^2+4b}}{2} \quad \because \ \lambda_{(a,b)}^2 - a \cdot \lambda_{(a,b)} - b = 0. \tag{2.3}$$

For example, if $a = n$ and $b = 1$, we can show $\lambda_{(n,1)}$ as $n$-th primary metallic ratios [25] such as golden ratio ($n = 1$), silver ratio ($n = 2$), and bronze ratio ($n = 3$). If $a = 1$ and $b = n$, we can show $\lambda_{(1,n)}$ as $n$-th secondary metallic ratios [25]. Both two metallic ratios are historically named and defined by de Spinadel in 1990s [25]. Therefore, since we know the horizontal increasing ratio is $(a+b)$ and the increasing ratio of the diagonal summation is also $\lambda_{(a,b)}$ in Equation (2.3), we would like to investigate the tendencies based on the condition that the increasing ratio $(a+b)$ is removed from the weighted Pascal's triangle as the normalized Pascal's triangle shown on the bottom in Figure 1. Moreover, we can estimate the various diagonal summations to be the constant instead of the other increasing ratios if we get the weighted Pascal's triangles numerically. This treatment and concept should be called the horizontal normalizations of weighted Pascal's triangle in this study shown on the bottom in Figure 1. At this time, the various diagonal summations can also show meaningful values or

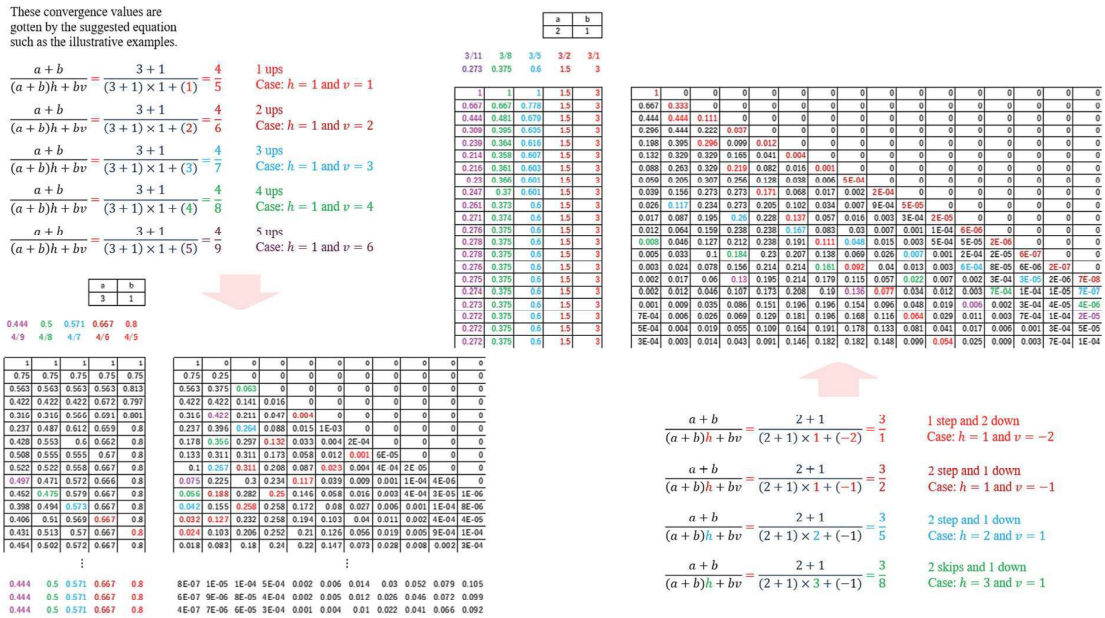


**Figure 3** Numerical results and these tendencies of several upward and right sift diagonals on weighted Pascal's triangle with the weights $a = 3$ and $b = 1$ and that of several downward diagonals on weighted Pascal's triangle with $a = 2$ and $b = 1$.

results the readers expect. For this concept, we would like to show you that there are several types for getting various diagonal summations shown in Figure 2.

After investigating the types of diagonals displayed on the left side in Figure 2, we got the numerical results shown as the meaningful fractions on the right side in Figure 2. From the numerical results, we can clarify the values as the diagonal summation using the parameters $a$, $b$, $h$, and $v$ as follows.

$$d = \frac{a+b}{(a+b)h+bv} \tag{2.4}$$

In Equation (2.4), $a$ and $b$ are weights of the weighted Pascal's triangle shown in Figure 2. $h$ is the horizontal step number and $v$ is the vertical step number on the weighted Pascal's triangle in Figure 2.

From the various results shown in Figure 3, we can understand that the tendencies of $d$ can display the visual constants about the behaviors of parameters $a, b, h,$ and $v$ in Equation (2.4) concretely as the designated diagonal summations on the normalized Pascal's triangles throughout the several illustrative examples.

## 3. 1, 2 and 3 skipped versions of generalized Padovan sequences on the extended and modified Pascal's triangles

As readers know, Padovan sequence is defined as

$$P_{(1,1),0}^{(1,1,1)} = P_{(1,1),1}^{(1,1,1)} = P_{(1,1),2}^{(1,1,1)} = 1, P_{(1,1),j}^{(1,1,1)} = P_{(1,1),j-2}^{(1,1,1)} + P_{(1,1),j-3}^{(1,1,1)} \quad (j \geq 3). \tag{3.1}$$

In equation (3.1), we would like to define the descriptions based on that the super script means the set of the initial constants of the sequence in this study. The subscript also means each weight of the terms of the equation. In this case, the initial conditions are shown as $(1,1,1)$ and the weights are $(1,1)$ in Equation (3.1). This type of sequence would like to be named 1 skipped sequence in this paper. Because the difference between the ordinal number of the sequence on the left side and that of the first term on the right side is equal to 2 (=1+1). That meaning of the value is shown the addition both one counting order and one skipping. If we use this description method as several skipped sequences when we show the structures of the various sequences, it must be much more convenient and efficient than we thought or imagined that.

On the other hand, we know the relationship between Padovan sequence and plastic ratio $\rho$ as the following standard models

$$\rho^3 = \rho + 1, \rho^5 = \rho^4 + 1 = \rho^2 + \rho + 1 \text{ and } \rho^8 = \rho^3 \cdot \rho^5 = (\rho + 1)(\rho^2 + \rho + 1) = \rho^4 + \rho^3 + \rho^2 + \rho + 1. \tag{3.2}$$

From the second description in Equation (3.2), we can redefine the equation between plastic ratio and Padovan sequence in the same way based on Equation (3.1). That is

$$P^{(1,1,1,2,2)}_{(1,1,1),0} = P^{(1,1,1,2,2)}_{(1,1,1),1} = P^{(1,1,1,2,2)}_{(1,1,1),2} = 1, P^{(1,1,1,2,2)}_{(1,1,1),3} = P^{(1,1,1,2,2)}_{(1,1,1),4} = 2,$$
$$P^{(1,1,1,2,2)}_{(1,1,1),j} = P^{(1,1,1,2,2)}_{(1,1,1),j-3} + P^{(1,1,1,2,2)}_{(1,1,1),j-4} + P^{(1,1,1,2,2)}_{(1,1,1),j-5} \quad (j \geq 5). \tag{3.3}$$

as the fundamentally 2 skipped Padovan sequence before changing initial constants. Since the difference between the ordinal number of the sequence on left side and that of the first term on the right side is equal to 3 (=1+2). There are one counting order and two skipping as the addition. According to this description in Equation (3.2), the sequence can be transformed into the following illustration. That is

$$P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),0} = P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),1} = P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),2} = 1, P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),3} = P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),4} = 2,$$
$$P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),5} = 3, P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),6} = 4, P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),7} = 5,$$
$$P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),j} = P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),j-4} + P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),j-5} + P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),j-6} + P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),j-7} + P^{(1,1,1,2,2,3,4,5)}_{(1,1,1,1,1),j-8} \quad (j \geq 8). \tag{3.4}$$

as the fundamentally 3 skipped Padovan sequence before changing initial constants. The difference between the ordinal number of the sequence on left side and that of the first term on the right side is equal to 4 (=1+3). These are shown as one counting order and three skipping as the difference.

In the same way, firstly, we would like to visualize and define the generalized Padovan sequences such as the 1 skipped and standard model based on the arbitrary whole numbers $g_0, g_1, g_2$. That is

$$P^{(g_0,g_1,g_2)}_{(1,1),0} = g_0, P^{(g_0,g_1,g_2)}_{(1,1),1} = g_1, P^{(g_0,g_1,g_2)}_{(1,1),2} = g_2, P^{(g_0,g_1,g_2)}_{(1,1),j} = P^{(g_0,g_1,g_2)}_{(1,1),j-2} + P^{(g_0,g_1,g_2)}_{(1,1),j-3} \quad (j \geq 3). \tag{3.5}$$

described shown in Figure 4. Figure 4 is illustrated as the 1 skipped sequence and the bottom figure by using the illustrative initial constants ($g_0 = 8, g_1 = 2, g_2 = 5$). For another well-known model such as the Perrin sequence, we can utilize its initial constants ($g_0 = 3, g_1 = 0, g_2 = 2$) with its flexibly initial constants to demonstrate that visually [1,4,7]. At the same time, we can confirm that the top graph in Figure 4 shows the 4 upward types of diagonal summations on the same conditions precisely. Generally, instead of describing Equation (3.5), this equation can be defined as

$$P^{(g_0,g_1,g_2,g_3,g_4)}_{(1,0,0,0,1),0} = g_0 \left(= P^{(g_0,g_1,g_2)}_{(1,1),0}\right), P^{(g_0,g_1,g_2,g_3,g_4)}_{(1,0,0,0,1),1} = g_1 \left(= P^{(g_0,g_1,g_2)}_{(1,1),1}\right), P^{(g_0,g_1,g_2,g_3,g_4)}_{(1,0,0,0,1),2} = g_2 \left(= P^{(g_0,g_1,g_2)}_{(1,1),2}\right),$$
$$P^{(g_0,g_1,g_2,g_3,g_4)}_{(1,0,0,0,1),3} = g_3 \left(= P^{(g_0,g_1,g_2)}_{(1,1),3}\right), P^{(g_0,g_1,g_2,g_3,g_4)}_{(1,0,0,0,1),4} = g_4 \left(= P^{(g_0,g_1,g_2)}_{(1,1),4}\right),$$
$$P^{(g_0,g_1,g_2,g_3,g_4)}_{(1,0,0,0,1),j} = P^{(g_0,g_1,g_2,g_3,g_4)}_{(1,0,0,0,1),j-1} + P^{(g_0,g_1,g_2,g_3,g_4)}_{(1,0,0,0,1),j-5} \quad (j \geq 5). \tag{3.6}$$

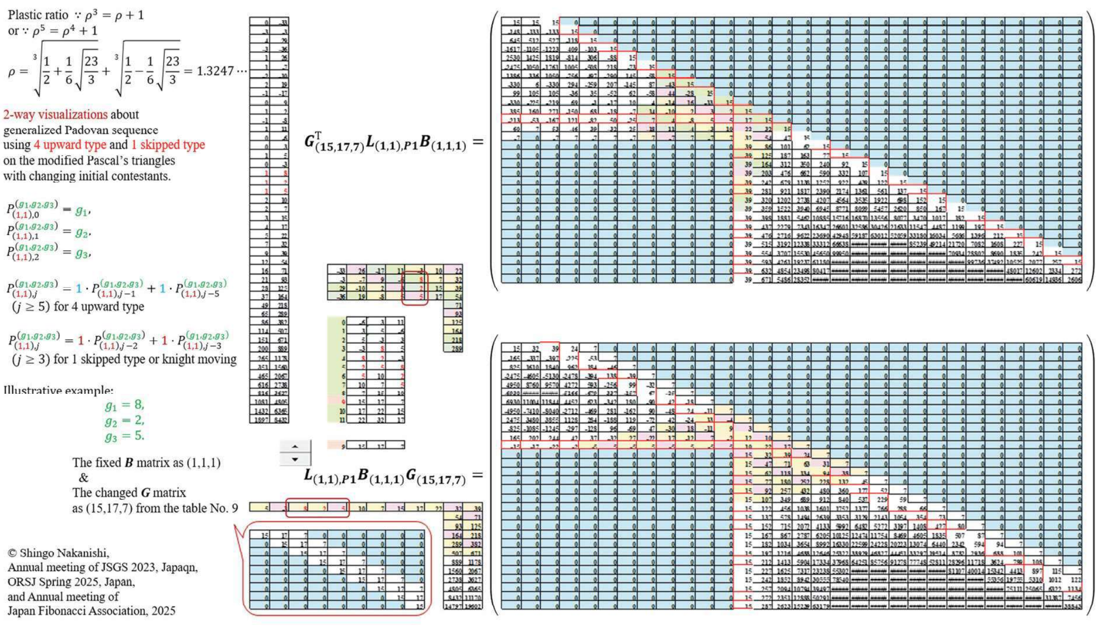


**Figure 4** Illustrative visualization of generalized Padovan sequence with initial constants ($g_0 = 8, g_1 = 2, g_2 = 5$) and several steps as flexibly changing initial constants according to the table shown in this figure as 4 downward diagonals and 1skipped diagonals (knight moving diagonals) on modified Pascal's triangle.

Secondly, Equation (3.3) can be also rewritten based on Equation (3.5) as the following 2 skipped generalized Padovan sequences. That is

$$P_{(1,1,1),0}^{(g_0,g_1,g_2,g_3,g_4)} = g_0\left(= P_{(1,1),0}^{(g_0,g_1,g_2)}\right), P_{(1,1,1),1}^{(g_0,g_1,g_2,g_3,g_4)} = g_1\left(= P_{(1,1),1}^{(g_0,g_1,g_2)}\right),$$
$$P_{(1,1,1),2}^{(g_0,g_1,g_2,g_3,g_4)} = g_2\left(= P_{(1,1),2}^{(g_0,g_1,g_2)}\right), P_{(1,1,1),3}^{(g_0,g_1,g_2,g_3,g_4)} = g_3\left(= P_{(1,1),3}^{(g_0,g_1,g_2)}\right), P_{(1,1,1),4}^{(g_0,g_1,g_2,g_3,g_4)} = g_4\left(= P_{(1,1),4}^{(g_0,g_1,g_2)}\right),$$
$$P_{(1,1,1),j}^{(g_0,g_1,g_2,g_3,g_4)} = P_{(1,1,1),j-3}^{(g_0,g_1,g_2,g_3,g_4)} + P_{(1,1,1),j-4}^{(g_0,g_1,g_2,g_3,g_4)} + P_{(1,1,1),j-5}^{(g_0,g_1,g_2,g_3,g_4)} \quad (j \geq 5). \tag{3.7}$$

Thirdly from Equation (3.4), we can describe the 3 skipped generalized Padovan sequences as follows.

$$P_{(1,1,1,1,1),0}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} = g_0\left(= P_{(1,1),0}^{(g_0,g_1,g_2)}\right), P_{(1,1,1,1,1),1}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} = g_1\left(= P_{(1,1),1}^{(g_0,g_1,g_2)}\right),$$
$$P_{(1,1,1,1,1),2}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} = g_2\left(= P_{(1,1),2}^{(g_0,g_1,g_2)}\right), P_{(1,1,1,1,1),3}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} = g_3\left(= P_{(1,1),3}^{(g_0,g_1,g_2)}\right),$$
$$P_{(1,1,1,1,1),4}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} = g_4\left(= P_{(1,1),4}^{(g_0,g_1,g_2)}\right), P_{(1,1,1,1,1),5}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} = g_5\left(= P_{(1,1),5}^{(g_0,g_1,g_2)}\right),$$
$$P_{(1,1,1,1,1),6}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} = g_6\left(= P_{(1,1),6}^{(g_0,g_1,g_2)}\right), P_{(1,1,1,1,1),7}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} = g_7\left(= P_{(1,1),7}^{(g_0,g_1,g_2)}\right),$$
$$P_{(1,1,1,1,1),j}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} = P_{(1,1,1,1,1),j-4}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} + P_{(1,1,1,1,1),j-5}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} + P_{(1,1,1,1,1),j-6}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)}$$
$$+P_{(1,1,1,1,1),j-7}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} + P_{(1,1,1,1,1),j-8}^{(g_0,g_1,g_2,g_3,g_4,g_5,g_6,g_7)} \quad (j \geq 8). \tag{3.8}$$

From Equations (3.5), (3.7) and (3.8), we can describe the 1, 2 and 3 skipped sequences for generalized Padovan sequence concisely. Therefore, we would like to display these definitions as illustrative figures concretely shown in Figures from 5 thought 7. To begin with, we would like to show you the 1, 2, and 3 skipped original Padovan sequences at the same time in Figure 5. From the two bottom equations in Figure 5, we can understand the idea in Equations (3.5) and (3.7) displayed as the flexibly changing initial conditions of 1 skipped and 2 skipped sequences about Padovan sequence using the table on the right side in Figure 5. One is used by No. 7 (5 steps from table No. 2) as 1skipped Padovan sequence with initial constants. The other is utilized by No. 9 (7 steps from table No. 2). Now, we suppose the following matrices

$$\boldsymbol{B}_{(1\cdots1)} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 & \ddots & 1 \\ & & & & & \ddots & \vdots \\ 0 & & \cdots & & & 0 & 1 \end{pmatrix}, \quad \boldsymbol{G}_{(\cdot)} = \begin{pmatrix} g_j & g_{j+1} & g_{j-1} & 0 & \cdots & \cdots & & 0 \\ 0 & g_j & g_{j+1} & g_{j-1} & \ddots & & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & & \ddots & g_{j-1} & 0 \\ \vdots & & & \ddots & & \ddots & g_{j+1} & g_{j-1} \\ \vdots & & & & & \ddots & g_j & g_{j+1} \\ 0 & 0 & 0 & 0 & & \cdots & 0 & g_j \end{pmatrix}. \tag{3.9}$$

To decide the steps, we can use the band matrices $\boldsymbol{G}_{(\cdot)}$ $(\in \mathbb{R}^{m\times m})$. $(\cdot)$ shown in Figures from 5 through 7 include several numbers according to the table in each Figure. In the same way, $\boldsymbol{B}_{(\cdot)}$ $(\in \mathbb{R}^{m\times m})$ shown in Figures 5 through 7 mean the base band matrices including the part $(\cdot)$ with (1,1) as the case of 1 skipped sequences, (1,1,1,1) as that of 2 skipped sequences, (1,1,1,2,2,2,2,2,1,1,1) as that of 3 skipped sequences based on the creating way displayed on the top illustrations in Figures 5 and 6.

Next, we would like to define the following matrix

$$\boldsymbol{L}_{(\cdot)} = \begin{pmatrix} l_{(1,1),P(k-2)} & \cdots & l_{(1,m),P(k-2)} \\ \vdots & \ddots & \vdots \\ l_{(m,1),P(k-2)} & \cdots & l_{(m,m),P(k-2)} \end{pmatrix}. \tag{3.10}$$

The subscript $(\cdot)$ of $\boldsymbol{L}_{(\cdot)}$ $(\in \mathbb{R}^{m\times m})$ should be more complex than that of above symbols $\boldsymbol{G}_{(\cdot)}$ and $\boldsymbol{B}_{(\cdot)}$. We would like to describe the idea using the elements of $\boldsymbol{L}_{(\cdot)} = (l_{(i,j),P(k-2)})$, $\boldsymbol{L}_{(\cdot)} = (l_{(i,j),P(k-2)})$, concretely to explain that in detail. If we would like to get the values as the elements about $i$-th row, $j$-th column, and $k-2$ skipped sequence $l_{(i,j),P(k-2)}$ on $\boldsymbol{L}_{(\cdot)} = (l_{(i,j),P(k-2)})$, we should use the proper calculations according to the number of skipped types $P(k-2)$. In cases from 1 to 3, the number of skipped types, we can get that respectively as follows.

$$l_{(i,j),P1} = \boldsymbol{w}_{P1}^T \boldsymbol{l}_{P1}, \quad \because \boldsymbol{w}_{P1} = (1,1)^T, \boldsymbol{l}_{P1} = \left(l_{(i-1,j-1),P1}, l_{(i-1,j),P1}\right)^T, \tag{3.11}$$

$$l_{(i,j),P2} = \boldsymbol{w}_{P2}^T \boldsymbol{l}_{P2}, \quad \because \boldsymbol{w}_{P2} = (1,1,1)^T, \boldsymbol{l}_{P2} = \left(l_{(i-1,j-2),P2}, l_{(i-1,j-1),P2}, l_{(i-1,j),P2}\right)^T, \tag{3.12}$$

$$l_{(i,j),P3} = \boldsymbol{w}_{P3}^T \boldsymbol{l}_{P3}, \quad \because \boldsymbol{w}_{P3} = (1,1,1,1,1)^T, \boldsymbol{l}_{P3} = \left(l_{(i-1,j-3),P3}, l_{(i-1,j-3),P3}, l_{(i-1,j-2),P3}, l_{(i-1,j-1),P3}, l_{(i-1,j),P3}\right)^T. \tag{3.13}$$

And the elements of the top rows $(i = 1)$ are defined respectively as follows

$$\left(l_{(1,1),P1}, l_{(1,2),P1}, \cdots, l_{(1,m),P1}\right) = (1,0,\cdots,0), \left(l_{(1,1),P2}, l_{(1,2),P1}, \cdots, l_{(1,m),P2}\right) = (1,0,\cdots,0), \tag{3.14}$$

$$\left(l_{(1,1),P3}, \cdots, l_{(1,11),P3}, l_{(1,12),P3}, \cdots, l_{(1,m),P3}\right) = (1,1,1,2,2,2,2,2,1,1,1,0,\cdots,0). \tag{3.15}$$

From Equation (3.14), we can estimate the leg extensions from lower triangles concretely and simply. On the other hand, from Equation (3.15), we cannot understand why the top rows need 11 cells as $(1,1,1,2,2,2,2,2,1,1,1)$ despite the above successful descriptions in Equation (3.14). This remains unknown before submitting this paper to RIMS. It might be clarified more easily and efficiently. If the element is located at the out of $\boldsymbol{L}_{(\cdot)}$ for carrying out the successful calculation, we can understand that the element can be considered as 0 without skipping or ignoring them to confirm that basically. Therefore, the vacant cells considered as 0 should be important to compute them effectively and precisely.

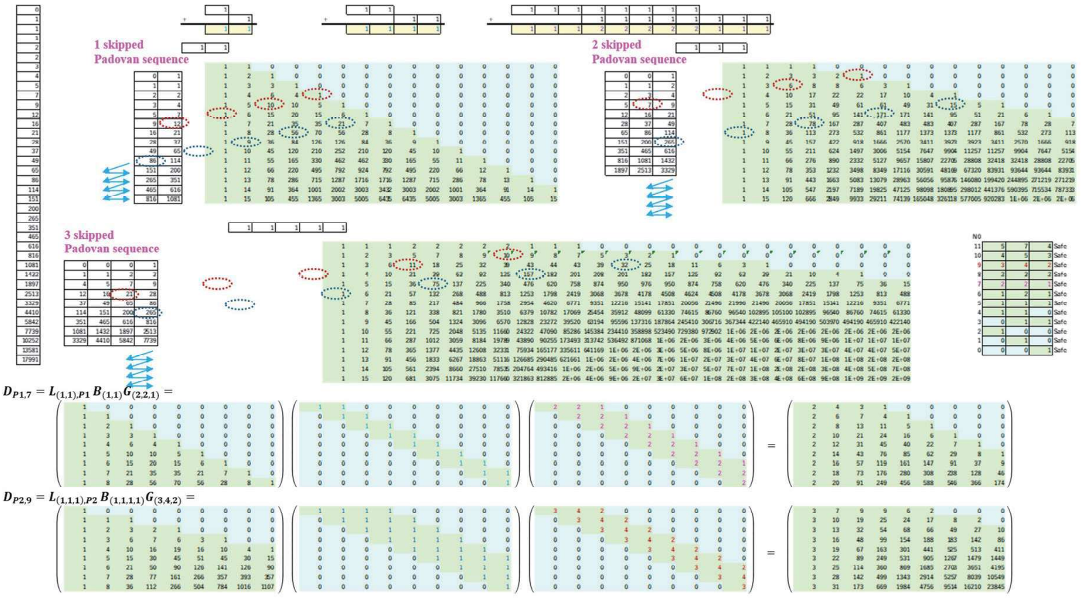


**Figure 5** Illustrative examples of 1, 2, and 3 skipped Padovan sequences on extended weighed Pascal's triangle and with changing initial conditions.

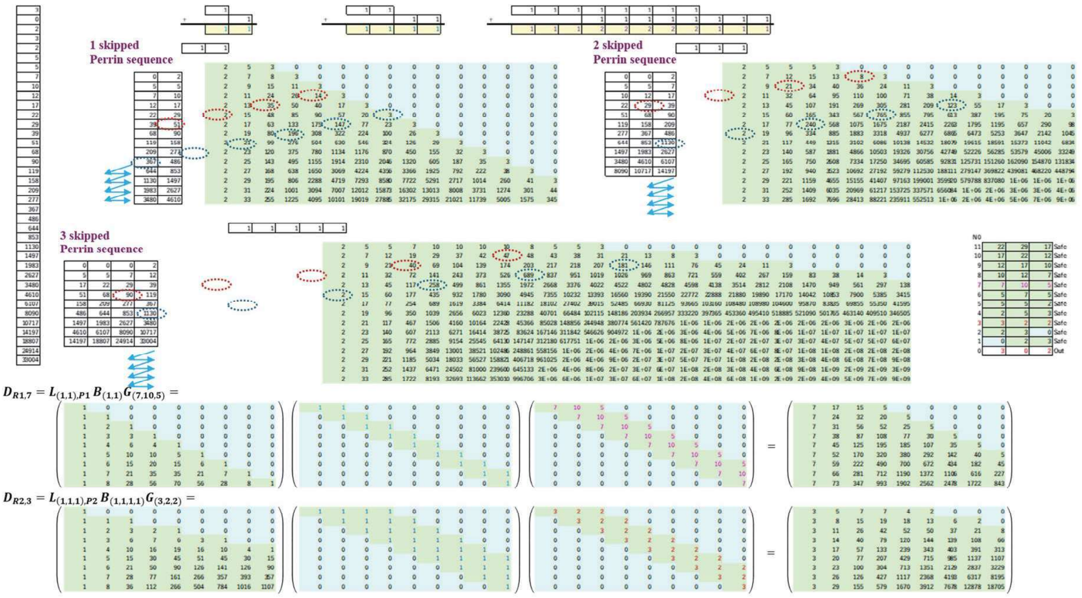


**Figure 6** Numerical examples of 1, 2, and 3 skipped Perrin sequences on extended weighed Pascal's triangle and with changing initial conditions.

These elements of the vectors $\boldsymbol{w}_{P(\cdot)}$ are equal to the terms on the right side in Equations (3.5), (3.7) and (3.8) respectively. That is, the meaning 1 of $\boldsymbol{w}_{P1}$ is indicated the number of skips in Equation (3.5). That of 2 as $\boldsymbol{w}_{P2}$ is equal to the skipping number in Equation (3.7). The number 3 about $\boldsymbol{w}_{P3}$ is equivalent to that in Equation (3.8) respectively. By using these calculations, we can also get the extensions of lower triangle matrices. Even if we utilize the extensional forms of $\boldsymbol{L}_{(\cdot)}$, we might compute the several skipped Padovan sequences somehow. However, the proofs of that are not shown in this paper. We would also like to notice the readers of that as the first rapid announcement in this study. Since the author is not a mathematician, I would be happy if mathematicians who are interested in that can complete that more beautifully and perfectly than I can to inform from generation to generation about that.

About the subscripts of $\boldsymbol{L}_{(1,1),P1}$, the digits within parentheses $(1,1)$ means each coefficient of terms on the right side in Equation (3.5) and P1 after the parentheses means using the 1 skipped Padovan sequence. Therefore, $\boldsymbol{L}_{(1,1,1),P2}$ is used as the 2 skipped Padovan sequence based on the coefficients are the similar trinomial term $(1,1,1)$ from the above element to the left direction. That is, (1,1) means the addition of 2 cells from the number on the above cell to that of 1 left stepped cell. $(1,1,1)$ indicates the addition by the values choosing 3 cells from the above cell to the left direction continuously.

In the same way, $(1,1,1,1,1)$ means the weighted vector. In addition to that, the vector with 5 cells with $(1,1,1,1,1)$ shown in Figures 5 and 6 is used as the summation from the value of the above cell to the left direction continuously. The concept of them is illustrated in Figures 5 and 6. These illustrations indicate the number of skipping and the length of cells we would like to use. If we also focus on the number of terms of the right side of Equations (3.5), (3.7) and (3.8), we can understand the extension numbers according to the difference between the number and 2. Basically, that of 1 skipped sequence in Equation (3.5) means 0 extension (0=2-2), that of Equation (3.7) is 1 extensions (1=3-2), and so on.

Therefore, from the matrices of $D_{(\cdot)}$, we can get the Padovan sequence as the designated diagonal summations shown in Figure 5. In Figure 5, the subscript of $D_{P1,7}$ means P is the initial letter of Padovan sequence to distinguish from the types of others, 1 as 1 skipped and 7 as No.7 chosen that meaning is 5 steps from initial conditions of Padovan sequence. In the same manner, that of $D_{P2,9}$ clarifies 2 skipped Padovan sequence with 7 (No.9) steps from the initial condition simply.

We would also like to deal with the other sequence shown in Figure 6 if the idea can be used concretely and visually. In Figure 6, the sequence is shown Perrin sequence selected as one of the well-known generalized Padovan sequences. As the illustrative examples, $D_{R1,7}$ means 1 skipped Perrin sequence with the condition No.7 on Table in Figure 6 and $D_{R2,3}$ means 2 skipped Perrin sequence with the condition No.3 on Table in Figure 6 at the same time to confirm that easily by choosing the letter R because of the same initial letter of Padovan sequence. Even though the initial constants of Perrin sequence are generally known ($g_0 = 3, g_1 = 0, g_2 = 2$) such as shown in Figure 7, The case No.0 in this table in Figure 9 are forbidden to make the

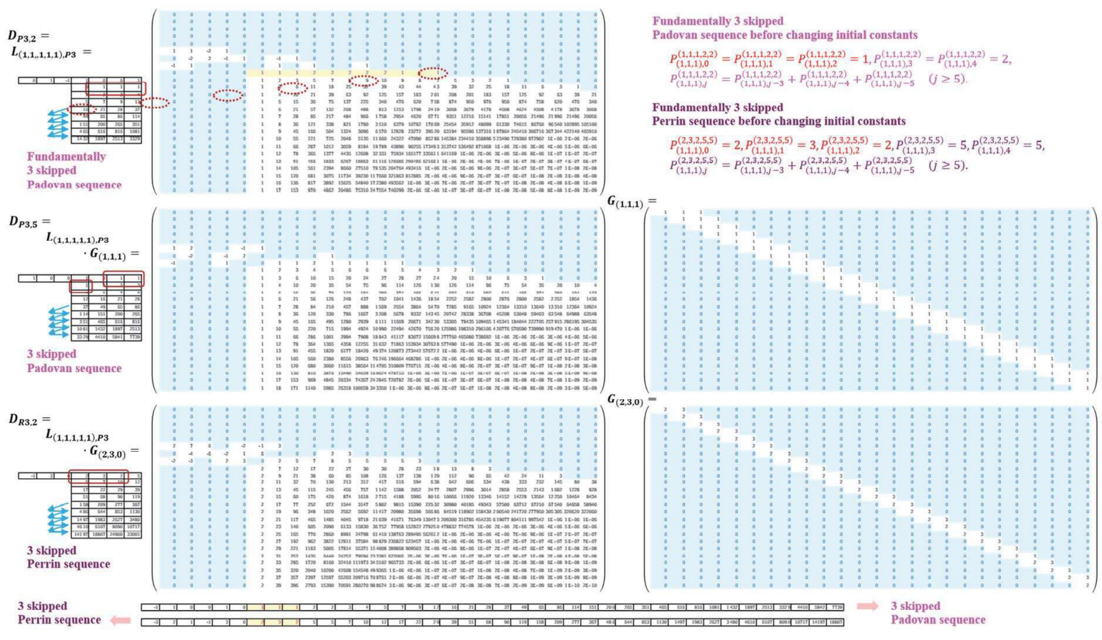


**Figure 7** Numerical examples of 3 skipped Padovan and Perrin sequences on extended weighed Pascal's triangle and with changing initial conditions.

sequence precisely. To tell the truth, from the left side on the table, 3, 0, 2 are used ($g_0 = 3, g_1 = 0, g_{-1} = 2$). If we use the negative order of Perrin sequence to make the sequence using this idea, we should confirm that in more detail to clarify that effectively and correctly in practice.

In Figure 7, 3 skipped Padovan and Perrin sequences are gotten respectively based on this idea. In Figure 7, $D_{P3,2}$ means 3 skipped Padovan sequence based on condition No. 2 on Table with no changing initial conditions as the original Padovan sequence. $D_{P3,5}$ means 3 skipped Padovan sequence based on condition No. 5 on Table with changing initial conditions as 3 (=5-2) steps from the original Padovan sequence. $D_{R3,2}$ means 3 skipped Perrin sequence based on condition No. 2 on Table. We can display the 1, 2 and 3 skipped generalized Padovan sequences using $\boldsymbol{L}_{(\cdot)}$, $\boldsymbol{B}_{(\cdot)}$, and $\boldsymbol{G}_{(\cdot)}$ effectively according to the designated conditions shown in Figures 5 through 7 as the extended models on the modified Pascal's triangles.

## 4. Generations of skipped weighted Gibonacci sequences based on the extensional models of modified Pascal's triangles

As shown in Equation (2.2), we would like to illustrate the skipped weighted Gibonacci sequences based on the idea of Section 3 instead of describing generalized Padovan sequences. When we use the addition theorem of weighted Fibonacci sequence with the weights $(a, b)$, we can also think Equation (2.2) as the following equation.

$$G_{(a,b),0} = g_0, G_{(a,b),1} = g_1, G_{(a,b),2} = g_2 = a \cdot g_1 + b \cdot g_0, \cdots, G_{(a,b),k+2} = g_{k+2} = a \cdot g_{k+1} + b \cdot g_k,$$
$$G_{(a,b),j} = F_{(a,b),k+2} \cdot G_{(a,b),j-k-1} + b \cdot F_{(a,b),k+1} \cdot G_{(a,b),j-k-2} \quad (j \geq k+3). \tag{4.1}$$

$$\boldsymbol{B}_{(F_{(a,b),1} \cdots F_{(a,b),k+1})} = \begin{pmatrix} F_{(a,b),1} & \cdots & F_{(a,b),k+1} & 0 & \cdots & 0 \\ 0 & F_{(a,b),1} & \cdots & F_{(a,b),k+1} & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & F_{(a,b),1} & \ddots & F_{(a,b),k+1} \\ & & & & \ddots & \vdots \\ 0 & & \cdots & & 0 & F_{(a,b),1} \end{pmatrix},$$

$$\boldsymbol{G}_{(g_j, g_{j-1})} = \begin{pmatrix} g_j & g_{j-1} & 0 & \cdots & & 0 \\ 0 & g_j & g_{j-1} & \ddots & & \vdots \\ & & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & g_{j-1} \\ 0 & & \cdots & & 0 & g_j \end{pmatrix}. \tag{4.2}$$

In Equations (4.1) and (4.2), $k-2$ means the skipped number about the sequence. Therefore, if we would like to confirm the $k-2$ skipped weighted Gibonacci sequences from some reasons, we should put the $k+1$ as the number of initial constants $G_{(a,b),j} = g_j$ $(0 <= j <= k)$ and 2 terms $F_{(a,b),k+2}$ as first terms, $b \cdot F_{(a,b),k+1}$ as second terms previously. Moreover, we can decide the changing order of the sequences by choosing $g_j$ and $g_{j-1}$ of $\boldsymbol{G}_{(\cdot)}$ in Equation (4.2) easily. In the same way as Section 3, we would like to visualize generalized Fibonacci sequences in Equation (4.1) as the skipped models based on the addition theorem shown in Figure 8. In Figure 8, we can display the example of 4 skipped model as follows

$$P^{(g_0,\cdots,g_5)}_{(120,2\times44),0} = g_0 = 0, P^{(g_0,\cdots,g_5)}_{(120,2\times44),1} = g_1 = 1, P^{(g_0,\cdots,g_5)}_{(120,2\times44),2} = g_2 = 2 = a,\ P^{(g_0,\cdots,g_5)}_{(120,2\times44),3} = g_3 = 6,$$
$$P^{(g_0,\cdots,g_5)}_{(120,2\times44),4} = g_4 = 16,\ P^{(g_0,\cdots,g_5)}_{(120,2\times44),5} = g_5 = 44,$$
$$P^{(g_0,\cdots,g_5)}_{(100,128,2\times40),j} = 120 \cdot P^{(g_0,\cdots,g_5)}_{(120,2\times44),j-5} + 2 \times 44 \cdot P^{(g_0,\cdots,g_5)}_{(120,2\times44),j-6} \quad (j \geq 6). \tag{4.3}$$

After confirming that concretely, we can understand several things and visualize that successfully. Certainly, even if we use the matrix forms to get the diagonals effectively and easily, the calculated values positioned the upper left and lower right side might include some errors. We should take care of them to estimate the sequences from the side of gotten data. Based on the above things, we can usually get the sequences after expanding the first term in Equation (4.1) such as

$$F_{(a,b),k+2} \cdot G_{(a,b),j-k-1} = a \cdot F_{(a,b),k+1} \cdot G_{(a,b),j-k-1} + b \cdot F_{(a,b),k} \cdot G_{(a,b),j-k-1} \tag{4.4}$$

and the second term of that as

$$b \cdot F_{(a,b),k+1} \cdot G_{(a,b),j-k-2} = ab \cdot F_{(a,b),k} \cdot G_{(a,b),j-k-2} + b^2 \cdot F_{(a,b),k} \cdot F_{(a,b),k-1} \cdot G_{(a,b),j-k-2} \tag{4.5}$$

easily. From Equations (4.4) and (4.5), we can reconstruct the form of Equation (4.1) to get the sequences with multi coefficient models such as Section 3. We can confirm and reconstruct the extended models throughout numerical examples such as $n$-Pell sequences, $n$-Pell Lucas sequences, $n$-Jacobsthal sequences, $n$-Jacobsthal Lucas sequences, and several other types of the sequences with the weights $(a, b)$. Especially, from the case of Equation (4.3), we would like to show the 4 skipped weighted Fibonacci or Lucas sequences with the weights ($a = 2, b = 2$) in Figures from 8 through 11 to confirm the extensions in practice. In Figure 8, these matrices and sequences from that have already been proposed as the skipped modeling of the matrices [1] based on Equation (4.2). In Figures From 8 through 11, we can understand the small difference of the number of terms and the values little by little to change the forms like lower triangular matrices with leg extensions to the right on that.

The number of initial constants from $g_j$ and the number of terms of the sequences are also increasing simultaneously according to the leg extensions. However, the skipped number remains the meaning of 4 skipping consistently even if these numbers are changing according to the conditions in Figures 8 to 11.

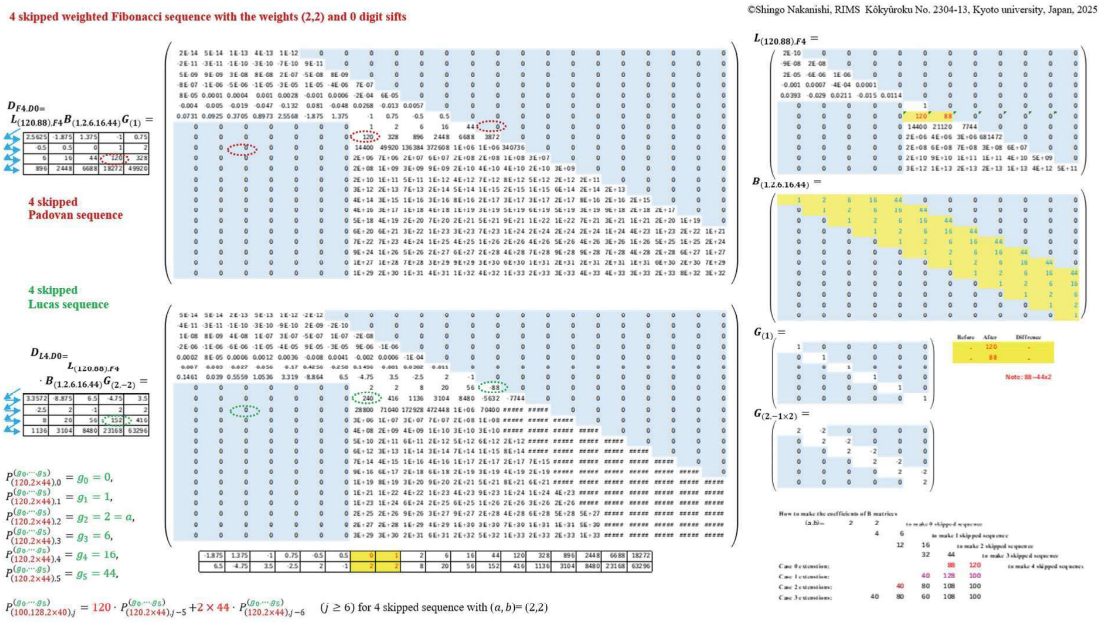


**Figure 8** Illustrative visualization of 4 skipped weighted Fibonacci and Lucas sequences on modified Pascal's triangles with 0 extension.

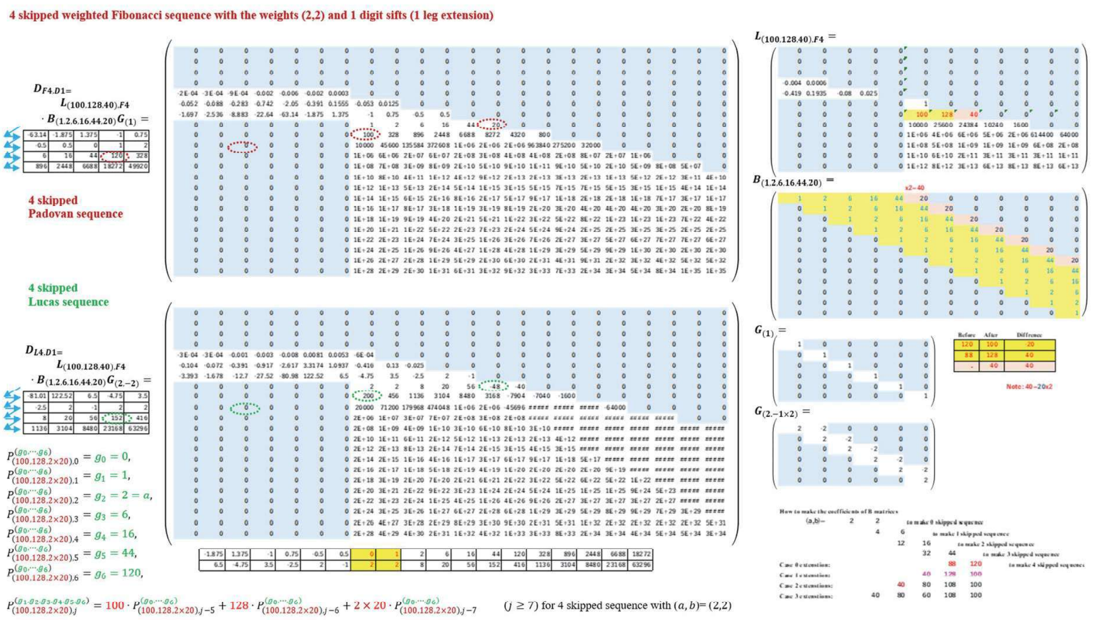


**Figure 9** Illustrative visualization of 4 skipped weighted Fibonacci and Lucas sequences on modified Pascal's triangles with 1 extension.

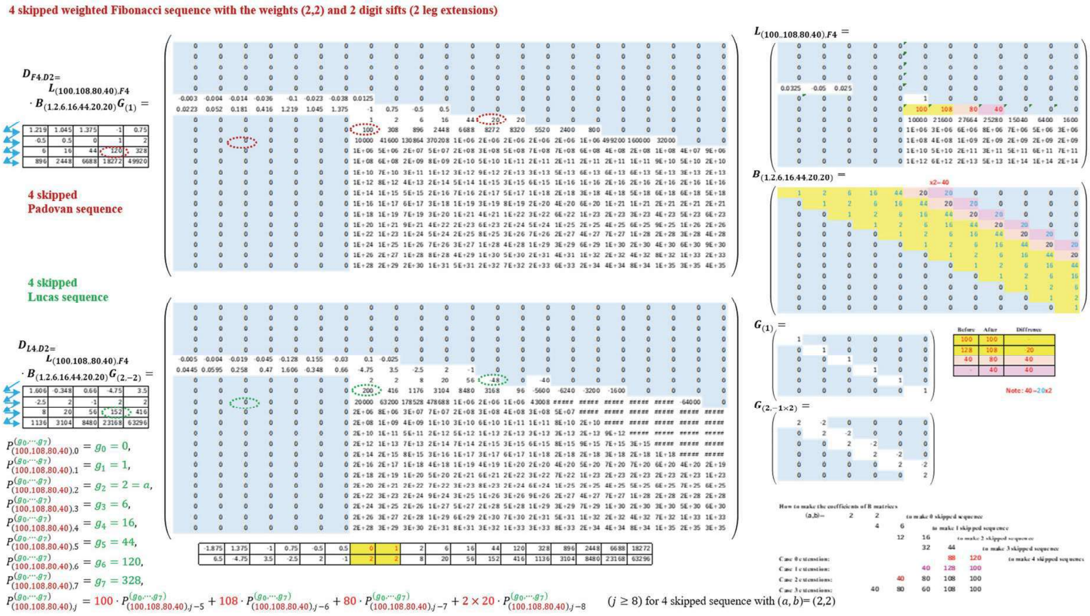


**Figure 10** Illustrative visualization of 4 skipped weighted Fibonacci and Lucas sequences on modified Pascal's triangles with 2 extensions.

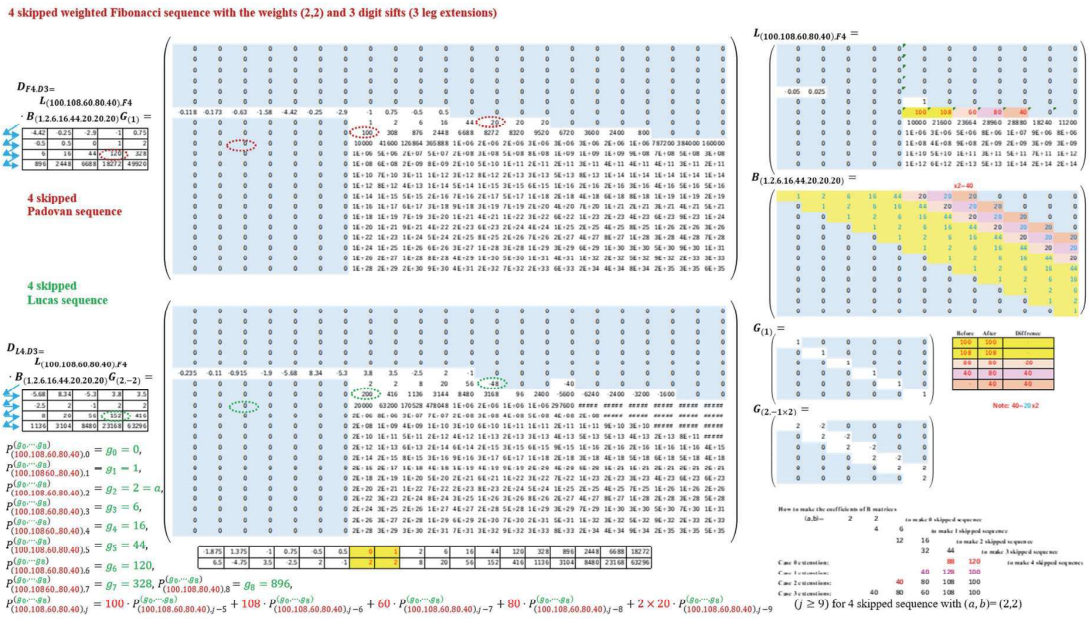


**Figure 11** Illustrative visualization of 4 skipped weighted Fibonacci and Lucas sequences on modified Pascal's triangles with 2 extensions.

In Figure 9, we can transform the equation in Equation (4.3) into the following equation. That is

$$P^{(g_0,\cdots,g_6)}_{(100,128,\,2\times20),0} = g_0 = 0 P^{(g_0,\cdots,g_6)}_{(100,128,\,2\times20),1} = g_1 = 1, P^{(g_0,\cdots,g_6)}_{(100,128,\,2\times20),2} = g_2 = 2 = a,$$

$$P^{(g_0,\cdots,g_6)}_{(100,128,\,2\times20),3} = g_3 = 6,\ P^{(g_0,\cdots,g_6)}_{(100,128,\,2\times20),4} = g_4 = 16,\ P^{(g_0,\cdots,g_6)}_{(100,128,\,2\times20),5} = g_5 = 44,\ P^{(g_0,\cdots,g_6)}_{(100,128,\,2\times20),6} = g_6 = 120,$$

$$P^{(g_1,g_2,g_3,g_4,g_5,g_6)}_{(100,128,\,2\times20),j} = 100\cdot P^{(g_0,\cdots,g_6)}_{(100,128,\,2\times20),j-5} + 128\cdot P^{(g_0,\cdots,g_6)}_{(100,128,\,2\times20),j-6} + 2\times20\cdot P^{(g_0,\cdots,g_6)}_{(100,128,\,2\times20),j-7} \quad (j \geq 7). \quad (4.6)$$

From Equation (4.6), we can understand that even though the number of initial constants and that of terms of the equation are increased than that of Equation (4.3), 4 skipped sequence are kept on this equation and shown in Figure 9 with some changing operations with $\boldsymbol{L}_{(\cdot)}$, $\boldsymbol{B}_{(\cdot)}$, and $\boldsymbol{G}_{(\cdot)}$ effectively according to the designated conditions. I am happy if the readers notice the modified way of the changing values about $\boldsymbol{L}_{(\cdot)}$, $\boldsymbol{B}_{(\cdot)}$, and $\boldsymbol{G}_{(\cdot)}$ in Equation (4.6) and in Figures 8 ad 9, and understand that meaning of some changing operations with $\boldsymbol{L}_{(\cdot)}$, $\boldsymbol{B}_{(\cdot)}$, and $\boldsymbol{G}_{(\cdot)}$ concretely. Based on this idea, Equation (4.6) can be rewritten as the following equations. Those are

$$P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),0} = g_0 = 0, P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),1} = g_1 = 1, P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),2} = g_2 = 2 = a,$$
$$P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),3} = g_3 = 6,\ P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),4} = g_4 = 16,\ P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),5} = g_5 = 44,$$
$$P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),6} = g_6 = 120,\ P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),7} = g_7 = 328,$$
$$P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),j} = 100 \cdot P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),j-5} + 108 \cdot P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),j-6}$$
$$+80 \cdot P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),j-7} + 2 \times 20 \cdot P^{(g_0,\cdots,g_7)}_{(100,108,\ 80,40),j-8} \qquad (j \geq 8) \tag{4.7}$$

and

$$P^{(g_0,\cdots,g_8)}_{(100,108,\ 60,80,40),0} = g_0 = 0, P^{(g_0,\cdots,g_8)}_{(100,10860,,\ 80,40),1} = g_1 = 1, P^{(g_0,\cdots,g_8)}_{(100,108,\ 60,80,40),2} = g_2 = 2 = a,$$
$$P^{(g_0,\cdots,g_8)}_{(100,108,\ 60,80,40),3} = g_3 = 6,\ P^{(g_0,\cdots,g_8)}_{(100,10860,,\ 80,40),4} = g_4 = 16,\ P^{(g_0,\cdots,g_8)}_{(100,108,60,\ 80,40),5} = g_5 = 44,$$
$$P^{(g_0,\cdots g_8)}_{(100,108,\ 60,80,40),6} = g_6 = 120,\ P^{(g_0,\cdots,g_8)}_{(100,10860,,\ 80,40),7} = g_7 = 328,\ P^{(g_0,\cdots,g_8)}_{(100,108,\ 80,40),8} = g_8 = 896,$$
$$P^{(g_0,\cdots,g_8)}_{(100,108,\ 60,80,40),j} = 100 \cdot P^{(g_0,\cdots,g_8)}_{(100,108,\ 60,80,40),j-5} + 108 \cdot P^{(g_0,\cdots,g_8)}_{(100,108,\ 60,80,40),j-6} + 60 \cdot P^{(g_0,\cdots,g_8)}_{(100,108,\ 60,80,40),j-7}$$
$$+80 \cdot P^{(g_0,\cdots,g_8)}_{(100,108,\ 60,80,40),j-8} + 2 \times 20 \cdot P^{(g_0,\cdots,g_8)}_{(100,108,\ 60,80,40),j-9} \qquad (j \geq 9). \tag{4.8}$$

About Equations (4.7) and (4.8) with some changing operations based on the idea, we can display the visual illustrations shown in Figures 10 and 11 respectively and clearly to confirm that of Figures 8 and 9. From this investigation, we can understand those sequences in Figures 8 through 11 are kept 4 skipped sequences using the summations of the designated diagonals on the leg extensions of modified Pascal's triangles. This visualization can be used for wide range types of weighted Gibonacci sequences throughout the numerical simulations. According to Prof. Yasuda's kind comments [26], these philosophies were likely to be suggested and discussed in problem B101 in Fibonacci Quarterly in 1969. Therefore, we need to investigate this idea based on that again to inform the next generations about that precisely and historically. In this study, the author would like to emphasize the idea as the first rapid announcement and necessity for the detailed investigations with getting the mathematical evidence by mathematicians about that.

## 5. Conclusions

In this paper, we dealt with the visualizations for horizontal normalizations of weighted Pascal's triangles and generations of several skipped generalized Padovan sequences and weighted Gibonacci sequences with leg extensions on the modified Pascal's triangles.

First, based on the normalizations of the weighted Pascal's triangles, we can estimate the converging the values as the various designated diagonal summations with the weights $a$, $b$, the horizontal step number $h$, and vertical step number $v$ and show that the meaningful fractions.

Second, for various types of sequences on modified Pascal's triangles or using the related Pascal matrices with leg extensions, we can also use the several inner products to illustrate that simply and concretely. In addition to that, we can devise the changing initial conditions to move the sequences on Pascal's triangle naturally, smoothly and visually based on the extensions of the modified Pascal's triangles.

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