

# INFINITELY TWISTED VERSION OF THE MÖBIUS BAND

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ABSTRACT. We construct an infinitely-twisted version of the Möbius band which we call the “Möbius Cascade”  $\mathbb{M}$ . In order to carry out infinitely many half-twists we call upon a “Cantor-set framework” to get a “pseudo”-surface with a Cantor set of wild points, and our geometric infinite-twist will be confined into the topological one in  $\mathbb{M}$ . Precisely,  $\mathbb{M}$  is not locally flat around these wild points in the sense that none of those points has neighborhood embeddable into the 2-dimensional plane. Natural flows on  $\mathbb{M}$  turn out to be unknotted knots in  $\mathbb{R}^3$ , winding around  $\mathbb{M}$  twice.  $\mathbb{M}$  can be expanded as the inverse limit, in either way, of orientable surfaces or non-orientable surfaces. We also show that we can get a kind of hollow sphere if two disjoint Möbius Cascades were glued together along their boundary.

## 1. INTRODUCTION

We present an infinitely-twisted version of the Möbius band, by repeating half-twists infinitely many times through a Cantor-set structure. Consider the surface  $M_n$  formed by gluing the two ends of a rectangular strip together with  $n$ -many half-twists. Then  $M_1$  is the Möbius band, while  $M_0$  is an annulus, and the following homeomorphisms hold:

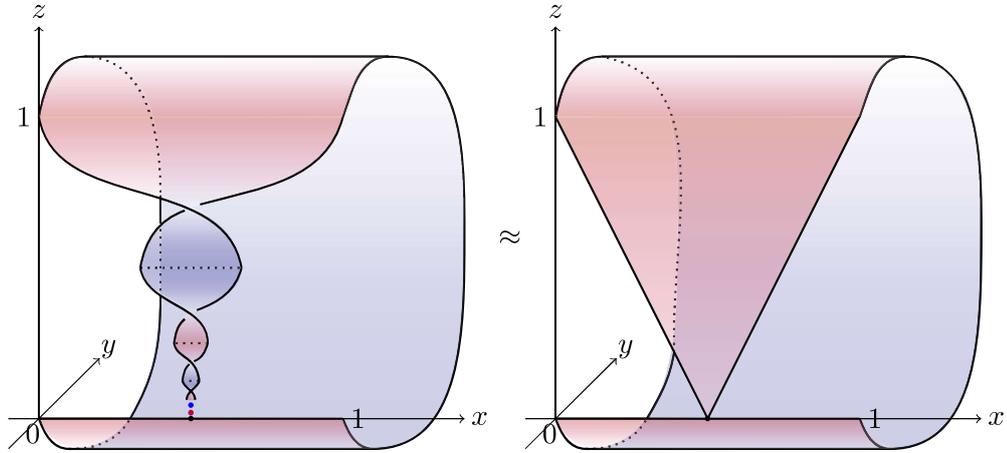
$$M_1 \approx M_3 \approx M_5 \approx \cdots ; \quad M_0 \approx M_2 \approx M_4 \approx \cdots .$$

This well-known fact arises some difficulties in making a “topological” version of an “infinitely-twisted” Möbius band, and the first try, everyone can think of, to get around such difficulties, is to converge infinitely many geometric twists to a point, like the left figure in Fig.1. But it fails to be twisted “topologically” since this figure is obviously homeomorphic to the right one which is locally flat. Then, how should we confine “geometric twist” into “topological one”? Our idea is to employ a Cantor-set framework, and what we could get was a “pseudo”-surface with wild points forming the Cantor set, as illustrated in Fig.6, which looks like a cascade falling down to the bottom Cantor set. We call it the “Möbius Cascade”  $\mathbb{M}$ , and its precise construction would be carried out in §2 by introducing a three-dimensional framework, the “Cantor Icicle”  $\mathbb{C}\mathbb{I}$  illustrated in Fig.3. We can show at the end of §2

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**Fig. 1.** The left figure looks infinitely twisted, but is not twisted at all topologically, since it is actually homeomorphic to the right figure, locally flat.

that  $\mathbb{M}$  is not locally flat around any wild point (where “flat” stands for “embeddable into the 2-dimensional plane”), which we interpret as evidence of being “infinitely twisted topologically” in view of Fig.1. In §3 we investigate the natural flows on  $\mathbb{M}$ , each of which is an unknotted knot in  $\mathbb{R}^3$  winding around  $\mathbb{M}$  twice. The structure of the boundary of  $\mathbb{M}$  is examined in §4. We point out in §5 that  $\mathbb{M}$  can be expressed in the form of inverse limit, in either way, of orientable surfaces or non-orientable surfaces. In §6 we show that, if two disjoint Möbius Cascades were glued together along their boundary, we get a pseudo-surface which looks like a hollow sphere with the Cantor set of wild points, as illustrated in Fig.12. It is not an easy task to define a proper topological notion for a membrane to “be infinitely-twisted”, and in this paper we adopt the following definition ( $\star$ ), below, taking account of the example in Fig.1. In general, let  $p$  be a point in a topological space  $X$ . We simply say  $p$  is *flat in  $X$*  if  $X$  is locally flat around  $p$ , i.e., we can find a neighborhood of  $p$  in  $X$  which is embeddable into  $\mathbb{R}^2$ ; otherwise, we say  $p$  is *non-flat in  $X$* . Let  $M$  be a membrane in  $\mathbb{R}^3$ , homeomorphic to the 2-dimensional disc  $\mathbb{D}^2$ , and let  $J_1$  and  $J_0$  be two disjoint proper arcs in  $\mathbb{R}^3$  such that  $J_1$  is included in the boundary  $\partial M$  of  $M$ , while  $J_0 \cap M$  is some non-empty subset of  $\partial M$ , where we regard that the membrane  $M$  lies in between the arcs  $J_1$  and  $J_0$ . Consider the space  $X = M \cup J_1 \cup J_0 = M \cup J_0$  in  $\mathbb{R}^3$ . It is then obvious that all points of  $X \setminus (J_0 \cap M)$  are flat in  $X$  so that non-flat points of  $X$  can live only in  $J_0 \cap M$ . We define that:

( $\star$ )  $M$  or  $M \cup J_0$  is *infinitely-twisted topologically between  $J_1$  and  $J_0$*  if all points of  $J_0 \cap M$  are non-flat in  $X$ .

For example, in the left figure of Fig.1, consider the geometrically twisted, front membrane  $M \approx \mathbb{D}^2$  in between  $I_1 = [0, 1] \times \{(0, 1)\}$  and  $I_0 = [0, 1] \times$

$\{(0,0)\}$  such that  $I_1 \subseteq M$ , and  $I_0 \cap M$  is a singleton. Then  $M$  is *not* infinitely-twisted topologically between  $I_1$  and  $I_0$ , since all points of  $M \cup I_0$  are flat. As for the Möbius Cascade  $\mathbb{M}$  in Fig.6, we will show that it *is* infinitely-twisted topologically between  $I_1$  and  $I_0$ .

We note that in this paper the term “surface” means a two-dimensional manifold with or without boundary. The term “pseudo”-surface intuitively suggests it is almost a surface together with some singular points, and we assume in this paper at least that any “pseudo”-surface  $X$  is a quotient image of some surface, and the set  $\text{Int}(X)$  forms an open dense subset of  $X$ , where  $\text{Int}(X)$  consists of every point with an open neighborhood in  $X$  homeomorphic to the open disc. For a pseudo-surface  $X$  we can define its “boundary” by  $\partial X = X \setminus \text{Int}(X)$ .

## 2. CONSTRUCTION OF THE MÖBIUS CASCADE

We aim to construct the geometric figure as in Fig.6. First, let  $C$  be the standard Cantor set on the unit interval  $I = [0, 1]$ , and recall its property that each  $x \in C$  is expressed uniquely as the triadic expansion:

$$x = \frac{x_1}{3} + \frac{x_2}{3^2} + \frac{x_3}{3^3} + \frac{x_4}{3^4} + \cdots = .x_1x_2x_3x_4 \cdots_{(3)} \quad \text{where } x_i = 0, 2.$$

Let  $FS(0, 2)$  denote the set of all finite strings of the two digits 0, 2, that is,  $\sigma \in FS(0, 2)$  iff  $\sigma : \{1, 2, \dots, k\} \rightarrow \{0, 2\}$  for some  $k \in \omega = \{0, 1, 2, \dots\}$ . We include the case  $k = 0$  so that the empty string  $\emptyset$  belongs to  $FS(0, 2)$ . Define  $|\sigma| = k$  if  $\text{dom}(\sigma) = \{1, 2, \dots, k\}$ ; in particular,  $|\emptyset| = 0$ . Put

$$\begin{aligned} I(\sigma) &= \left[ \frac{\sigma(1)}{3} + \cdots + \frac{\sigma(k)}{3^k}, \frac{\sigma(1)}{3} + \cdots + \frac{\sigma(k)}{3^k} + \frac{1}{3^k} \right] \\ &= [.\sigma(1) \cdots \sigma(k) 000 \cdots_{(3)}, .\sigma(1) \cdots \sigma(k) 222 \cdots_{(3)}] \end{aligned}$$

for  $\sigma \in FS(0, 2)$ . For example,  $I(\emptyset) = I = [0, 1]$  and

$$I(0) = [.000 \cdots_{(3)}, .022 \cdots_{(3)}] = [0, 1/3], \quad I(2) = [.200 \cdots_{(3)}, .222 \cdots_{(3)}] = [2/3, 1].$$

Then

$$C = \bigcap_{k \in \omega} \bigcup \{I(\sigma) : \sigma \in FS(0, 2), |\sigma| = k\}.$$

Now consider the following compact figure  $\mathbb{K}$  included in the unit square on the vertical  $xz$ -plane in  $\mathbb{R}^3$ :

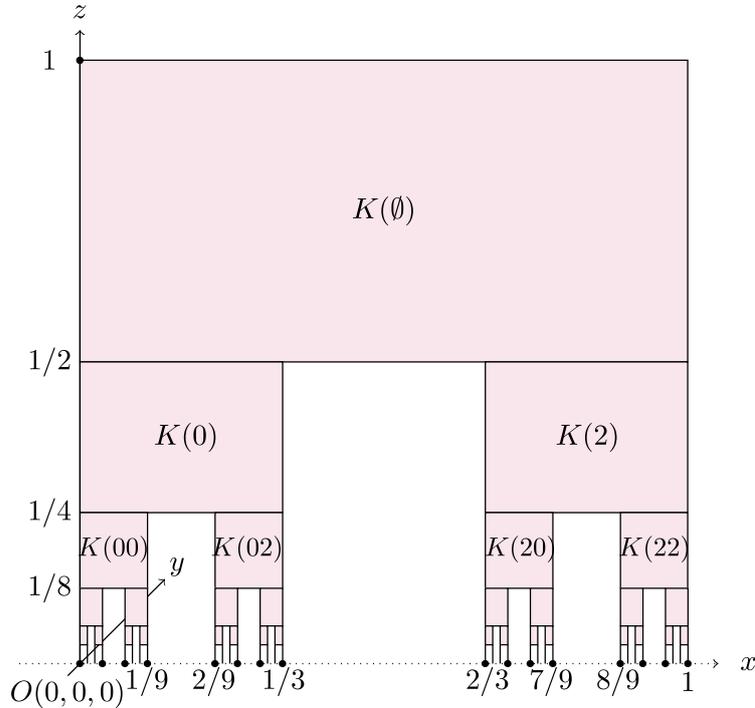
$$\mathbb{K} = C_0 \cup \{K(\sigma) : \sigma \in FS(0, 2)\} \subseteq [0, 1] \times \{0\} \times [0, 1]$$

where  $C_0 = C \times \{(0,0)\}$  is the Cantor set on the  $x$ -axis, and each  $K(\sigma)$  is a rectangular block such that

$$K(\sigma) = I(\sigma) \times \{0\} \times [2^{-|\sigma|-1}, 2^{-|\sigma|}].$$

This figure  $\mathbb{K}$  is illustrated in Fig.2 and looks like a cascade, so we call it *the Cantor Cascade*. We note that this  $\mathbb{K}$  is topologically just a two-dimensional closed disc surrounded by a Jordan curve since we can shrink (deformation retract) the unit square  $[0, 1] \times \{0\} \times [0, 1]$  to  $\mathbb{K}$ ; for instance, we can push up the interval  $[1/3, 2/3] \times \{(0,0)\}$  gradually to the  $\Pi$ -shaped

arc  $(1/3, 0, 0) \rightarrow (1/3, 1/2, 0) \rightarrow (2/3, 1/2, 0) \rightarrow (2/3, 0, 0)$ . We are going



**Fig. 2.** The Cantor Cascade  $\mathbb{K}$  whose bottom is the Cantor set.

to twist this  $\mathbb{K}$  infinitely many times, and for this purpose we set a three-dimensional framework. Let  $D(\sigma)$  denote the closed disc on the  $xy$ -plane with the diameter  $I(\sigma) \times \{0\}$  on the  $x$ -axis, in particular,  $D(\emptyset)$  is the disc  $(x - 1/2)^2 + y^2 \leq 1/4$  with the diameter  $[0, 1] \times \{0\}$ . Consider the solid cylinder  $D(\sigma) \times [2^{-|\sigma|-1}, 2^{-|\sigma|}]$  with the base  $D(\sigma)$  and height  $2^{-|\sigma|-1}$ , which we call  $B(\sigma)$ , *the block of  $\sigma$* . Assembling these blocks, we build the following solid figure  $\mathbb{C}\mathbb{I}$  in  $\mathbb{R}^3$ :

$$\mathbb{C}\mathbb{I} = C_0 \cup \cup \{B(\sigma) : \sigma \in FS(0, 2)\} \subseteq D(\emptyset) \times [0, 1] \subseteq \mathbb{R}^3$$

$$\text{where } B(\sigma) = D(\sigma) \times [2^{-|\sigma|-1}, 2^{-|\sigma|}].$$

This looks like an icicle generated by the blocks falling down to the Cantor set  $C_0$ , and its cross-section through the  $xz$ -plane  $y = 0$  coincides with the Cantor cascade  $\mathbb{K}$ . So, we call it *the Cantor Icicle*  $\mathbb{C}\mathbb{I}$ ; see Fig.3. (In some literature, this figure is described as “the Cantor tree, upside down and thickened”.) This  $\mathbb{C}\mathbb{I}$  is topologically just a three-dimensional ball. Note that both  $\mathbb{C}\mathbb{I}$  and  $\mathbb{K}$  are fractal in the following sense. Let  $\Phi_0, \Phi_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be contractions such that

$$\Phi_0(x, y, z) = (x/3, y/3, z/2), \quad \Phi_2(x, y, z) = (2/3 + x/3, y/3, z/2),$$

and define their compositions

$$\Phi_\sigma = \Phi_{\sigma(k)} \circ \cdots \circ \Phi_{\sigma(2)} \circ \Phi_{\sigma(1)} : \mathbb{R}^3 \rightarrow \mathbb{R}^3.$$

for  $\sigma \in FS(0, 2)$ ,  $|\sigma| = k$ , with the convention  $\Phi_\emptyset = id$ . Then

$$\Phi_\sigma(B(\emptyset)) = B(\sigma), \quad \Phi_\sigma(K(\emptyset)) = K(\sigma)$$

and

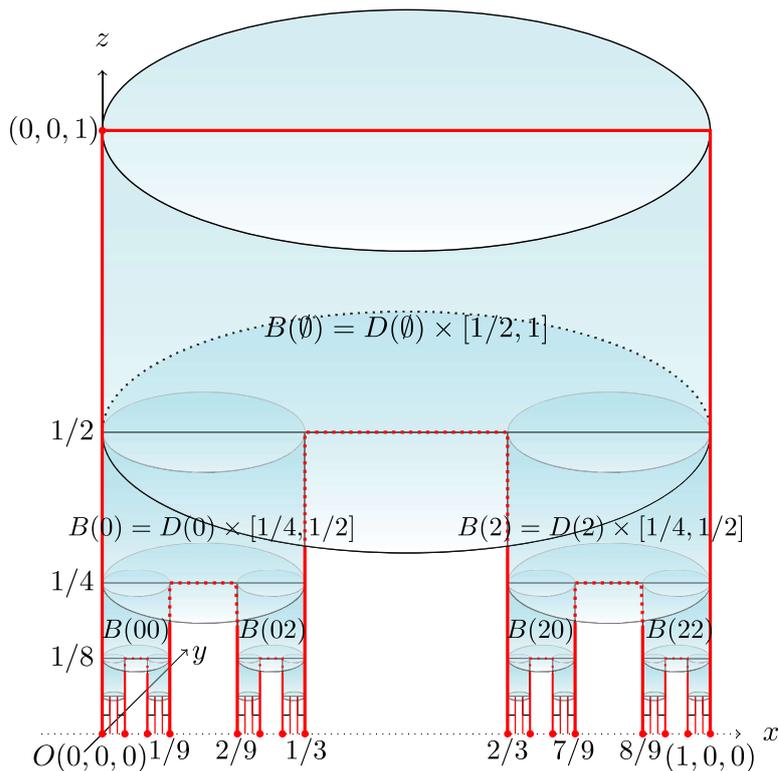
$$\mathbb{CI} = cl(\cup\{\Phi_\sigma(B(\emptyset)) : \sigma \in FS(0, 2)\}) = (\cup\{B(\sigma) : \sigma \in FS(0, 2)\}) \cup C_0,$$

$$\mathbb{K} = cl(\cup\{\Phi_\sigma(K(\emptyset)) : \sigma \in FS(0, 2)\}) = (\cup\{K(\sigma) : \sigma \in FS(0, 2)\}) \cup C_0,$$

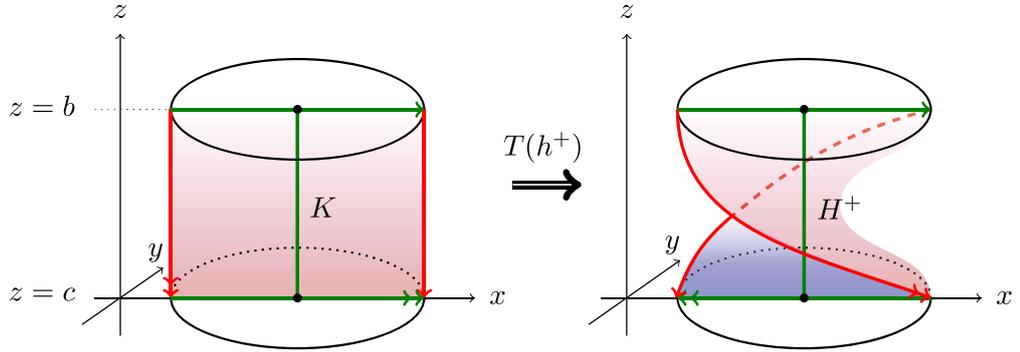
where  $cl$  means the closure in  $\mathbb{R}^3$ . So, we may say that  $\mathbb{CI}$  is a fractal set generated by the initiator  $B(\emptyset) = D(\emptyset) \times [1/2, 1]$  and contractions  $\Phi_0, \Phi_2$ . Consequently,  $\mathbb{K}$  is a fractal set generated by the initiator  $K(\emptyset)$  and contractions  $\Phi_0, \Phi_2$ . Note also that for each point  $(x, 0, 0) \in C_0$ , where  $x = .x_1x_2x_3 \cdots_{(3)} \in C$ , the collection

$$(D(x_1x_2 \cdots x_k) \times [0, 2^{-k}]) \cap \mathbb{CI} \quad \text{for } k = 1, 2, \dots$$

forms a neighborhood base at the point  $(x, 0, 0) \in \mathbb{CI}$ .



**Fig. 3.** The Cantor Icicle  $\mathbb{CI}$  whose cross-section through the  $xz$ -plane  $y = 0$  is the Cantor Cascade  $\mathbb{K}$ .



**Fig. 4.** The map  $T(h^+)$  transforms the rectangular cross-section  $K$  of  $D^2 \times [c, b]$  into the right-handed helicoid  $H^+$ .

Next, we twist/turn this  $\mathbb{C}\mathbb{I}$  in order to twist/turn  $\mathbb{K}$ . Consider in general a horizontal *twist-turn* of a cylinder  $D^2 \times [a, b]$  in  $\mathbb{R}^3$  where  $D^2$  is some closed disc on the  $xy$ -plane with its diameter on the  $x$ -axis. For any continuous function  $h : [a, b] \rightarrow [-\pi, \pi]$  define a transformation  $T(h) : D^2 \times [a, b] \rightarrow D^2 \times [a, b]$  by

$$T(h)(r, \theta, z) = (r, \theta + h(z), z),$$

where  $(r, \theta)$  stands for the polar coordinates around the center of  $D^2$ . This  $T(h)$  rotates each horizontal disc at height  $z$  by the angle  $h(z)$ . Specifically, we use functions  $h = h^+$  or  $h = h^-$  such that

$$h^+(z) = \pi(b - z)/(b - c) \text{ for } c \leq z \leq b, \text{ and } h^+(z) = \pi \text{ for } a \leq z \leq c$$

where  $c \in [a, b]$ , and  $h^-$  is defined by  $h^- = -h^+$ . We call the corresponding transformation  $T(h^+)$  or  $T(h^-)$  as *the twist-turn of  $D^2 \times [a, b]$* , which half-twists the upper cylinder  $D^2 \times [c, b]$  fixing the top  $D^2 \times \{b\}$ , and half-turns the lower cylinder  $D^2 \times [a, c]$ . We may denote them precisely as  $T(h^+) = T_R(D^2; a, c, b)$  and  $T(h^-) = T_L(D^2; a, c, b)$  where the subscript  $R$  or  $L$  indicates *right-handed* or *left-handed*, respectively. We may denote either of them simply as  $T(D^2; a, c, b)$  when we don't care about right- or left-handedness. Note that  $T(h^+)$  and  $T(h^-)$  are identical on the intermediate disc  $D^2 \times \{c\}$  just as the reflection through its center. Let  $K$  be the rectangular cross-section of  $D^2 \times [c, b]$  through the  $xz$ -plane, then  $T(h^+)(K)$  or  $T(h^-)(K)$  is the right- or left-handed helicoid, respectively, so, let us put  $T(h^+)(K) = H^+$  and  $T(h^-)(K) = H^-$ . Observe that the correspondence  $(r, h(z), z) \rightarrow (r, -h(z), z)$  provides the natural homeomorphism  $H^+ \approx H^-$ , fixing the top and bottom of the helicoids, which is essentially generated by the conjugation on the  $xy$ -plane regarded as the complex plane. So, we may say that the helicoids  $H^+$  and  $H^-$  are *conjugate*. The right-handed helicoid  $H^+$  is shown in Fig.4.

Now, we want to half-twist every block  $B(\sigma) = D(\sigma) \times [2^{-|\sigma|-1}, 2^{-|\sigma|}]$  of  $\mathbb{C}\mathbb{I}$  to define a continuous deformation of the whole  $\mathbb{C}\mathbb{I}$ , so we adopt the

following consecutive process. Define  $\mathbb{C}\mathbb{I}_k$  for  $k \in \omega$  by

$$\mathbb{C}\mathbb{I}_k = \cup\{B(\sigma) : |\sigma| < k\} \cup \cup\{D(\sigma) \times [0, 2^{-k}] : |\sigma| = k\},$$

which includes also  $\cup\{B(\sigma) : |\sigma| = k\}$ ; Then

$$\mathbb{C}\mathbb{I}_0 \supseteq \mathbb{C}\mathbb{I}_1 \supseteq \cdots \supseteq \mathbb{C}\mathbb{I} = \cap_{k \in \omega} \mathbb{C}\mathbb{I}_k,$$

where, for example,  $\mathbb{C}\mathbb{I}_0 = D(\emptyset) \times [0, 1] \supseteq B(\emptyset)$ , and  $\mathbb{C}\mathbb{I}_1 = B(\emptyset) \cup ((D(0) \cup D(2)) \times [0, 1/2]) \supseteq B(\emptyset) \cup B(0) \cup B(2)$ .

First, we define a deformation  $T_0$  of  $\mathbb{C}\mathbb{I}_0$  by

$$T_0 = T(D^2; 0, 1/2, 1) : \mathbb{C}\mathbb{I}_0 = D(\emptyset) \times [0, 1] \rightarrow \mathbb{C}\mathbb{I}_0$$

which half-twists the upper block  $B(\emptyset) = D(\emptyset) \times [1/2, 1]$ , half-turns the lower cylinder  $D(\emptyset) \times [0, 1/2]$ , and is essentially derived by the reflection of the intermediate disc  $D(\emptyset) \times \{1/2\}$  through its center. Next, define a deformation  $T_1$  of  $\mathbb{C}\mathbb{I}_1$

$$T_1 : \mathbb{C}\mathbb{I}_1 = B(\emptyset) \cup ((D(0) \cup D(2)) \times [0, 1/2]) \rightarrow \mathbb{C}\mathbb{I}_1$$

by setting  $T_1 \upharpoonright B(\emptyset) = id$ , and

$$T_1 \upharpoonright D(i) \times [0, 1/2] = T(D(i); 0, 1/4, 1/2) \text{ for } i = 0, 2.$$

So, for each  $i = 0, 2$ ,  $T_1$  half-twists  $B(i) = D(i) \times [1/4, 1/2]$ , and half-turns  $D(i) \times [0, 1/4]$ , reflecting the disc  $D(i) \times \{1/4\}$ . In general, define, for each  $k \in \omega$ , a deformation  $T_k$  of  $\mathbb{C}\mathbb{I}_k$

$$T_k : \mathbb{C}\mathbb{I}_k = \cup\{B(\sigma) : |\sigma| < k\} \cup \cup\{D(\sigma) \times [0, 2^{-k}] : |\sigma| = k\} \rightarrow \mathbb{C}\mathbb{I}_k$$

by setting  $T_k \upharpoonright \cup\{B(\sigma) : |\sigma| < k\} = id$ , and for each  $\sigma$  with  $|\sigma| = k$ ,

$$T_k \upharpoonright D(\sigma) \times [0, 2^{-k}] = T(D(\sigma); 0, 2^{-k-1}, 2^{-k})$$

which half-twists  $B(\sigma) = D(\sigma) \times [2^{-k-1}, 2^{-k}]$  and half-turns  $D(\sigma) \times [0, 2^{-k-1}]$ , reflecting the disc  $D(\sigma) \times \{2^{-k-1}\}$ . Then, we define  $T_\infty : \mathbb{C}\mathbb{I} \rightarrow \mathbb{C}\mathbb{I}$  by

$$T_\infty = \lim_{k \rightarrow \infty} T_k \circ \cdots \circ T_1 \circ T_0 : \mathbb{C}\mathbb{I} = \cap_{k \in \omega} \mathbb{C}\mathbb{I}_k \rightarrow \mathbb{C}\mathbb{I} = \cap_{k \in \omega} \mathbb{C}\mathbb{I}_k.$$

Observe that though this definition apparently takes the form of infinite composition,  $T_\infty$  on each block  $B(\sigma)$  for  $\sigma$  with  $|\sigma| = k$  is actually identical to the finite composition  $T_k \circ \cdots \circ T_1 \circ T_0$ . Therefore we can see that  $T_\infty \upharpoonright \mathbb{C}\mathbb{I} \setminus C_0$  is a homeomorphism, made of half-twists and half-turns. We need to examine the behavior of  $T_\infty$  on the bottom Cantor set  $C_0$ , in particular, its continuity. Take any point  $(x, 0, 0) \in C_0$  where  $x = .x_1 x_2 x_3 \cdots_{(3)} \in C$ .  $T_0$  half-turns the cylinder  $D(\emptyset) \times [0, 1/2]$ , reflecting the diameter  $[0, 1] \times \{(0, 0)\} = I(\emptyset) \times \{(0, 0)\}$  of its bottom disc. So,  $T_0(x, 0, 0)$  is expressed as

$$T_0(x, 0, 0) = (x^{(0)}, 0, 0), \quad x^{(0)} = .\hat{x}_1 \hat{x}_2 \hat{x}_3 \cdots_{(3)},$$

where the ‘‘hat’’ notation means the alternation of digits 0 and 2 :  $\hat{\alpha} = 2 - \alpha$  for  $\alpha = 0, 2$ . Then this point  $(x^{(0)}, 0, 0)$  lies on the diameter  $I(\hat{x}_1) \times \{(0, 0)\}$

of the bottom disc of the cylinder  $D(\widehat{x}_1) \times [0, 1/4]$ , and this diameter is reflected by  $T_1$ . Therefore

$$T_1 \circ T_0(x, 0, 0) = T_1(x^{(0)}, 0, 0) = (x^{(1)}, 0, 0) \text{ where } x^{(1)} = .\widehat{x}_1 x_2 x_3 \cdots_{(3)}.$$

Similarly,  $T_2$  reflects  $I(\widehat{x}_1 x_2) \times \{(0, 0)\}$  so that

$$T_2 \circ T_1 \circ T_0(x, 0, 0) = T_2(x^{(1)}, 0, 0) = (x^{(2)}, 0, 0); \quad x^{(2)} = .\widehat{x}_1 x_2 \widehat{x}_3 \widehat{x}_4 \cdots_{(3)}.$$

Repeating this way, we finally get

$$T_\infty(x, 0, 0) = (x^*, 0, 0)$$

where  $x^*$  stands for

$$x^* = .\widehat{x}_1 x_2 \widehat{x}_3 x_4 \widehat{x}_5 x_6 \cdots_{(3)}.$$

Thus, the map  $T_\infty$  on  $C_0$  is essentially the same as the map on the Cantor set  $C$  such that

$$x = .x_1 x_2 x_3 x_4 x_5 x_6 \cdots_{(3)} \rightarrow x^* = .\widehat{x}_1 x_2 \widehat{x}_3 x_4 \widehat{x}_5 x_6 \cdots_{(3)},$$

which is an *involution* because

$$(x^*)^* = .\widehat{\widehat{x}}_1 x_2 \widehat{\widehat{x}}_3 x_4 \widehat{\widehat{x}}_5 x_6 \cdots_{(3)} = x.$$

This involution naturally induces an involution of strings

$$FS(0, 2) \ni \sigma \rightarrow \sigma^* \in FS(0, 2)$$

such that  $\sigma^* = (x_1, x_2, \dots, x_k)^*$  is defined by

$$\sigma^* = (\widehat{x}_1, x_2, \widehat{x}_3, x_4, \dots, \widetilde{x}_k)$$

where  $\widetilde{x}_k = \widehat{x}_k$  if  $k$  is odd and  $\widetilde{x}_k = x_k$  if  $k$  is even. We assume  $\emptyset^* = \emptyset$ . Using this  $\sigma^*$  we have

$$T_\infty(D(\sigma) \times [0, 2^{-k}]) = D(\sigma^*) \times [0, 2^{-k}],$$

and

$$T_\infty(B(\sigma)) = B(\sigma^*) \quad \text{or} \quad T_\infty(B(\sigma^*)) = B(\sigma).$$

Since the collection  $D(x_1, x_2, \dots, x_k) \times [0, 2^{-k}]$ ,  $k = 1, 2, \dots$  forms the neighborhood base at the point  $(x, 0, 0) = (.x_1 x_2, \dots_{(3)}, 0, 0)$  of  $C_0$ , we see that  $T_\infty : \mathbb{C}\mathbb{I} \rightarrow \mathbb{C}\mathbb{I}$  is continuous at every point of  $C_0$ , and consequently, it is continuous on the whole  $\mathbb{C}\mathbb{I}$ . Noting that an involution is a bijection, we can conclude that:

**Assertion 2.1.**  $T_\infty$  is a homeomorphism  $T_\infty : \mathbb{C}\mathbb{I} \approx \mathbb{C}\mathbb{I}$  that exchanges points  $(x, 0, 0)$  and  $(x^*, 0, 0)$  on the bottom Cantor set  $C_0$ , for every  $x \in C$ .

In the above construction of  $T_\infty$  we did not care about the right- or left-handedness of twist-turn of blocks or cylinders since it would not affect the topology of our final resultant, though will affect if we live only inside  $\mathbb{R}^3$ .

*Remark 2.2.* Let us compute some  $x^*$  :

$$0^* = .\widehat{0}0\widehat{0}0\widehat{0}0 \cdots_{(3)} = .202020 \cdots_{(3)} = 3/4,$$

$$(1/3)^* = .\widehat{0}2\widehat{2}2\widehat{2}2 \cdots_{(3)} = .220202 \cdots_{(3)} = 11/12.$$

For  $x = .x_1 x_2 x_3 x_4 \cdots_{(3)} \in C$ , we have  $1 - x = .\widehat{x}_1 \widehat{x}_2 \widehat{x}_3 \widehat{x}_4 \cdots_{(3)}$  and

$$x^* + (1 - x)^* = 1$$

since  $.\widehat{x}_1 x_2 \widehat{x}_3 x_4 \cdots_{(3)} + .x_1 \widehat{x}_2 x_3 \widehat{x}_4 \cdots_{(3)} = .2222 \cdots_{(3)} = 1$ . Utilizing this formula together with  $(x^*)^* = x$ , we get

$$0^* = 3/4, \quad (3/4)^* = 0, \quad 1^* = 1/4, \quad (1/4)^* = 1,$$

$$(1/3)^* = 11/12, \quad (11/12)^* = 1/3, \quad (2/3)^* = 1/12, \quad (1/12)^* = 2/3.$$

In general, for two points  $x = .x_1 x_2 x_3 \cdots_{(3)}$  and  $y = .y_1 y_2 y_3 \cdots_{(3)}$  on  $C$ , we can compute the difference of  $x^*$  and  $y^*$  by

$$\begin{aligned} x^* - y^* &= \left( \frac{\widehat{x}_1}{3} + \frac{x_2}{3^2} + \frac{\widehat{x}_3}{3^3} + \frac{x_4}{3^4} + \cdots \right) - \left( \frac{\widehat{y}_1}{3} + \frac{y_2}{3^2} + \frac{\widehat{y}_3}{3^3} + \frac{y_4}{3^4} + \cdots \right) \\ &= \frac{\widehat{x}_1 - \widehat{y}_1}{3} + \frac{x_2 - y_2}{3^2} + \frac{\widehat{x}_3 - \widehat{y}_3}{3^3} + \frac{x_4 - y_4}{3^4} + \cdots \\ &= \frac{y_1 - x_1}{3} + \frac{x_2 - y_2}{3^2} + \frac{y_3 - x_3}{3^3} + \frac{x_4 - y_4}{3^4} + \cdots \end{aligned}$$

hence

$$|x^* - y^*| \leq \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{3^k}.$$

But note that it is not necessarily true that  $|x^* - y^*| \leq |x - y|$ ; for example,

$$(1/3)^* - (2/3)^* = 11/12 - 1/12 = 5/6 = (2/3 - 1/3) \cdot 5/2,$$

$$(1/4)^* - (3/4)^* = 1 - 0 = 1 = (3/4 - 1/4) \cdot 2.$$

Though geometrically it would be obvious that  $x^* \neq x$  for every  $x \in C$ , we can actually evaluate  $|x^* - x|$  as follows:

$$|x^* - x| = \left| \frac{\widehat{x}_1 - x_1}{3} + \frac{\widehat{x}_3 - x_3}{3^3} + \cdots \right| = \left| \frac{(\pm 2)}{3} + \frac{(\pm 2)}{3^3} + \cdots \right| = \frac{2}{3} \left( 1 + \frac{(\pm 1)}{3^2} + \frac{(\pm 1)}{3^4} + \cdots \right),$$

hence

$$1/2 < 7/12 = \frac{2}{3} \left( 1 - \frac{1}{3^2} - \frac{1}{3^4} - \cdots \right) \leq |x^* - x| \leq \frac{2}{3} \left( 1 + \frac{1}{3^2} + \frac{1}{3^4} + \cdots \right) = 3/4.$$

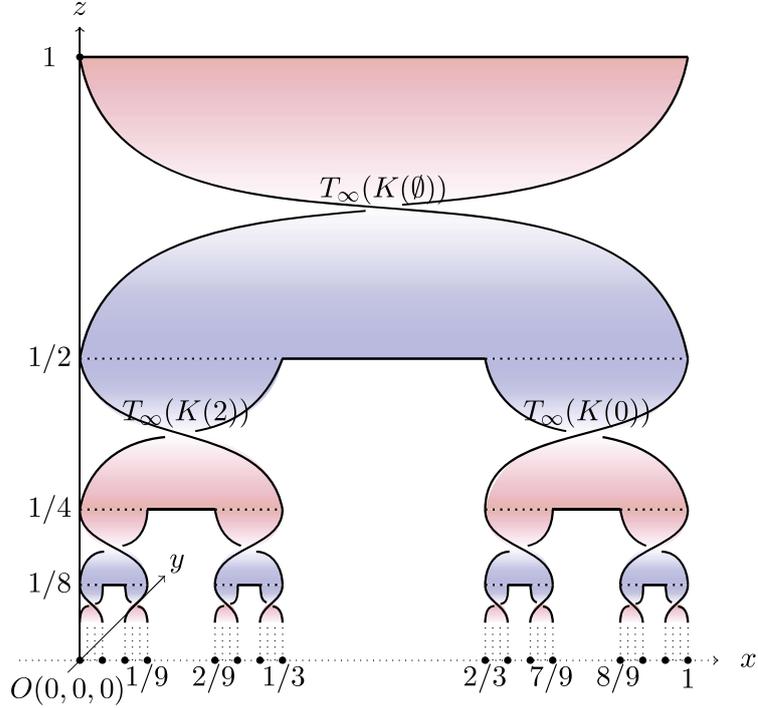
□

Since  $\mathbb{K} \subseteq \mathbb{C}\mathbb{I}$ , the homeomorphism  $T_\infty : \mathbb{C}\mathbb{I} \approx \mathbb{C}\mathbb{I}$  induces the homeomorphism

$$T_\infty \upharpoonright \mathbb{K} : \mathbb{K} \approx \widetilde{\mathbb{K}}$$

where  $\widetilde{\mathbb{K}}$  denotes the image

$$\widetilde{\mathbb{K}} = T_\infty(\mathbb{K}) = C_0 \cup \cup \{T_\infty(K(\sigma)) : \sigma \in FS(0, 2)\}.$$



**Fig. 5.**  $\tilde{\mathbb{K}} = T_\infty(\mathbb{K})$ , which is well twisted “geometrically” by  $T_\infty$ , but not yet twisted “topologically” because  $\tilde{\mathbb{K}} \approx \mathbb{K} \approx \mathbb{D}^2$ .

Let  $|\sigma| = k$ , then since  $K(\sigma) \subseteq B(\sigma)$ , we have

$$T_\infty(K(\sigma)) = T_k \circ T_{k-1} \circ \cdots \circ T_1 \circ T_0(K(\sigma)).$$

Since  $T_k$  works on  $B(\sigma)$  as a half-twist, and  $T_{k-1}, \dots, T_1, T_0$  work on  $B(\sigma)$  as half-turns, each  $T_\infty(K(\sigma))$  is a helicoid as shown in Fig.4, so that  $\tilde{\mathbb{K}} = T_\infty(\mathbb{K})$  would look like Fig.5. Though this  $\tilde{\mathbb{K}}$  is quite twisted “geometrically”, the fact  $\tilde{\mathbb{K}} \approx \mathbb{K}$  tells that it is *flat* (i.e., embeddable into the 2-dimensional plane), not twisted “topologically”. So, we carefully call  $\tilde{\mathbb{K}}$  the *geometrically twisted*  $\mathbb{K}$ . In order to make the geometric twist into the topological one, we need to confine the bottom Cantor set into the interval  $[0, 1]$ . Let  $S$  be the round surface defined by  $y^2 - y + z^2 - z = 0$ ,  $y \geq 0$ ,  $0 \leq x \leq 1$ , which is the part  $y \geq 0$  of a tube with the axis  $y = z = 1/2$ , connecting the top interval  $I_1 = [0, 1] \times \{(0, 1)\}$  and the bottom one  $I_0 = [0, 1] \times \{(0, 0)\}$ . Join this  $S$  to the geometrically twisted  $\tilde{\mathbb{K}}$ , and denote the resultant by  $\mathbb{M}$ :

$$\mathbb{M} = \tilde{\mathbb{K}} \cup S \subseteq \mathbb{R}^3.$$

Put  $M(\sigma) = B(\sigma) \cap \mathbb{M}$ , which is a helicoid such that

$$M(\sigma) = B(\sigma) \cap \tilde{\mathbb{K}} = T_\infty(B(\sigma^*)) \cap \tilde{\mathbb{K}}$$



Möbius Cascade  $\mathbb{M}$  is not locally flat around any point of the bottom Cantor set  $C_0$ .

*Proof.* Let  $U$  be any open set in  $\tilde{\mathbb{K}} \cup I_0$  containing some point of  $C_0$ . Then we need to show that this  $U$  is not embeddable into  $\mathbb{R}^2$ . Due to the fractal structure of  $\tilde{\mathbb{K}}$ , it is easy to see that  $U$  includes a homeomorphic copy of  $\tilde{\mathbb{K}} \cup I_0$ . Therefore, we need only to show that

(\*)  $\tilde{\mathbb{K}} \cup I_0$  is not embeddable into  $\mathbb{R}^2$ .

Put  $A = (1/3, 0, 0)$ ,  $B = (2/3, 0, 0)$ ,  $C = (7/9, 0, 0)$ ,  $D = (8/9, 0, 0)$ , and consider the subintervals  $\overline{AB}$ ,  $\overline{CD}$  of  $I_0$ . We will show that the subspace  $\tilde{\mathbb{K}} \cup \overline{AB} \cup \overline{CD}$  of  $\tilde{\mathbb{K}} \cup I_0$  is not embeddable into  $\mathbb{R}^2$ . Let  $A_*$ ,  $B_*$ ,  $C_*$ ,  $D_*$  be points such that

$$A = T_\infty(A_*), B = T_\infty(B_*), C = T_\infty(C_*), D = T_\infty(D_*),$$

i.e.,

$$A_* = (11/12, 0, 0), B_* = (1/12, 0, 0), C_* = (1/36, 0, 0), D_* = (11/36, 0, 0),$$

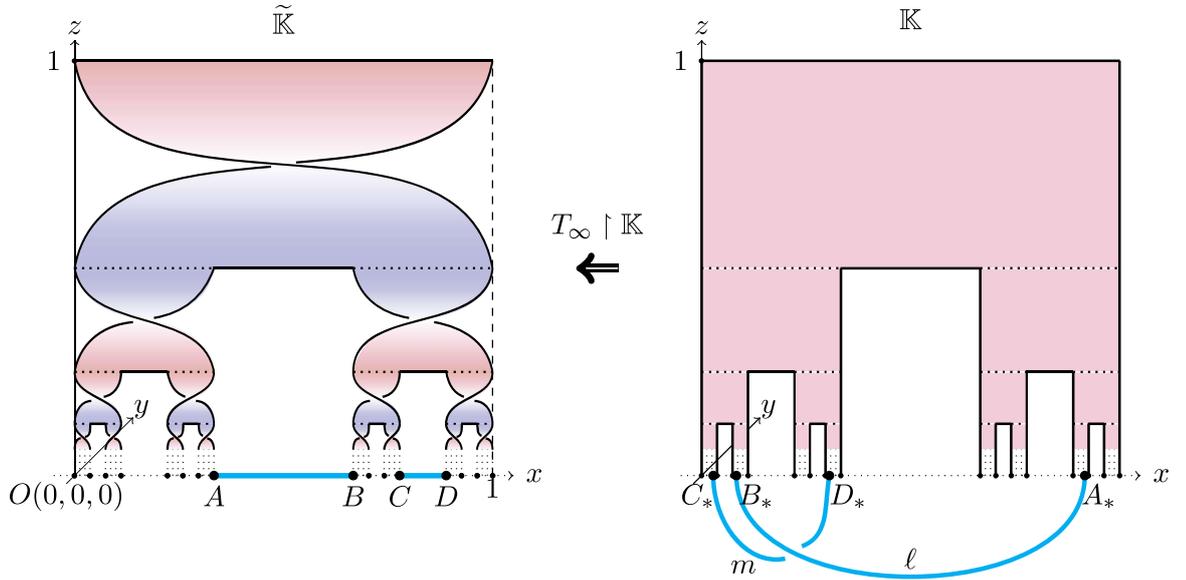
since  $1/3 = (11/12)^*$ ,  $2/3 = (1/12)^*$ ,  $7/9 = (1/36)^*$ ,  $8/9 = (11/36)^*$ ; see Remark 2.2. Note here that the involution  $T_\infty = T_\infty^{-1}$  on the bottom Cantor set changes the order  $A < B < C < D$  on  $I_0$  into  $C_* < B_* < D_* < A_*$ , as shown in Fig.7. (To see this change of order, we need to take care only of  $T_2 \circ T_1 \circ T_0$  instead of  $T_\infty$ .) Since  $\mathbb{K}$  is placed in the three-dimensional space  $\mathbb{R}^3$ , we can connect  $A_*$ ,  $B_*$  by an arc  $\ell$ , and  $C_*$ ,  $D_*$  by an arc  $m$  such that  $\ell, m$  are disjoint and outside of  $\mathbb{K}$ , i.e.,  $\ell \cap \mathbb{K} = \{A_*, B_*\}$ ,  $m \cap \mathbb{K} = \{C_*, D_*\}$ . Then, since  $\ell, m$  are disjoint, it is easy to extend the homeomorphism  $T_\infty \upharpoonright \mathbb{K} : \mathbb{K} \approx \tilde{\mathbb{K}}$  to the homeomorphism

$$\mathbb{K} \cup \ell \cup m \approx \tilde{\mathbb{K}} \cup \overline{AB} \cup \overline{CD}.$$

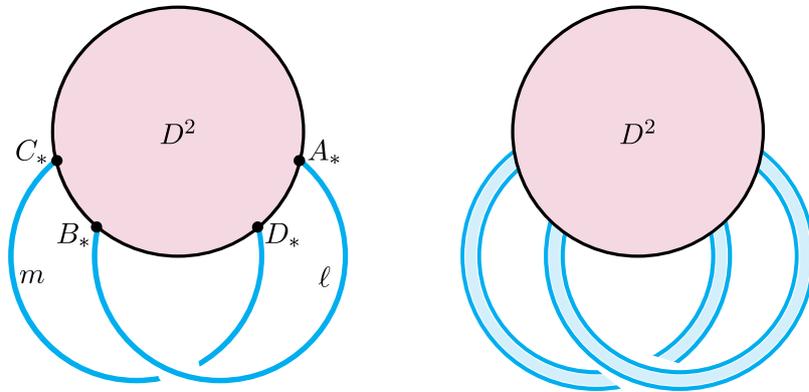
So, it suffices to show that  $\mathbb{K} \cup \ell \cup m$  is not planar. Since  $\mathbb{K} \approx D^2$ , we can “inflate”  $\mathbb{K}$  to get  $D^2 \cup \ell \cup m \approx \mathbb{K} \cup \ell \cup m$  as the left figure in Fig.8, where  $S^1 = \partial D^2$  is partitioned into four open arcs  $A_* \hat{C}_*$ ,  $C_* \hat{B}_*$ ,  $B_* \hat{D}_*$ ,  $D_* \hat{A}_*$  by the points  $A_*$ ,  $C_*$ ,  $B_*$ ,  $D_*$ , counter-clockwise ordered. Though this figure is well known to be non-planar as the Crossed Arcs Lemma or an application of the complete bipartite graph  $K_{3,3}$ , we present here an alternative proof: Suppose this figure were planar, then, so would be the right figure in Fig.8, where the arcs  $\ell, m$  were a bit widened to tapes. But this right figure is homeomorphic to the punctured torus  $T^2 \setminus (disc)$ , which is non-planar.  $\square$

As mentioned in Remark 2.2 the homeomorphism  $T_\infty$  moves every point  $(x, 0, 0) \in C_0$  to  $(x^*, 0, 0) \in C_0$ , different from  $(x, 0, 0)$ . There could be other homeomorphisms between  $\mathbb{K}$  and  $\tilde{\mathbb{K}} = T_\infty(\mathbb{K})$ , different from  $T_\infty$ , and let  $\phi$  be any such homeomorphism  $\phi : \mathbb{K} \approx \tilde{\mathbb{K}}$ . Then we can show that  $\phi$  still moves “almost all” points of  $C_0$  in the following sense. Let  $F_0(\phi) = \{(x, 0, 0) \in C_0 : \phi(x, 0, 0) = (x, 0, 0)\}$  denote the set of all fixed points of  $\phi \upharpoonright C_0$ , which is obviously a closed subset of  $C_0$ . Then

**Assertion 2.4.**  $C_0 \setminus F_0(\phi)$  is an open dense subset of  $C_0$ .



**Fig. 7.** The subspace  $\tilde{\mathbb{K}} \cup \overline{AB} \cup \overline{CD}$  of  $\tilde{\mathbb{K}} \cup I_0$  is not embeddable into the plane  $\mathbb{R}^2$ , where  $A = (1/3, 0, 0)$ ,  $B = (2/3, 0, 0)$ ,  $C = (7/9, 0, 0)$ ,  $D = (8/9, 0, 0)$  are lined on  $I_0$  in this order.



**Fig. 8.** If the arcs  $\ell, m$  in the left figure were a bit widened to tapes, we get the right figure homeomorphic to the punctured torus  $T^2 \setminus (disc)$ , which, in turn, deformation retracts to the left figure.

*Proof.* Suppose otherwise; then we can find an open subinterval  $J$  of  $I_0$  such that  $\emptyset \neq J \cap C_0 \subseteq F_0(\phi)$ , which means that  $\phi$  is an identity map on  $J \cap C_0$ . Then  $\phi$  can be extended to the homeomorphism  $\Phi : \mathbb{K} \cup J \approx \tilde{\mathbb{K}} \cup J$  by setting

$\Phi$  to be identity on  $J$ . This fact implies that the open subset  $\widetilde{\mathbb{K}} \cup J$  of  $\widetilde{\mathbb{K}} \cup I_0$  is flat since  $\mathbb{K} \cup J$  is flat, contradicting Assertion 2.3.  $\square$

Note that the essential part of the proof of Assertion 2.3 is that  $\mathbb{M}$  or  $\widetilde{\mathbb{K}} \cup I_0$  includes a copy of the left figure in Fig.8. Though the topological difference between the left and right figures in Fig.8 is subtle (the right figure deformation retracts to the left one), we point out that the “pseudo”-surface  $\mathbb{M}$  can not include any copy of the right figure in Fig.8 which is  $T^2 \setminus (\text{disc})$ , a non-flat surface. This fact is deduced from the next assertion, which also implies that  $\mathbb{M}$  can not contain even a copy of the Möbius Band.

**Assertion 2.5.**  $\mathbb{M} \setminus C_0$  is homeomorphic to  $\mathbb{D}^2 \setminus C$  where  $C$  is a Cantor set lying on the boundary  $\partial \mathbb{D}^2$  of  $\mathbb{D}^2$ . Hence  $\mathbb{M} \setminus C_0$  is flat, and consequently,  $\mathbb{M}$  can not include any copy of a non-flat surface.

*Proof.* Note that  $S \cup (\widetilde{\mathbb{K}} \setminus C_0) \approx \widetilde{\mathbb{K}} \setminus C_0 \approx \mathbb{D}^2 \setminus (\text{a Cantor set on } \partial \mathbb{D}^2)$ . Hence

$$\mathbb{M} \setminus C_0 = (S \setminus C_0) \cup (\widetilde{\mathbb{K}} \setminus C_0) \approx \mathbb{D}^2 \setminus (\text{two disjoint Cantor sets on } \partial \mathbb{D}^2).$$

Since the union of two Cantor sets is homeomorphic to just one Cantor set, we can conclude

$$\mathbb{M} \setminus C_0 \approx \mathbb{D}^2 \setminus (\text{a Cantor set on } \partial \mathbb{D}^2).$$

This implies that  $\mathbb{M} \setminus C_0$  is flat. Suppose  $\mathbb{M}$  includes a copy  $N$  of a surface; then Assertion 2.3 implies  $N \subseteq \mathbb{M} \setminus C_0$ , hence  $N$  must be flat.  $\square$

### 3. NATURAL KNOT FLOWS ON THE MÖBIUS CASCADE

We now consider the flows on the Möbius Cascade  $\mathbb{M}$  naturally derived from the downward flows on the Cantor Cascade  $\mathbb{K}$ . For each  $x \in C$  the vertical line  $(x, 0) \times [0, 1]$  can be seen as the downward flow on the cascade  $\mathbb{K}$ , so that its homeomorphic image by  $T_\infty$  can be viewed as the natural downward flow on  $T_\infty(\mathbb{K})$  passing points

$$(x, 0, 1) \rightarrow (x^{(0)}, 0, 2^{-1}) \rightarrow (x^{(1)}, 0, 2^{-2}) \rightarrow (x^{(2)}, 0, 2^{-3}) \rightarrow \cdots \rightarrow (x^*, 0, 0).$$

Here, recall the notation in §2 that, for  $x = .x_1 x_2 x_3 \cdots_{(3)} \in C$ ,

$$\begin{aligned} x^{(0)} &= .\widehat{x}_1 \widehat{x}_2 \widehat{x}_3 \widehat{x}_4 \cdots_{(3)}, & x^{(1)} &= .\widehat{x}_1 x_2 x_3 x_4 \cdots_{(3)}, & x^{(2)} &= .\widehat{x}_1 x_2 \widehat{x}_3 \widehat{x}_4 \cdots_{(3)}, \\ & & & & & \cdots, & x^* &= .\widehat{x}_1 x_2 \widehat{x}_3 x_4 \widehat{x}_5 x_6 \cdots_{(3)}. \end{aligned}$$

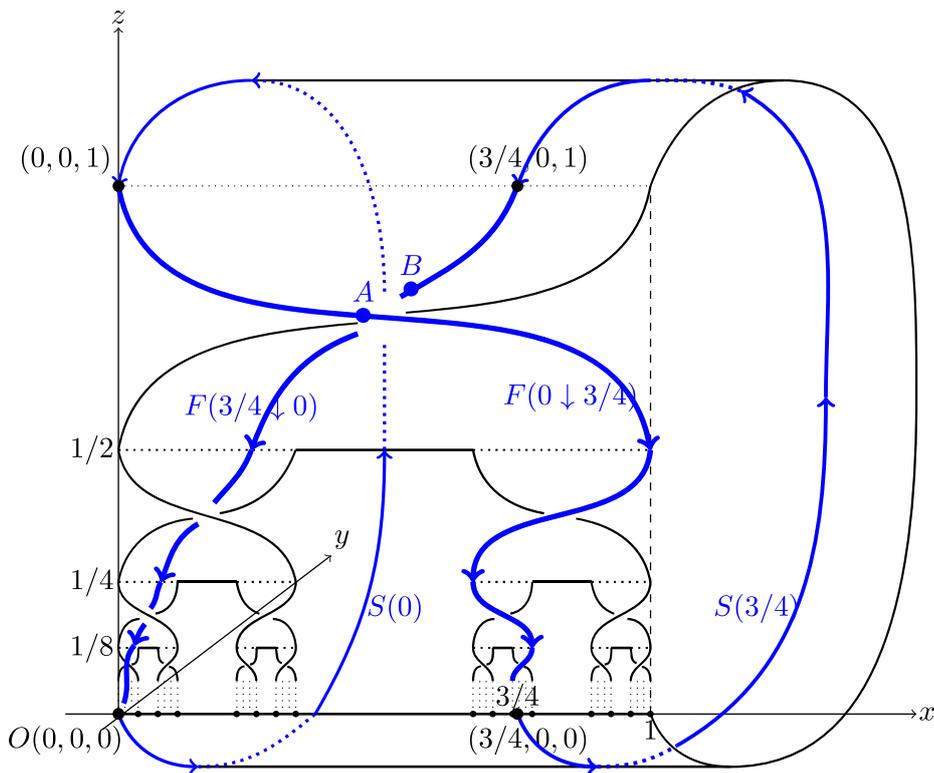
Let us denote this downward flow by  $F(x \downarrow x^*)$ , noting that such a flow can be defined only for  $x \in C$  since any downward flow from  $(x, 0, 1)$  for  $x \in [0, 1] \setminus C$  may stop on the way and does not reach the bottom Cantor set. Replacing  $x$  by  $x^*$ , we can get another flow  $F(x^* \downarrow x)$ . Viewing that these two flows  $F(x \downarrow x^*)$ ,  $F(x^* \downarrow x)$  are on  $\mathbb{M} = T_\infty(\mathbb{K}) \cup S \subseteq \mathbb{R}^3$ , connect them via the upward vertical flows  $S(x)$ ,  $S(x^*)$  on the surface  $S$ , where  $S(t)$  stands for  $S(t) = S \cap (\{t\} \times \mathbb{R}^2)$  for  $t \in [0, 1]$ . Then, we get a loop

$$F(x \downarrow x^*) \cup S(x^*) \cup F(x^* \downarrow x) \cup S(x)$$

passing four points  $(x, 0, 1)$ ,  $(x^*, 0, 0)$ ,  $(x^*, 0, 1)$ ,  $(x, 0, 0)$ . Let us denote this loop by  $F(x \circ x^*)$ , noting that  $F(x \circ x^*) = F(x^* \circ x)$ . Note also that the fact  $x \neq x^*$  (see Remark 2.2) implies  $F(x \downarrow x^*) \cap F(x^* \downarrow x) = \emptyset$ , so that  $F(x \circ x^*)$  is a simple closed curve, in other words, a *knot* in  $\mathbb{R}^3$  winding around  $\mathbb{M}$  twice. So, we may call each  $F(x \circ x^*)$  the *knot flow* on  $\mathbb{M}$ . Fig.9 illustrates  $F(0 \circ 3/4) = F(3/4 \circ 0)$ , a typical knot flow.

**Assertion 3.1.** *For every  $x \in C$  the flow  $F(x \circ x^*)$  is a knot winding around  $\mathbb{M}$  twice, and is unknotted in  $\mathbb{R}^3$ .*

*Proof.* We need to prove the unknottedness. Let  $x \in C$ , and consider the vertical mid plane  $\{1/2\} \times \mathbb{R}^2$ , which divides the knot into two arcs  $F(x \circ x^*) = L \cup R$  where  $L$  and  $R$  stand for the “left” arc and the “right” arc, respectively. Let  $A, B$  be the common end-points of  $L$  and  $R$ , i.e.,  $L \cap R = \{A, B\}$  (see Fig.9 for the case  $x = 0$  or  $3/4$ ). Connect  $A, B$  by the straight arc  $\overline{AB}$ . Then, by transforming the arc  $R$  into  $\overline{AB}$  in  $\mathbb{R}^3$ , the knot  $F(x \circ x^*)$  can be deformed isotopically in  $\mathbb{R}^3$  into  $L \cup \overline{AB}$ , which is unknotted.  $\square$



**Fig. 9.** The arrowed loop indicates the knot flow  $F(0 \circ 3/4) = F(3/4 \circ 0)$  on the Möbius Cascade  $\mathbb{M}$ .

Though, as noted in §2, the topology of  $\mathbb{M}$  does not depend on whether each helicoid  $M(\sigma)$  is a right- or left-handed one, some geometry of  $\mathbb{M}$

confined in  $\mathbb{R}^3$ , like the linking of two knot flows, would depend on the choice of right- or left-handedness. The unknottedness in Assertion 3.1 is independent of such choice, and this fact is noteworthy comparing with the well-known fact that the boundary of the Möbius band made by one half-twist is unknotted, but the boundary of the Möbius band made by *three* half-twists is the Trefoil, which is knotted in  $\mathbb{R}^3$  (cf.[1, 4]).

Note that the knot flow in Fig.9 has the feature that only its half  $S(0) \cup F(0 \downarrow 3/4)$  lies on the boundary of  $\mathbb{M}$ , and it would be interesting to see that the downward flow  $F(0 \downarrow 3/4)$  looks quite twisted compared with the downward flow  $F(3/4 \downarrow 0)$ , due to the fact that 0 is an end-point of the Cantor set  $C \subseteq [0, 1]$ , but  $3/4 = .202020 \dots_{(3)}$  is not; about which we will examine in the next §4.

#### 4. THE BOUNDARY OF $\mathbb{M}$

The boundary  $\partial\mathbb{M} = \mathbb{M} \setminus \text{Int}(\mathbb{M})$  of  $\mathbb{M}$ , as defined at the end of §1, can be expressed as

$$\partial\mathbb{M} = T_\infty(\partial\mathbb{K} \setminus I_1) \cup S(0) \cup S(1) \cup I_0,$$

where every point on  $\partial\mathbb{M} \setminus C_0$  is normal in the sense that it has an open neighborhood in  $\mathbb{M}$  homeomorphic to the upper half-plane  $\mathbb{R} \times [0, \infty)$ , but every point on  $C_0$  is singular in the sense that  $\mathbb{M}$  is neither locally flat nor locally simply connected around the point. Let us consider the  $\pi$ -shaped arc joining the points  $(1/3, 0, 0) \rightarrow (1/3, 0, 1/2) \rightarrow (2/3, 0, 1/2) \rightarrow (2/3, 0, 0)$  in  $\mathbb{K}$ , and call it  $\Pi$ . Its image by  $T_\infty$  looks like the capital omega, so we put  $\Omega = T_\infty(\Pi)$ . Using the contractions  $\Phi_0, \Phi_2$  in §2, for  $\sigma \in FS(0, 2)$ ,  $|\sigma| = k$ , define  $\Pi(\sigma)$  in  $\mathbb{K}$  and  $\Omega(\sigma)$  in  $\mathbb{M}$  by

$$\Pi(\sigma) = \Phi_{\sigma(k)} \circ \dots \circ \Phi_{\sigma(1)}(\Pi), \quad \Omega(\sigma) = \Phi_{\sigma(k)} \circ \dots \circ \Phi_{\sigma(1)}(\Omega),$$

respectively, setting in particular  $\Pi(\emptyset) = \Pi$ ,  $\Omega(\emptyset) = \Omega$ . Then we get

$$\Omega(\sigma^*) = T_\infty(\Pi(\sigma))$$

for every  $\sigma \in FS(0, 2)$ . Put

$$\Omega^+ = F(0 \downarrow 3/4) \cup F(1 \downarrow 1/4) \cup S(0) \cup S(1) \cup I_0$$

which is not an arc, but would look like  $\Omega$  if it were detached at the two points  $(1/4, 0, 0)$ ,  $(3/4, 0, 0)$ . Then the boundary  $\partial\mathbb{M}$  can be expressed as

$$\partial\mathbb{M} = \cup\{\Omega(\sigma) : \sigma \in FS(0, 2)\} \cup \Omega^+.$$

Note that all arcs  $\Pi(\sigma)$ ,  $\sigma \in FS(0, 2)$ , are disjoint in  $\mathbb{K}$ , so that the corresponding arcs  $\Omega(\sigma)$ ,  $\sigma \in FS(0, 2)$  are also disjoint in  $\mathbb{M}$ , as can be seen in Fig.10 where they are arranged in the way that  $\Omega(\tau)$  is behind  $\Omega(\sigma)$  if  $|\tau| > |\sigma|$ . Let *End* denote the set of all *end points* of  $C \subseteq [0, 1]$ , i.e., the set of all points  $x = .x_1x_2x_3 \dots_{(3)} \in C$  such that the digits  $x_1, x_2, x_3, \dots$  are eventually 0000... or 2222... Put *Inn* =  $C \setminus \text{End}$ , whose points are usually called *inner points* of  $C$ , but note that these terms “end/inner point” only indicate the placement of  $C$  in  $[0, 1]$ , and are not topological properties

of  $C$  since  $C$  itself is homogeneous. For  $x = .x_1x_2x_3\cdots_{(3)} \in \text{End}$  the digits of  $x^* = .\widehat{x}_1x_2\widehat{x}_3x_4\widehat{x}_5\cdots$  are eventually  $0202020\cdots$ , so that  $x^*$  is an inner point of  $C$ . Put

$$E_0 = \{(x, 0, 0) : x \in \text{End}\}, \quad E_0^* = \{(x^*, 0, 0) : x \in \text{End}\}$$

and  $IN_0 = C_0 \setminus E_0 = \{(x, 0, 0) : x \in \text{Inn}\} \supseteq E_0^*$ . Then,  $E_0$  coincides with the set of all end points of arcs  $\Pi(\sigma)$ ,  $\sigma \in FS(0, 2)$  together with  $(0, 0, 0)$ ,  $(1, 0, 0)$ , while  $E_0^*$  coincides with the set of all end points of arcs  $\Omega(\sigma)$ ,  $\sigma \in FS(0, 2)$  and  $\Omega^+$ , assuming that the points  $(1/4, 0, 0) = (1^*, 0, 0)$  and  $(3/4, 0, 0) = (0^*, 0, 0)$  are the “end points” of  $\Omega^+$ . In general, for  $\sigma = (x_1, x_2, \cdots, x_k)$ , let  $(a_\sigma, 0, 0)$ ,  $(b_\sigma, 0, 0)$ ,  $a_\sigma < b_\sigma$ , be the end points of  $\Pi(\sigma)$ , then

$$a_\sigma = .x_1 \cdots x_k 0222 \cdots, \quad b_\sigma = .x_1 \cdots x_k 2000 \cdots.$$

The corresponding end points of  $\Omega(\sigma^*)$  are  $(a_\sigma^*, 0, 0)$ ,  $(b_\sigma^*, 0, 0)$  such that

$$a_\sigma^* = .\widehat{x}_1x_2\widehat{x}_3\cdots x_k 220202 \cdots > b_\sigma^* = .\widehat{x}_1x_2\widehat{x}_3\cdots x_k 002020 \cdots$$

if  $k$  is even, and

$$a_\sigma^* = .\widehat{x}_1x_2\widehat{x}_3\cdots \widehat{x}_k 002020 \cdots < b_\sigma^* = .\widehat{x}_1x_2\widehat{x}_3\cdots \widehat{x}_k 220202 \cdots$$

if  $k$  is odd. In particular for  $\sigma = \emptyset$ ,  $|\emptyset| = 0$ ,

$$a_\emptyset = .0222 \cdots = 1/3 < b_\emptyset = .2000 \cdots = 2/3,$$

$$a_\emptyset^* = .22020 \cdots = 11/12 > b_\emptyset^* = .002020 \cdots = 1/12.$$

Note that  $\Omega(\sigma)$  is “wider” than the corresponding  $\Pi(\sigma)$  in the sense that

$$|a_\sigma^* - b_\sigma^*| = |a_\sigma - b_\sigma| \cdot 5/2.$$

(The case  $\sigma = \emptyset$  of this equality was already pointed out in Remark 2.2.) We say that  $\Omega(\sigma)$  *bridges over* a point  $(x, 0, 0) \in I_0$  when  $(x, 0, 0)$  lies in between the two end points of  $\Omega(\sigma)$ . Then we can prove the following assertion showing how the boundary  $\partial\mathbb{M}$  is interwoven around every point of  $E_0^*$ . See Fig.10.

**Assertion 4.1.** *Every point  $(x^*, 0, 0) \in E_0^*$  is bridged over by infinitely many  $\Omega(\sigma)$ 's.*

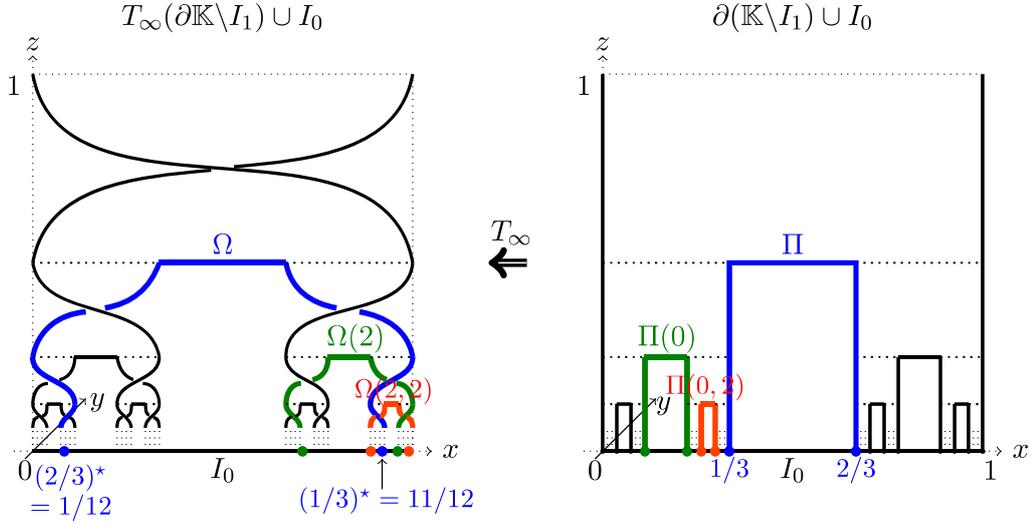
*Proof.* We prove the case  $x = 1/3$ , i.e.,  $((1/3)^*, 0, 0) = (11/12, 0, 0) \in E_0^*$ ; other cases can be proved in a similar way. We can show that all of

$$\Omega(\widehat{0}) = T_\infty(\Pi(0)), \quad \Omega(\widehat{0}2) = T_\infty(\Pi(02)), \quad \Omega(\widehat{0}2\widehat{2}) = T_\infty(022), \cdots,$$

determined by the expansion  $1/3 = .0222 \cdots_{(3)}$ , bridge over the point  $((1/3)^*, 0, 0)$ .

See Fig.10 in which  $\Omega(2) = \Omega(\widehat{0})$  and  $\Omega(22) = \Omega(\widehat{0}2)$  are illustrated. Let us prove the case  $\Omega(\widehat{0}2)$ ; other cases can be proved similarly. The end-points of  $\Omega(\widehat{0}2)$  are, as shown above,

$$a_{\widehat{0}2}^* = .\widehat{0}2\mathbf{22}0202 \cdots_{(3)} > b_{\widehat{0}2}^* = .\widehat{0}2\mathbf{00}2020 \cdots_{(3)}.$$



**Fig. 10.** In the right picture both arcs  $\Pi(0)$  and  $\Pi(0,2)$  lie on the left side of the end-point  $(1/3, 0, 0)$  of the arc  $\Pi$ , but in the left picture the corresponding  $\Omega(2) = T_\infty(\Pi(0))$  and  $\Omega(2,2) = T_\infty(\Pi(0,2))$  bridge over the end-point  $((1/3)^*, 0, 0)$  of the corresponding arc  $\Omega = T_\infty(\Pi)$ .

Comparing these with  $(1/3)^* = \widehat{0}2\widehat{2}2\widehat{2}2 \cdots_{(3)} = \widehat{0}2\mathbf{0}2\mathbf{0}202 \cdots_{(3)}$ , we get

$$a_{02}^* > (1/3)^* > b_{02}^*.$$

Hence  $\Omega(\widehat{0}2)$  bridges over  $((1/3)^*, 0, 0)$ .  $\square$

## 5. EXPANSION BY INVERSE LIMIT

We will show that the Möbius Cascade  $\mathbb{M}$  can be expanded by inverse limit of orientable or non-orientable surfaces, in either way.

For  $0 < a = 2^{-k} \leq 1$  let  $\mathbb{M}[a]$  be the surface obtained from  $\mathbb{M}$  by collapsing its front part inbetween  $z = a$  and  $z = 0$ , vertically down to  $I_0$ , and let  $\pi_{k,k+1} : \mathbb{M}[2^{-k-1}] \rightarrow \mathbb{M}[2^{-k}]$  be the map naturally induced by collapsing the part of  $\mathbb{M}$  inbetween  $z = 2^{-k}$  and  $z = 2^{-k-1}$ . Then we can see that

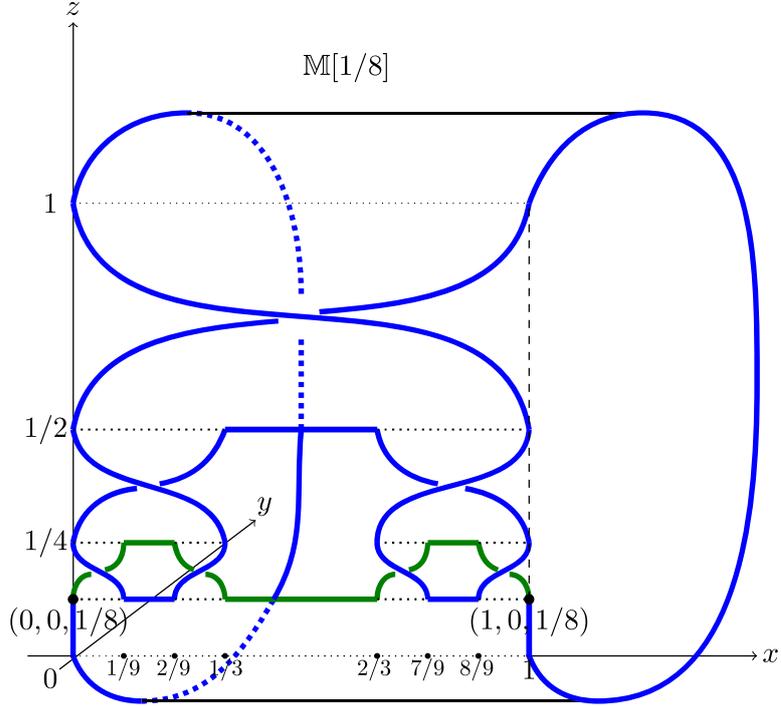
$$\mathbb{M} = \lim_{\leftarrow} \{\mathbb{M}[2^{-k}], \pi_{k,k+1}, k \in \omega\}.$$

Taking account of the parity of half-twists, we see that the surface  $\mathbb{M}[2^{-k}]$  for odd  $k$  is non-orientable, while  $\mathbb{M}[2^{-k}]$  for even  $k$  is orientable. Therefore,  $\mathbb{M}$  can be expanded by the inverse limit of non-orientable surfaces

$$\mathbb{M} = \lim_{\leftarrow} \{\mathbb{M}[2^{-k}], \pi_{k,k+2}, k \text{ odd}\},$$

and also by the inverse limit of orientable surfaces

$$\mathbb{M} = \lim_{\leftarrow} \{\mathbb{M}[2^{-k}], \pi_{k,k+2}, k \text{ even}\},$$



**Fig. 11.** The boundary of the surface  $\mathbb{M}[1/8]$  is a simple closed curve as the join of two arcs connecting at  $(0, 0, 1/8), (1, 0, 1/8)$ .

where  $\pi_{k,k+2} = \pi_{k,k+1} \circ \pi_{k+1,k+2}$ . Obviously,  $\mathbb{M}[1]$  is the annulus, and  $\mathbb{M}[2^{-1}]$  is the Möbius band. Let us identify the 2-dimensional surface  $\mathbb{M}[2^{-k}]$  in general. We use the standard notation that  $\Sigma(n)$  stands for the non-orientable surface of genus  $n$ , i.e., the connected sum of  $n$ -many projective planes  $P^2 \# \dots \# P^2$ , and  $S(n)$  stands for the orientable surface of genus  $n$ , i.e., the connected sum of  $n$ -many tori  $T^2 \# \dots \# T^2$ . We can observe that

- (1) each  $\mathbb{M}[2^{-k}]$  has only one boundary  $\approx S^1$ , and
- (2) the Euler characteristic  $\chi(\mathbb{M}[2^{-k}]) = 1 - 2^{k-1}$ .

So, the classification theorem for 2-dimensional surfaces (cf.[3, 5]) tells that

$$\mathbb{M}[2^{-k}] \approx \Sigma(2^{k-1}) \setminus (\text{disc}) \text{ for odd } k,$$

$$\mathbb{M}[2^{-k}] \approx S(2^{k-2}) \setminus (\text{disc}) \text{ for even } k.$$

For simplicity, let us explain this for  $\mathbb{M}[1/8]$ , the case for  $k = 3$ . That the boundary  $\partial(\mathbb{M}[1/8])$  is a simple closed curve would be observed from Fig.11. The Euler characteristic of  $\mathbb{M}[1/8]$  remains the same when every twisted building block of  $\mathbb{M}[1/8]$  is replaced by the corresponding flat one, so that  $\chi(\mathbb{M}[1/8])$  is equal to the Euler characteristic of the sphere with 5 holes, hence  $\chi(\mathbb{M}[1/8]) = 2 - 5 = -3$ . Cap off the boundary circle  $\partial(\mathbb{M}[1/8])$  with a disc to get a closed 2-dimensional surface. Then, this non-orientable closed

surface has the Euler characteristic  $-3 + 1 = -2$  so that it is homeomorphic to  $\Sigma(4)$ . Thus, we can conclude that

$$\mathbb{M}[1/8] \approx \Sigma(4) \setminus (\text{disc})$$

where

$$\Sigma(4) \approx P^2 \# P^2 \# P^2 \# P^2 \approx K^2 \# K^2 \approx T^2 \# K^2.$$

## 6. WHAT IF TWO MÖBIUS CASCADES ARE GLUED TOGETHER ALONG THEIR BOUNDARY?

Since the Klein's bottle  $K^2$  is the connected sum of two projective planes  $P^2 \# P^2$ , we get the Klein's bottle if two Möbius bands are glued together along their boundary circle. So, we can ask what we get if two Möbius Cascades are glued together along their boundary? Consider the surface of the Cantor icicle  $\mathbb{CI}$  in  $\mathbb{R}^3$ , and take away the ceiling open disc with the center  $(1/2, 1/2, 1)$ . Then we get the side boundary surface of  $\mathbb{CI}$ , which we denote  $S(\mathbb{CI})$ , and call the *side surface* of the Cantor icicle. Let us divide this side surface into two halves by the  $xz$ -plane  $y = 0$ , and let  $\mathbb{K}^{(+)}$  and  $\mathbb{K}^{(-)}$  denote the parts of  $S(\mathbb{CI})$  satisfying the condition  $y \geq 0$  and  $y \leq 0$ , respectively:

$$S(\mathbb{CI}) = \mathbb{K}^{(+)} \cup \mathbb{K}^{(-)}.$$

Then, accordingly, the top circle of  $S(\mathbb{CI})$  would be divided into two half circles such that:

$$\ell^+ = \{(x, g(x), 1) : x \in [0, 1]\}, \quad \ell^- = \{(x, -g(x), 1) : x \in [0, 1]\}$$

where  $g(x)$  satisfies  $g(x) \geq 0$ ,  $(x - 1/2)^2 + g(x)^2 = 1/4$ . To confine the bottom Cantor set  $C_0$ , attach the interval  $I_0 = [0, 1] \times \{(0, 0)\}$  to  $S(\mathbb{CI})$ , and identify  $I_0$  and the arcs  $\ell^+$ ,  $\ell^-$  by the rule

$$(x, 0, 0) \sim (x, g(x), 1) \sim (x, -g(x), 1)$$

for each  $x \in [0, 1]$ . Then, we will show that the quotient space

$$(S(\mathbb{CI}) \cup I_0) / \sim$$

would be what we can get by glueing two Möbius Cascades. To see this, define  $\mathbb{M}^{(+)}$  and  $\mathbb{M}^{(-)}$  by

$$\mathbb{M}^{(+)} = (T_\infty(\mathbb{K}^{(+)}) \cup I_0) / \sim, \quad \mathbb{M}^{(-)} = (T_\infty(\mathbb{K}^{(-)}) \cup I_0) / \sim$$

where  $\ell^+$  is identified to  $I_0$  in  $\mathbb{M}^{(+)}$ , and  $\ell^-$  is identified to  $I_0$  in  $\mathbb{M}^{(-)}$ , so that  $\partial(\mathbb{M}^{(+)}) = \partial(\mathbb{M}^{(-)})$ . It would be not so difficult to see that both  $\mathbb{M}^{(+)}$  and  $\mathbb{M}^{(-)}$  are homeomorphic to the Möbius Cascade  $\mathbb{M}$ . Thus, the quotient space

$$(S(\mathbb{CI}) \cup I_0) / \sim = \mathbb{M}^{(+)} \cup \mathbb{M}^{(-)}$$

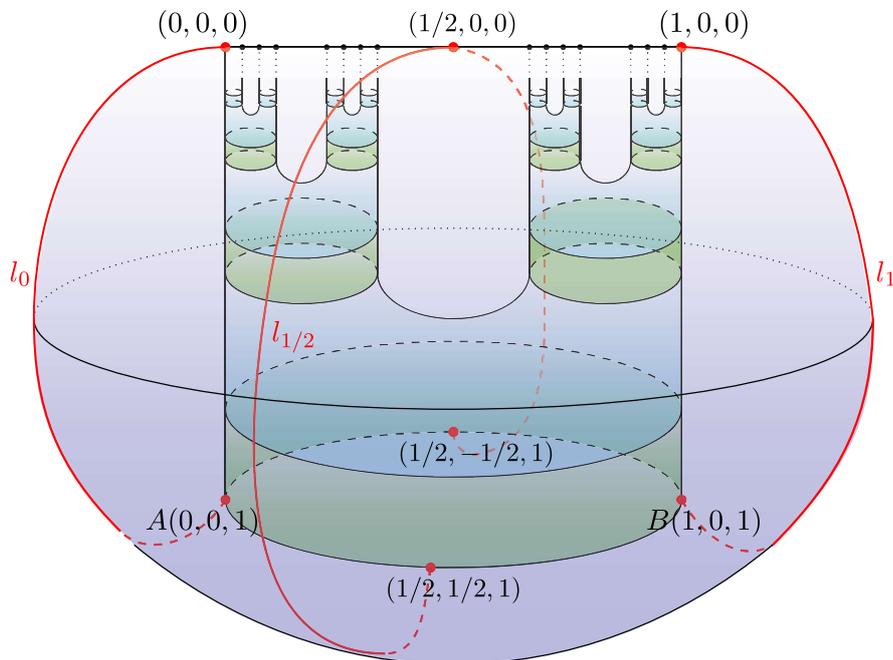
is what we get by glueing two Möbius Cascades. We have realized this quotient space geometrically in  $\mathbb{R}^3$  as the hollow sphere  $\check{S}(\mathbb{CI}) \cup W$  in Fig.12, where we adopted the upside-down view for easier illustration.  $\check{S}(\mathbb{CI})$  denotes the hollow inside (a smooth surface with saddle points) which is a

homeomorphically modified version of  $S(\mathbb{CI})$ .  $W$  is the smooth surface wrapping  $\check{S}(\mathbb{CI}) \cup I_0$ , and can be parametrized as the union  $W = \cup_{0 \leq x \leq 1} l_x$  of disjoint arcs  $l_x$  ( $0 \leq x \leq 1$ ) such that  $l_x$  passes the outside of  $\check{S}(\mathbb{CI})$  and connects the points in the equivalence class

$$\{(x, g(x), 1), (x, -g(x), 1), (x, 0, 0)\}$$

for  $0 \leq x \leq 1$ . This whole figure  $\check{S}(\mathbb{CI}) \cup W$  can be viewed as a sphere with the hollow  $\check{S}(\mathbb{CI})$  so that it could be a mold for the Cantor icicle.

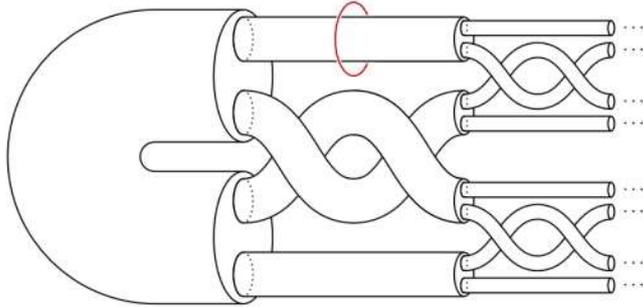
It would be interesting to note that our gluing of two Möbius Cascades was realized in  $\mathbb{R}^3$ , comparing with the fact that  $\mathbb{R}^4$  is needed for the realization of the gluing of two Möbius bands, i.e., the Klein's bottle.



**Fig. 12.** The hollow sphere  $\check{S}(\mathbb{CI}) \cup W$  where  $\check{S}(\mathbb{CI}) \approx S(\mathbb{CI})$  is the surface of the hollow inside, and  $W$  is a smooth surface which wraps  $\check{S}(\mathbb{CI}) \cup I_0$  and consists of arcs  $l_x$  ( $0 \leq x \leq 1$ ) connecting points  $(x, g(x), 1), (x, -g(x), 1), (x, 0, 0)$ . For example,  $l_{1/2}$  connects the three points  $(1/2, 0, 0), (1/2, \pm 1/2, 1)$ , while  $l_0$  connects the two points  $(0, 0, 0)$  and  $A(0, 0, 1)$ . Here, we adopted the upside-down view for easier illustration; compare with Fig.3

Finally, we want to point out that the geometric structure like the Cantor Icicle  $\mathbb{CI}$  in Fig.3 appears in the well-known construction of the Alexander

horned sphere as can be seen in Fig.13. The fact that the Alexander horned sphere is homeomorphic to the standard sphere can be proved in the same way as we showed in §2 that  $T_\infty$  is a homeomorphism.



**Fig. 13.** An Alexander horned sphere. Picture from p.72 in [2].

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