

The tableaux algebra, Kashiwara crystals, and toric geometry

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Abstract

This is the report on the eponymous talk given at the conference “Interactions of Representation Theory and Related Topics” held at the Research Institute for Mathematical Sciences. The aforementioned talk and this report are based on the arXiv preprint 2510.27209 and contain joint work with S. Daugherty, N. González, B. Muniz, J. Pan, and J. Torres.

1 The tableaux algebra

Our main object of interest is a very explicitly defined algebra studied in [DGM⁺26], where we refer the reader for a detailed explanation of the results presented here. We now recall the definition and state some of the basic properties of the tableaux algebra.

Definition 1.1. Let SSYT_m^n be the set of semistandard Young tableaux of any shape having at most n rows and entries in $\{1, \dots, m\}$. Given S and T in SSYT_m^n of shapes λ and μ , respectively, we define $S \star T$ as the semistandard Young tableaux of shape $\lambda + \mu$ obtained by horizontally concatenating the rows of S and T , and then sorting the entries in each row in weakly increasing order from left to right.

Note that we must have $m \geq n$ for SSYT_m^n to be non-empty, and that when $S, T \in \text{SSYT}_m^n$ then $S \star T \in \text{SSYT}_m^n$.

Example 1.2. Let $m = 4$ and $n = 3$. Consider

$$S = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 & \\ \hline \end{array} \quad \text{and} \quad T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array}$$

which have shapes $\lambda = (2, 2, 1)$ and $\mu = (2, 1, 1)$, respectively. Then

$$S \star T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & \\ \hline 3 & 4 & & \\ \hline \end{array}$$

has shape $\lambda + \mu = (4, 3, 2)$, and indeed all of S , T , and $S \star T$ are in SSYT_m^n .

This operation makes $(\text{SSYT}_m^n, \star, \emptyset)$ into a very well behaved monoid, as it is multigraded, cancellative, commutative, reduced, torsion-free, and is generated by the semistandard Young tableaux which are columns.

Definition 1.3. Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. We define the *tableaux algebra* as the monoid algebra of SSYT_m^n , and we denote it by $\mathcal{T}_m^n$.

The first non-trivial tableaux algebra already offers loads of intuition for the larger ones.

Example 1.4. Let $m = 3$ and $n = 2$. Then

$$\mathcal{T}_3^2 = \frac{\mathbb{k} \left[\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \right]}{\left(\begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 1 \\ \hline \end{array} - \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} \cdot \begin{array}{|c|} \hline 2 \\ \hline \end{array} \right)}.$$

Given how explicitly and combinatorially defined is $\mathcal{T}_m^n$, we would like to emphasize that it is possible to give explicit and combinatorial proofs of all the results concerning it. This is illustrated by how we leverage the good properties of SSYT_m^n to bookkeep some of the useful properties of $\mathcal{T}_m^n$, as follows.

Theorem 1.5. The tableaux algebra $\mathcal{T}_m^n$ is:

1. Finitely generated over $\mathbb{k}$ with the columns of SSYT_m^n forming a generating set.
2. An integral domain.
3. Noetherian.
4. Jacobson.

In particular, its Jacobson radical is zero.

An important observation is that the product $\star$ in $\mathcal{T}_m^n$ does not arise from an insertion algorithm, as is typically the case for algebras defined from Young tableaux. The first place we encountered this product was when working with combinatorial models for Kashiwara crystals (as we will discuss in Section 2), but after digging deeper we found it in many other areas, the other which we will be mentioning being toric geometry (as we will discuss in Section 3).

2 Kashiwara crystals

The first application of the tableaux algebra we consider is to Kashiwara crystals.

Intuitively, a crystal is the skeleton of a module of a quantum group. More precisely, given $\mathfrak{g}$ a (suitable) Lie algebra, denote its associated quantum group by $\mathcal{U}_q(\mathfrak{g})$. It is a Hopf algebra with a triangular decomposition (see [Lus93] for more details)

$$\mathcal{U}_q(\mathfrak{g}) \cong \mathbb{k}[F_\alpha] \otimes \mathbb{k}[K_\alpha^{\pm 1}] \otimes \mathbb{k}[E_\alpha]$$

where the indices α correspond to the simple roots of $\mathfrak{g}$, the F_α are polynomial generators in negative degree, the $K_\alpha^{\pm 1}$ are polynomial generators in degree zero, and the E_α are polynomial generators in positive degree. A particularity of this framework is that given any fixed irreducible module V_λ of $\mathfrak{g}$, we can always find such a decomposition satisfying that the actions of F_α , $K_\alpha^{\pm 1}$, and E_α are as lowering, diagonal, and raising operators, respectively. However, in general there does not exist a PBW basis of $\mathcal{U}_q(\mathfrak{g})$ where F_α , $K_\alpha^{\pm 1}$, and E_α act as lowering, diagonal, and raising operators simultaneously for all irreducible modules V_λ of $\mathfrak{g}$ (see for example [Jan96, Chapter 9] for more details). The celebrated work of Kashiwara [Kas90, Kas91] showed that it is possible to find a PBW basis with the desired actions in the limit $q \rightarrow 0$. Namely, in the limit $\mathcal{U}_{q \rightarrow 0}(\mathfrak{g})$ there exists a basis $F, K^{\pm 1}, E$ where F and E act as lowering and raising operators with scalar 1, and $K^{\pm 1}$ act as diagonal operators with scalar 0. This means that for our purposes we can think of crystals as colored graphs, although in full generality they carry much more data, such as a weight function and a recipe for taking tensor products of crystals.

In type A , the three main models used to work with crystals are string polytopes, Gelfand–Tsetling patterns, and semistandard Young tableaux. There are very explicit ways of moving between these models, in the aforementioned order.

$$\text{string polytopes} \longleftrightarrow \text{Gelfand–Tsetling patterns} \longleftrightarrow \text{semistandard Young tableaux}$$

The literature on string polytopes developed operations to multiply them, often with the goal of finding and understanding atomic decompositions of characters and crystals. We refer the reader to the work of Littelmann [Lit98], Lecouvey and Lenart [LL21], Patimo [Pat25], and Bossinger and Torres [BT25], just to name a few among the many authors that contributed to this approach. An important question of interest follows.

Question 2.1. *Does the multiplication of two string polytopes coincide with the tensor product of the associated crystals?*

We translated this question from a product of string polytopes into a product of semistandard Young tableaux within the tableaux algebra, we answered this question in the negative, and we obtained new embeddings of crystals that are similar in spirit and form to the ones the literature of string polytopes has been interested in.

Theorem 2.2. *Let T be the highest or lowest weight vector of $\mathcal{B}(\mu)$. Then the map*

$$\begin{aligned} \Phi_T: \mathcal{B}(\lambda) &\longrightarrow \mathcal{B}(\lambda + \mu) \\ S &\longmapsto S \star T \end{aligned}$$

is a crystal embedding.

Proof. Let A_μ and Z_μ be the highest and lowest weight vectors of $\mathcal{B}(\mu)$, respectively. Then

$$S \star A_\mu = \left[S \xleftarrow{\text{col}} w_{\text{row}}(A_\mu) \right] \quad \text{and} \quad S \star Z_\mu = \left[S \xleftarrow{\text{row}} w_{\text{col}}(Z_\mu) \right],$$

which are insertions and thus crystal embeddings. This argument also justifies why in general $S \star T$ does not come from an insertion algorithm. $\square$

3 Toric geometry

The second application of the tableaux algebra we consider is to toric geometry.

Note that $\mathcal{T}_m^n$ is always Koszul, which can be argued succinctly in terms of the operation $\star$. This allows us to decompose $\mathcal{T}_m^n$ as two disjoint pieces that do not interact. One piece is a polynomial ring in two or three variables, depending on whether $m = n$ or $m \geq n$, respectively. The other piece is comprised of the columns of semistandard Young tableaux where the operation $\star$ forces the so-called product relations.

Theorem 3.1. *Let $E_m^n := G_m^n \setminus F_m^n$ where $G_m^n := \{T \in \text{SSYT}_m^n(1^k), 1 \leq k \leq n\}$,*

$$F_n^n := \left\{ \begin{array}{|c|} \hline n \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right\} (m = n), \text{ and } F_m^n := \left\{ \begin{array}{|c|} \hline m \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right\} (m \neq n).$$

The tableaux algebra can be decomposed as

$$\mathcal{T}_m^n = \mathbb{k}\langle G_m^n \rangle \cong \frac{\mathbb{k}[E_m^n]}{\mathcal{P}_m^n} \otimes \mathbb{k}[F_m^n].$$

Moreover, $\mathcal{T}_m^n$ is a PBW algebra as well as a Koszul algebra.

This poses two interesting questions.

Question 3.2. *What is the ideal of product relations $\mathcal{P}_m^n$?*

Question 3.3. *What is the size of a minimal generating set of $\mathcal{P}_m^n$?*

Both of these questions are related to toric geometry, and we answer them cleanly. The following result answers the first question.

Theorem 3.4. *The zero locus $\mathcal{V}(\mathcal{P}_m^n)$ is a toric variety.*

Proof. It can be checked that $\mathcal{P}_m^n$ is a prime binomial ideal, so the claim follows by well known work of Eisenbud and Sturmfels. $\square$

In fact, the diagonal (or single-indexed) tableaux algebra is isomorphic to the Gelfand–Tsetlin semigroup. In turn, the Gelfand–Tsetlin semigroup is a flat degeneration of the Plücker algebra:

$$\mathcal{T}_n^n \cong \mathcal{GT}_n \cong \frac{\text{Plücker algebra}}{(\text{initial ideal of the ideal of Plücker relations})},$$

so it is natural to seek a similar description for $\mathcal{T}_m^n$. The following result achieves this.

Theorem 3.5. *Let $m > n$, let $Q_m^n := \bigcap_{k=1}^n P_k$ where $P_k \subsetneq SL_m$ is the maximal parabolic subgroup given by*

$$\left[\begin{array}{c|c} k \times k & \\ \hline & (m-k) \times (m-k) \end{array} \right]$$

and let $L^{\bar{a}}$ be the tensor product of the ample generators of $\text{Pic}(SL_m/P_k)$. Then the tableaux algebra $\mathcal{T}_m^n$ is a flat degeneration of the algebra of global sections $\bigoplus_a \Gamma(SL_m/Q_m^n, L^{\bar{a}})$, and the zero locus $\mathcal{V}(\mathcal{P}_m^n)$ is a toric degeneration of the partial flag variety SL_m/Q_m^n .

The proof heavily relies on work of Hibi [Hib87] and work of Gonciulea and Lakshmibai [GL96]. As for the second question, the following result deals with it.

Theorem 3.6.

1. *There exists an explicit construction of a minimal generating set of the product relations $\mathcal{P}_m^n$. It can be abbreviated in the following notation:*

$$B_m^n := \{T \cdot T' - L_{T,T'} \cdot R_{T,T'} \mid T, T' \in G_m^n\}.$$

2. *There exists an explicit formula for the minimal number of generators of $\mathcal{P}_m^n$, namely the cardinality $|B_m^n|$, which exclusively depends on m and n .*

As a consequence of the two theorems above, namely leveraging the relation between $\mathcal{T}_m^n$ and with toric geometry, we find that B_m^n is a Gröbner basis of $\mathcal{P}_m^n$. This means that we have in fact counted three sets: the minimal number of Plücker relations in the partial flag SL_m/Q_m^n , the incidence relations between the different Grassmannians in SL_m/Q_m^n , and the minimal number of Plücker–Grassmann relations within each Grassmannian $\text{Gr}(i, m)$. All the formulas we found are new, and for the first two cases just mentioned, they are the only known formulas. In particular, it is worth mentioning that we do not use the well known fact that the minimal number of Plücker relations is the sum of the incidence relations and the Plücker–Grassmann. That fact turns out to be a consequence of our computations.

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