FREE GROUPS WITH NO NON-TRIVIAL CLOSED SUBGROUPS

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Abstract. The free group F over a countably infinite set X admits a metric group topology that is topologically simple and satisfies the Algebraic Small Subgroup Generating Property (ASSGP). On one hand, the ASSGP ensures that F admits no non-trivial continuous homomorphisms to an NSS group. On the other, topological simplicity implies that it contains no closed normal subgroups other than the trivial group $\{e\}$ and F itself, thereby satisfying the *no small normal subgroup* (NSnS) property.

1. Basic notation and terminology

We denote by $\mathbb{N}$ the set of natural numbers and we let $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Given a set X we denote by $[X]^{<\omega}$ the family of finite subsets of X.

Let G be a group. For subsets A, B of G, we let

$$AB = \{ab : a \in A, b \in B\} \quad \text{and} \quad A^{-1} = \{a^{-1} : a \in A\}.$$

We say that a subset A of G is *symmetric* if and only if $A = A^{-1}$. For a subset A of G, we denote by $\langle A \rangle$ the smallest subgroup of G containing A. To simplify the notation, we write $\langle x \rangle$ instead of $\langle \{x\} \rangle$ for $x \in G$. Given a group G and an arbitrary subset X of G we set $\mathrm{Cyc}(X) := \{x \in G : \langle x \rangle \subseteq X\}$.

Definition 1.1. *Given a set X we denote by $F(X)$ the free group over X. In the context of $F(X)$, the set X shall be referred to as the* alphabet *of $F(X)$, and its elements shall be called its* letters*. The group operation $\cdot$ for $F(X)$ is considered in the standard fashion via the concatenation of reduced words, making $(F(X), \cdot)$ a group. When $Z \subseteq X$ is a subset (a* sub-alphabet *of X), we consider the free group $F(Z)$ over Z as a natural subgroup of $F(X)$ in the obvious way.*

2. Introduction

The topological spaces appearing in this manuscript shall be assumed to be Hausdorff. A topological group G is said to have *no small normal subgroup (*NSnS*)* if G contains an open neighbourhood of the identity which contains no other normal subgroup of G other than $\{e\}$. We say instead that G has *no small subgroup (*NSS*)* whenever the open neighbourhood can be selected to contain no subgroups of G other than $\{e\}$. The NSS property plays an important historical role in the solution to the fifth problem of Hilbert due to Gleason, Montgomery-Zippin, and Yamabe for the characterization of Lie groups.

2.1. Minimal Almost Periodicity and Extreme Amenability.

A topological group G is said to be *minimally almost periodic* (MinAP) if it admits no non-trivial continuous homomorphism to a compact group. In the opposite direction, a topological group G is *maximally almost periodic* (MAP) if its points can be separated by its family of continuous homomorphisms to compact groups. These two families of groups were introduced in the context of Functional Analysis by von Neumann as a generalization of Bohr's work.

Constructing MinAP groups is historically known to be difficult. In topological dynamics, MinAP groups naturally appear when considering *extremely amenable groups*: topological groups G for which any continuous action of G on a compact Hausdorff space admits a fixed point. Every extremely amenable group is minimally almost periodic. The Kechris-Pestov-Todorčević [6] correspondence provides a beautiful framework to fully describe extreme amenability for certain automorphism groups, providing an immense selection of MinAP groups.

2.2. **Topologically Simple Groups.**

A topological group G is said to be *topologically simple* whenever the only closed and normal subgroups of G are the trivial group and G itself. Historically, these groups have been powerful tools for producing a variety of counter-examples. Modern examples usually belong to one of two categories: locally compact and totally disconnected (t.d.l.c.) groups, and extremely amenable (or MinAP) groups.

For simplicial trees, combinatorial constructions of simple groups by Tits [10] have profoundly influenced t.d.l.c. group theory, with recent generalizations by Banks, Elder, and Willis [1]. However, local compactness is quite often incompatible with minimal almost periodicity, and directly incompatible with extreme amenability due to Veech's theorem [11]: any locally compact group acts freely on a compact space.

In the non-Abelian realm, the infinite permutation group S_∞ equipped with the point-wise convergence topology is a Polish topologically simple group which is not NSS. Consequently, it admits no non-trivial continuous homomorphism to an NSS group, making it MinAP. Recent results by Etedadialiabadi, Gao, and Li [5] further prove that the isometry group of the Urysohn space $\mathbb{U}_\Delta$ is extremely amenable and topologically simple.

2.3. **The** ASSGP **and the Main Result.**

The following notion was first defined explicitly in [9]:

Definition 2.1. *A topological group G satisfies the* algebraic small subgroup generating property (ASSGP) *whenever $G = \langle \mathrm{Cyc}(U) \rangle$ holds for each open neighbourhood U of the identity of G.*

Any group with this property admits no non-trivial continuous homomorphism to an NSS group, placing ASSGP groups firmly within the class of MinAP groups:

$$\text{ASSGP} \to \text{MinAP} \leftarrow \text{extremely amenable.}$$

There exist groups which admit topologically simple MinAP group topologies, but no ASSGP group topologies, such as finite powers of the integers $\mathbb{Z}$ [2, 4] and the symmetric group S_∞ [3]. The main result of the upcoming [12] is to establish that the free group $F(X)$ of countably infinite rank bridges this gap:

Theorem 2.2 ([12])**.** *A free group $F(X)$ of countably infinite rank admits a metrizable* ASSGP *group topology which is topologically simple.*

Assuming a set of fairly complex combinatorial lemmas proven in the upcoming [12], we provide a draft of the proof of Theorem 2.2 in Section 5.

3. The topologization framework

To topologize the free group $F(X)$ over a countably infinite set X, we employ a partial-order framework developed by Shakhmatov and the author [9]. This forcing-like technique induces specific topological properties by identifying suitable dense sets within a partially ordered set of approximations of a neighbourhood system for a future topology. By systematically intersecting these countably many dense sets via the Rasiowa-Sikorski lemma, we build a filter that dictates the final properties of the topology. This allows us to strictly induce global algebraic traits from local combinatorial approximations without requiring any axiomatic extensions.

3.1. **Finite Neighbourhood Systems.** The construction begins with the concept of a *finite neighbourhood system* (FNS):

Definition 3.1. *For a finite sub-alphabet* $Z \subseteq X$*, an FNS is a finite sequence of subsets* $\mathscr{U} = \{U_i : i \leq n\}$ *of words in* $F(Z)$ *that are symmetric, contain the identity element, and satisfy the following conjugation containment condition:*

$$\bigcup_{z \in \bar{Z}} z \cdot U_{i+1} \cdot U_{i+1} \cdot z^{-1} \subseteq U_i \tag{1}$$

where $\bar{Z} = Z \cup Z^{-1} \cup \{e\}$.

Intuitively, these finite sequences approximate a future "full" neighbourhood system for the infinitely generated free group $F(X)$. To consistently expand these local systems, we introduce the concept of an *extension*:

Definition 3.2. *If* $\mathscr{V} = \{V_i : i \leq m\}$ *is an FNS over a larger alphabet* Y $(X \subseteq Y)$*, we say that it* extends $\mathscr{U} = \{U_i : i \leq n\}$ *if* $V_i \cap F(X) = U_i$ *holds for all* $i \leq n$.

This crucial restriction ensures that when the larger system is restricted back to the native alphabet X, only the original words from U_i appear, despite the closure under conjugation by **foreign letters** from $Y \setminus X$.

3.2. **The Poset $\mathbb{P}$ and Dense Subsets.**

Definition 3.3. *Let* X *be a countably infinite alphabet. We define a poset* $(\mathbb{P}, \leq)$ *consisting of all valid approximations* $p = \langle\langle X^p, n^p, \mathscr{U}^p \rangle\rangle$*, where* X^p *is a finite sub-alphabet of* X*,* $n^p \geq 1$ *is an integer, and* $\mathscr{U}^p$ *is an FNS for* $F(X^p)$*. The ordering* $q \leq p$ *means that the system* q *is a valid extension of* p *in the sense of Definition 3.2.*

Because the total alphabet X is countable, we can use the Rasiowa-Sikorski lemma to select a countable linear order (a filter) $\mathbb{F} \subseteq \mathbb{P}$ that intersects various (at most countable) dense subsets of $\mathbb{P}$. Essentially, the filter $\mathbb{F}$ acts as a coherent, non-contradictory path of finite approximations. It safely "glues" the local Finite Neighbourhood Systems together into a valid neighbourhood basis for a group topology on the entire free group. Given the linear order $\mathbb{F}$, we then produce a corresponding sequence $\{U_n : n \in \mathbb{N}\}$ by setting:

$$U_n = \bigcup \{U_n^p : p \in \mathbb{F} \text{ and } n \leq n^p\}. \tag{2}$$

4. Dense Subsets and their Application

By carefully curating dense subsets, we ensure that the resulting topology (defined by employing the union of the systems along the filter as in (2) as a neighbourhood basis) possesses desired characteristics. In [9] we introduced specific dense subsets which first allow the resulting neighbourhood system to be a group topology.

Lemma 4.1 ([9, Lemma 8.6])**.** *Given a countably infinite set* X *and* $(\mathbb{P}, \leq)$ *the partial order from Definition 3.1, the following holds:*

Quantifier	*Name*		*Set*	*Property*
$\forall n \in \mathbb{N}$	A_n	=	$\{q \in \mathbb{P} : n \leq n^q\}$	*is dense and downward-closed in* $\mathbb{P}$.
$\forall S \in [X]^{<\omega}$	B_S	=	$\{q \in \mathbb{P} : S \subseteq X^q\}$	*is dense in* $\mathbb{P}$.
$\forall g \in F(X) \setminus \{e\}$	C_g	=	$\{q \in \mathbb{P} : g \in F(X^q) \setminus U^q_{n^q}\}$	*is dense in* $\mathbb{P}$.
$\forall g \in F(X)$	D_g	=	$\{q \in \mathbb{P} : g \in \langle \mathrm{Cyc}(U^q_{n^q}) \rangle\}$	*is dense in* $\mathbb{P}$.

Let us summarize the role each of the above subsets play for the construction of the topology. Let us consider $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ to be the sequence as defined in (2) (which, naturally, depends on the choice of the filter $\mathbb{F}$ and the dense subsets that $\mathbb{F}$ intersects). To ensure that the sequence $\mathcal{U}$ becomes a true system of neighbourhoods of identity for a group topology on $F(X)$ (which in addition satisfies additional properties), we check the following:

- **Nested symmetric neighbourhoods:** The condition (1) set upon the FNS ensures that the resulting system $\mathcal{U}$ is nested; i.e, for each $n \in \mathbb{N}$ the inclusion $U_{n+1} = U_{n+1}^{-1} \subseteq U_n$ holds.
- **Hausdorffness and Metrizability:** Dense sets of the form A_n (ensuring the sequence $\mathcal{U}$ is infinite), B_S (ensuring conjugation laws for a neighbourhood system apply to the entire alphabet X), and C_g (separating any non-trivial element g from the identity via $\mathcal{U}$) guarantee that the resulting system $\mathcal{U}$ forms a valid, metrizable (due to countability), and Hausdorff basis of open neighbourhoods of the identity for a group topology on $F(X)$.
- **The ASSGP:** We ensure that the filter $\mathbb{F}$ intersects with dense sets $D_g = \{q \in \mathbb{P} : g \in \langle \mathrm{Cyc}(U^q_{n^q})\rangle\}$ for all non-trivial $g \in F(X)$. This guarantees that every element of the group is algebraically generated by *cyclic elements* contained within arbitrary basic neighbourhoods, strictly realizing the ASSGP.

Summarizing all of the above, it then suffices to construct the sequence $\mathcal{U}$ with respect to a filter $\mathbb{F}$ which intersects all members of a countably infinite family of dense subsets such as

$$\mathscr{D}_1 = \{C_g : g \in F(X) \setminus \{e\}\} \cup \{A_n \cap D_g : n \in \mathbb{N}, g \in F(X)\} \cup \{B_S : S \in [X]^{<\omega}\}. \tag{3}$$

This then yields the main result of [9]:

Theorem 4.2 ([9])**.** *A free group $F(X)$ of countably infinite rank admits a metrizable group* ASSGP *group topology.*

We note that the main combinatorial complexity lies in proving that each of the sets in Lemma 4.1 are dense in $(\mathbb{P}, \leq)$. This density requires meticulously tracking the cancellation of "foreign" letters introduced during the "enrichment" of finite neighbourhood systems. For instance, given two finite alphabets $Z \subseteq Y$ and a specific word $g \in F(Z)$, one must guarantee that when a FNS is extended to conjugate the specific target word g with respect to the letters in the larger alphabet Y, the intricate irreducible word mechanics of the free group do not inadvertently introduce unwanted elements into the pre-existing FNS taken with respect to the native alphabet Z.

4.1. Topological Simplicity for the countably generated free group $F(X)$.

Definition 4.3. *For each $g \in F(X)$ we denote the* conjugacy class *of g as*

$$\mathrm{Conj}(g) = \{w \cdot g \cdot w^{-1} : w \in F(X)\}. \tag{4}$$

The technical heart of the upcoming [12] lies in proving the following novel result:

Lemma 4.4. *For every $n \in \mathbb{N}$, $S \in [X]^{<\omega}$, $g \in F(X) \setminus \{e\}$ and $h \in F(X)$ the set*

$$E_{n,S,g,h} = \{q \in \mathbb{P} : n \leq n^q, S \subseteq X^q \text{ and } (\mathrm{Conj}(g) \cdot h) \cap U^q_{n^q} \neq \emptyset\}. \tag{5}$$

is dense in $(\mathbb{P}, \leq)$.

In plain terms, intersecting this dense set guarantees that the conjugacy class of any non-trivial element g will eventually intersect every basic neighbourhood U_n when translated by an arbitrary element h. This is the exact mechanical requirement to ensure the conjugacy class is topologically dense within the constructed group topology. We now demonstrate this procedure, which may be applied by the interested reader to prove the validity of Theorem 4.2 from the validity of Lemma 4.1.

Consider the family:

$$\mathscr{D}_2 = \mathscr{D}_1 \cup \{E_{n,S,g,h} : n \in \mathbb{N}, S \in [X]^{<\omega}, g \in F(X) \setminus \{e\}, h \in F(X)\}. \tag{6}$$

From this point forward, let us consider $\mathbb{F}$ to be a linear order of $\mathbb{P}$ which intersects all members of $\mathscr{D}_2$. By Theorem 4.2 the system $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a neighbourhood basis for a metrizable group topology $\mathcal{T}$ for $F(X)$ (this naturally holds because the filter $\mathbb{F}$ once more intersects all members of $\mathscr{D}_1$). Assuming Lemma 4.4, we now check the following:

Lemma 4.5. *The topology $\mathcal{T}$ for $F(X)$ satisfies the following property: for each $g \in F(X) \setminus \{e\}$ the set* $\mathrm{Conj}(g)$ *is topologically dense.*

Proof. It suffices to prove the following: for any $n \in \mathbb{N}$ and $h \in F(X)$ the intersection $\mathrm{Conj}(g) \cap (U_n \cdot h)$ is non-empty. Let us consider a finite set $Z \subseteq X$ such that $g, h \in F(Z)$. Consider the corresponding set $E_{n,Z,g,h^{-1}}$ as in (5). The set $\mathbb{F}$ intersects all members of $\mathscr{D}_2$. Therefore, there exists $p \in \mathbb{F} \cap E_{n,Z,g,h^{-1}}$. By the definition of $E_{n,Z,g,h^{-1}}$, we have

$$(\mathrm{Conj}(g) \cdot h^{-1}) \cap U^p_{n^p} \neq \emptyset.$$

Given $w \in (\mathrm{Conj}(g) \cdot h^{-1}) \cap U^p_{n^p}$ we have that $w = u \cdot h^{-1} \in U^p_{n^p}$ with $u \in \mathrm{Conj}(g)$. Therefore $w \cdot h \in (U^p_{n^p} \cdot h) \subseteq (U_{n^p} \cdot h)$ by (2). This proves that

$$\mathrm{Conj}(g) \cap (U_{n^p} \cdot h) \neq \emptyset. \tag{7}$$

We now employ the fact that the system $\mathcal{U}$ is nested. The inequality $n \leq n^p$ implies $U_{n^p} \subseteq U_n$. Therefore, (7) implies that

$$\mathrm{Conj}(g) \cap (U_n \cdot h) \neq \emptyset.$$

Since $n \in \mathbb{N}$ and $h \in F(X)$ were arbitrary, we conclude that $\mathrm{Conj}(g)$ is topologically dense in $(F(X), \mathscr{T})$. □

The above lemma proves that by simultaneously intersecting the filter $\mathbb{F}$ with all the aforementioned dense families in $\mathscr{D}_2$, the resulting topology successfully synthesizes metrizability, the ASSGP, and topological simplicity into a single structure on the free group $F(X)$ (as we shall see in the following proof).

5. Proof of Theorem 2.2

The system $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a neighbourhood basis for a metrizable group topology $\mathcal{T}$ for $F(X)$, which additionally satisfies the ASSGP. It only suffices to prove that this topology makes $F(X)$ topologically simple.

Let N be an arbitrary and non-trivial normal subgroup of $F(X)$. Our goal is to prove that it is dense in $F(X)$. This now follows elegantly from Lemma 4.5: for any non-trivial $g \in N$, its conjugacy class $\mathrm{Conj}(g)$ is topologically dense in $F(X)$. Because N is a normal subgroup of $F(X)$, it must fully contain $\mathrm{Conj}(g)$. Consequently, N itself is topologically dense in $F(X)$.

We have proven that the only closed and normal subgroups are $\{e\}$ and $F(X)$ itself.

6. Concluding Remarks and Open Questions

The topology constructed on $F(X)$ highlights a delicate boundary in the classification of minimally almost periodic groups. The infinite symmetric group S_∞, equipped with the topology of point-wise convergence, serves as a prototypical example of a Polish topologically simple group that admits no non-trivial continuous homomorphisms to NSS groups. However, S_∞ inherently cannot admit an ASSGP group topology.

Our main result demonstrates that the free group $F(X)$ of countably infinite rank mirrors S_∞ with respect to its continuous homomorphic images and topological simplicity, with the clear dividing line being Polishness (in the case of S_∞) versus the rather “extreme” algebraic generation of the ASSGP (in the case of $F(X)$).

Furthermore, while it is well-established that extreme amenability implies minimal almost periodicity, the converse remains a major area of exploration, particularly for commutative groups.

This naturally leads to an open problem, essentially due to Comfort and Gould [2], which our framework on $F(X)$ may eventually help illuminate:

Question 6.1. *Is every* ASSGP *topological group also extremely amenable?*

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