

TORSION THEORY OF FINITE DIMENSIONAL ALGEBRAS AND TRIM LATTICES

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ABSTRACT. In recent joint works with Sota Asai, Osamu Iyama, and Charles Paquette, we have introduced the notion of brick-splitting torsion pairs as a generalization of the classical notion of splitting torsion pairs. After discussing some properties of these pairs, we give a full characterization of them in terms of some important lattice-theoretical phenomena, known as left-modular elements. This leads to the notion of brick-directed algebras, which itself is a novel and vast generalization of an important family of classical algebras, the so-called representation-directed algebras. We show that brick-directed algebras have many interesting properties and use them to strengthen and generalize some lattice-theoretical studies. In particular, for a brick-finite algebra A , we prove that A is brick-directed if and only if the lattice of torsion classes in $\text{mod-}A$ is trim.

1. INTRODUCTION

Torsion classes and their lattice structure have become central in the modern representation theory of associative algebras, closely interacting with τ -tilting theory and lattice theory. More specifically, let A denote a finite dimensional algebra and $\text{mod } A$ denote the set of all finitely generated left A -modules. Then, the set $\text{tors } A$ of torsion classes in $\text{mod } A$, ordered by inclusion, is known to form a complete lattice and satisfies many good properties that are of interest in τ -tilting theory and lattice theory; for instance, see [AIR, BCZ, DI+] and the references therein. On the other hand, inspired by several fundamental open problems related to the behavior of bricks in $\text{mod } A$, these important indecomposables have been the focus of many studies in recent years; for example, see [MP1, MP2] and the references therein.

This manuscript summarizes some of our recent results centered on the novel brick-analogues of some classical notions. As our first modern notion, we say a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod } A$ is *brick-splitting* if every brick in $\text{mod } A$ lies in $\mathcal{T}$ or in $\mathcal{F}$. Every classical splitting torsion pair is evidently brick-splitting, but the converse fails already for hereditary algebras. In particular, the new notion is a genuine refinement of the classical one, and it leads to an interesting generalization of the splitting phenomenon. One of our first main results in [AIMP1] shows that brick-splitting torsion classes coincide exactly with certain elements of the lattice $\text{tors } A$, known as *left modular* elements; these are some important elements in lattice theory, but they are originally introduced and treated only in the context of finite lattices. As the second new notion introduced and studied in our work, we say A is a *brick-directed algebra* if $\text{mod } A$ admits no cyclic sequence of non-zero non-isomorphisms between bricks. This is a brick-analogue and conceptual generalization of the classical representation-directed algebras and allows us to realize several lattice-theoretical properties of $\text{tors } A$ in terms of bricks, even when $\text{tors } A$ is an infinite lattice. In fact, by adopting a more abstract and general setting beyond the lattices of torsion classes, we generalize some of the earlier lattice-theoretical studies conducted for finite lattices. More specifically, in [AIMP2] we treat the equivalence of left modularity and extremality for some infinite lattices.

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After some preparation in earlier parts, in Section 5 we mostly restrict ourselves to brick-finite algebras; equivalently, to the setting where $\text{tors } A$ is a finite lattice. In particular, for a brick-finite algebra A , we prove that brick-directedness is equivalent to $\text{tors } A$ being a *left modular* lattice, to $\text{tors } A$ being an *extremal* lattice, and to $\text{tors } A$ being a *trim* lattice; these are known to be three distinct lattice-theoretical properties that give different generalizations of distributivity for finite lattices (see [BS, Mu, Th]). This sequence of equivalences develops some earlier work and gives a purely lattice-theoretic characterization of brick-directedness. In Section 6 we briefly explain how the equivalence between left modularity and extremality persists beyond the brick-finite setting, once one works with a suitable class of (possibly infinite) semidistributive lattices. This shows that the phenomenon uncovered for brick-finite algebras is not an artifact of finiteness, and it produces an abundance of infinite left modular and extremal lattices of torsion classes coming from brick-directed, brick-infinite algebras.

Throughout, k is a field and A is a basic, connected, finite dimensional associative k -algebra with multiplicative identity. Moreover, $\text{mod } A$ denotes the category of finitely generated left A -modules, and $\text{ind } A$ the set of isomorphism classes of its indecomposable objects. Unless otherwise specified, modules in $\text{mod } A$ are always considered up to isomorphism. We write $\{e_1, \dots, e_n\}$ for a complete set of primitive orthogonal idempotents of A , $\{S_1, \dots, S_n\}$ for the corresponding simple modules, and $P_i = Ae_i$ for the indecomposable projective covering S_i . The *Ext-quiver* Ω_A of A has vertex set $\{S_1, \dots, S_n\}$, with an arrow $S_i \rightarrow S_j$ whenever $\text{Ext}_A^1(S_i, S_j) \neq 0$, and note that connectedness of A is equivalent to the connectedness of Ω_A .

2. BRICKS AND THE LATTICE OF TORSION CLASSES

We briefly recall the main notions needed below. For all the standard materials in representation theory, see [ASS, ARS]; and for the rudiments of lattice theory, see [Gr, DP].

2.1. Torsion classes and bricks. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\text{mod } A$ is said to be a *torsion pair* if $\mathcal{T} \cap \mathcal{F} = 0$, $\mathcal{T}$ is closed under quotients, $\mathcal{F}$ is closed under submodules, and every $M \in \text{mod } A$ has a canonical short exact sequence $0 \rightarrow t(M) \rightarrow M \rightarrow M/t(M) \rightarrow 0$ with $t(M) \in \mathcal{T}$ and $M/t(M) \in \mathcal{F}$. In this case $\mathcal{T}$ is a *torsion class*, $\mathcal{F} = \mathcal{T}^\perp$ is the associated *torsion-free class*, and each of $\mathcal{T}, \mathcal{F}$ determines the other; for more details, see [ASS, VI].

We write $\text{tors } A$ for the set of torsion classes in $\text{mod } A$, and consider it under the inclusion-order. Then, $\text{tors } A$ forms a complete lattice, with meet given by the intersection, and join is given by

$$\bigvee_{i \in I} \mathcal{T}_i = \bigcap \{ \mathcal{T} \in \text{tors } A \mid \mathcal{T} \supseteq \mathcal{T}_i \text{ for all } i \}.$$

It is known that $\text{tors } A$ is weakly atomic and semidistributive [DI+, GM]. In particular, for each cover relation $\mathcal{V} < \mathcal{U}$ in $\text{Hasse}(\text{tors } A)$, the set $\{ \mathcal{T} \in \text{tors } A \mid \mathcal{T} \vee \mathcal{V} = \mathcal{U} \}$ has a unique minimal element.

Recall that $M \in \text{mod } A$ is a *brick* if $\text{End}_A(M)$ is a division algebra. By *brick* A we denote the set of (isomorphism classes of) bricks, and A is called *brick-finite* if $\text{brick } A$ is a finite set. By [DIJ, DI+], A is brick-finite if and only if it is τ -tilting finite, if and only if $\text{tors } A$ is a finite lattice.

Each cover relation in $\text{Hasse}(\text{tors } A)$ is labeled by a unique brick. That is, if $\mathcal{U} \subsetneq \mathcal{T}$ in $\text{tors } A$ such that $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{T}$ implies $\mathcal{U} = \mathcal{V}$ or $\mathcal{V} = \mathcal{T}$, then $\text{brick}[\mathcal{U}, \mathcal{T}] := \{ X \in \text{brick } A \mid X \in \mathcal{T} \cap \mathcal{U}^\perp \}$ contains a unique element. That being the case, $\mathcal{T} \rightarrow \mathcal{U}$ is an arrow of $\text{Hasse}(\text{tors } A)$ exactly when $\text{brick}[\mathcal{U}, \mathcal{T}]$ is a singleton, and this element is the *label*. We remark that a single brick may label more than one arrow. This brick-labeling of the cover relation in $\text{tors } A$ underlies the following fundamental bijection.

Theorem 2.1 ([DI+, Thm. 3.4]) *For any algebra A , there is a bijection between $\text{brick } A$, the set $\text{JI}(\text{tors } A)$ of completely join-irreducible elements of $\text{tors } A$, and the set $\text{MI}(\text{tors } A)$ of completely meet-irreducible elements of $\text{tors } A$.*

A closely related tool freely used below is the description of the smallest torsion class generated by a subcategory $\mathcal{C}$ of $\text{mod } A$. More precisely, we consider $\text{Filt}(\text{Fac } \mathcal{C})$, the smallest extension-closed subcategory of $\text{mod } A$ containing all quotients of finite direct sums of objects of $\mathcal{C}$. In particular, every torsion class is uniquely determined by the bricks it contains, and to every subset $\mathcal{S} \subseteq \text{brick } A$ one may associate the smallest torsion class $\mathcal{T}(\mathcal{S})$ containing $\mathcal{S}$.

We also use a totally ordered refinement of the brick labeling due to Demonet [KD, Appendix]. In particular, a *chain of bricks* of A is an injective map $\Psi : I \rightarrow \text{brick } A$ from a totally ordered set I such that $\text{Hom}_A(\Psi(i), \Psi(j)) = 0$ whenever $i < j$ in I . Every chain of bricks Ψ gives rise to a chain of torsion classes $\{\mathcal{T}(\Psi(J))\}_J$, indexed by the down-closed subsets J of I , and this construction identifies the maximal chains of bricks of A with the maximal chains of $\text{tors } A$.

2.2. Lattice-theoretic notions. Let $(L, \leq)$ be a complete lattice with least element 0 and greatest element 1. Recall that $x < z$ is a *cover relation*, denoted by $x < z$, if $x \leq y \leq z$ forces $y \in \{x, z\}$. The *Hasse quiver* $\text{Hasse}(L)$ has vertex set L and an arrow $x \rightarrow z$ for each cover relation $x < z$. The lattice L is said to be *weakly atomic* if every $x < y$ in L contains a cover relation, namely, $\text{Hasse}[x, y]$ has at least one arrow. Every finite lattice is obviously weakly atomic. An element $x \in L$ is *join-irreducible* if $x \neq y \vee z$ for all $y, z < x$, and *completely join-irreducible* if $x \neq \bigvee_{y \in S} y$ for every $S \subseteq \{y \in L \mid y < x\}$. The (completely) meet-irreducible elements are defined dually. By $\text{JI}(L)$ and $\text{MI}(L)$, we, respectively, denote the sets of join-irreducible and meet-irreducible elements.

Recall that L is *semidistributive* if $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$, and dually for meets, for all $x, y, z \in L$. Notably, this forces the set $\{a \in L \mid x \vee a = z\}$ to have a unique minimal element for every cover relation $x < z$.

An element $x \in L$ is *left modular* if $(y \vee x) \wedge z = y \vee (x \wedge z)$ for all $y \leq z$ in L . For a weakly atomic lattice it suffices to check this on cover relations $y < z$. Following [Ma], a finite lattice L of length m is called *extremal* if $|\text{JI}(L)| = |\text{MI}(L)| = m$. We also recall that, following [BS], a finite lattice L is said to be *left modular* if it has a maximal chain of left modular elements. Finite extremal lattices and finite left modular lattices give two distinct generalizations of finite distributive lattices. In particular, unlike distributive lattices, a (finite) extremal lattice may admit maximal chains shorter than m . However, the *spine* of an extremal lattice L , which is defined to be the union of all chains of maximal length, is always a distributive sublattice. As a more recent notion, following [Th], a finite lattice is called *trim* if it is both extremal and left modular. Every finite distributive lattice is trim, but the converse fails. Building on recent lattice-theoretic work, we will use the following.

Theorem 2.2 ([TW, Mu]) *Let L be a finite semidistributive lattice. Then, the following three notions are equivalent: (1) L is left modular; (2) L is extremal; (3) L is trim.*

We emphasize that the semidistributivity of L is an essential condition in the above theorem; see [Mu]. We also remark that, since $\text{tors } A$ is always semidistributive, Theorem 2.2 applies to $\text{tors } A$ whenever A is brick-finite. That being the case, in Section 5 we translate the three conditions of such a lattice $\text{tors } A$ into representation-theoretic statements about bricks in $\text{mod } A$.

3. BRICK-SPLITTING TORSION PAIRS

Definition 3.1 Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\text{mod } A$. Then, $(\mathcal{T}, \mathcal{F})$ is *splitting* if $\text{ind } A \subseteq \mathcal{T} \cup \mathcal{F}$, and it is *brick-splitting* if $\text{brick } A \subseteq \mathcal{T} \cup \mathcal{F}$. A torsion class $\mathcal{T}$ is called brick-splitting if $(\mathcal{T}, \mathcal{T}^\perp)$ is a brick-splitting torsion pair.

Every splitting pair is brick-splitting, since bricks are indecomposable. The pairs $(0, \text{mod } A)$ and $(\text{mod } A, 0)$ are always brick-splitting, thus called *trivial*. From the following example, which is a radical-square-zero string algebra with a loop at each of two vertices, one can easily conclude that the notions genuinely differ. This will serve as a running example throughout the note.

Example 3.2 Let Q be the quiver with vertices 1, 2, a loop α at 1, an arrow $\beta : 1 \rightarrow 2$, and a loop γ at 2, and let $A = kQ/I$ where I is generated by all paths of length 2. Writing modules by their radical filtration,

$$\text{ind } A = \left\{ 1, 2, \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}^1, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}^1, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}^1 \right\}, \quad \text{brick } A = \left\{ 1, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right\}.$$

Let $\mathcal{T} = \text{Fac} \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right)$, whose indecomposables are $1, \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}^1$, and let $\mathcal{F} = \mathcal{T}^\perp$, whose indecomposables are $2, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$. Then $(\mathcal{T}, \mathcal{F})$ is brick-splitting: all three bricks lie in $\mathcal{T} \cup \mathcal{F}$. However, $(\mathcal{T}, \mathcal{F})$ is not splitting, since the length-3 and length-4 indecomposables lie in neither $\mathcal{T}$ nor $\mathcal{F}$. One checks directly that $\text{mod } A$ admits no non-trivial splitting torsion pair at all, while the torsion classes $\text{Fac} \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right)$ and $\text{Fac} \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right)$ are both non-trivial brick-splitting torsion classes. By contrast, the torsion class $\mathcal{T}'' = \text{Fac} \left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right)$ is *not* brick-splitting: the brick $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ lies in neither $\mathcal{T}''$ nor $\mathcal{T}''^\perp$.

The full lattice $\text{tors } A$ for this algebra has five elements, and is depicted in Figure 1, with each non-trivial torsion class labeled by a generating module.

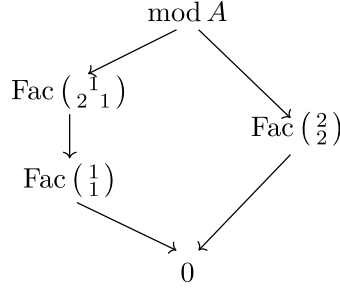


FIGURE 1. The lattice $\text{tors } A$ of Example 3.2.

The left chain $\text{mod } A \supset \text{Fac} \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right) \supset \text{Fac} \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) \supset 0$ passes via all three brick labels, while the right chain of length two, through $\text{Fac} \left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \right)$, does not.

The following theorem gives three equivalent descriptions of brick-splitting torsion classes, in terms interval description via labels, and also a lattice-theoretic description via left modularity.

Theorem 3.3 *For an algebra A and $\mathcal{T} \in \text{tors } A$, the following are equivalent:*

- (1) $(\mathcal{T}, \mathcal{T}^\perp)$ is a brick-splitting torsion pair;
- (2) every $X \in \text{brick } A$ appears as the label of an arrow of $\text{Hasse}[0, \mathcal{T}]$ or of $\text{Hasse}[\mathcal{T}, \text{mod } A]$;
- (3) $\mathcal{T}$ is a left modular element of the lattice $\text{tors } A$.

Sketch. (1) $\Leftrightarrow$ (2): if $X \in \mathcal{T}$, then X labels the minimal cover of $\mathcal{T}(X) \cap {}^\perp X$ inside $[0, \mathcal{T}]$, where $\mathcal{T}(X)$ is the smallest torsion class containing X ; the case $X \in \mathcal{T}^\perp$ is dual, using intervals in $[\mathcal{T}, \text{mod } A]$. Conversely, a label appearing in either interval forces the corresponding brick into $\mathcal{T}$ or $\mathcal{T}^\perp$.

(3) $\Rightarrow$ (1): if $\mathcal{T}$ is left modular and $L \in \text{brick } A \setminus \mathcal{T}$, let $\mathcal{V} \triangleleft \mathcal{T}(L)$ be the unique cover below the (completely join-irreducible) torsion class $\mathcal{T}(L)$. Left modularity gives $(\mathcal{V} \vee \mathcal{T}) \cap \mathcal{T}(L) = \mathcal{V} \vee (\mathcal{T} \cap \mathcal{T}(L)) = \mathcal{V}$, so $L \notin \mathcal{V} \vee \mathcal{T}$; a short argument with the canonical sequence $0 \rightarrow t(L) \rightarrow L \rightarrow L/t(L) \rightarrow 0$ then forces $t(L) = 0$, i.e. $L \in \mathcal{T}^\perp$.

(1) $\Rightarrow$ (3): using weak atomicity, it suffices to check $(\mathcal{U} \cap \mathcal{T}) \vee \mathcal{V} = \mathcal{U} \cap (\mathcal{T} \vee \mathcal{V})$ for each cover $\mathcal{V} \triangleleft \mathcal{U}$ with label L . If $L \in \mathcal{T}$ this follows since $\mathcal{T}(L) \vee \mathcal{V} = \mathcal{U}$; if $L \in \mathcal{T}^\perp$ this follows since $\mathcal{T}, \mathcal{V} \subseteq {}^\perp L$ and ${}^\perp L \cap \mathcal{U} = \mathcal{V}$. Full details are given in [AIMP1]. $\square$

As shown below, brick-splitting torsion pairs also control the extension quiver of A .

Proposition 3.4 *Let A be connected and let $(\mathcal{T}, \mathcal{F})$ be a non-trivial brick-splitting torsion pair. If S_1 is a simple module in $\mathcal{T}$ and S_2 is a simple module in $\mathcal{F}$, then $\text{Ext}_A^1(S_2, S_1) = 0$.*

Proof. Any non-split short exact sequence $0 \rightarrow S_1 \rightarrow E \rightarrow S_2 \rightarrow 0$ has E indecomposable of length two with non-isomorphic composition factors, so every non-zero endomorphism of E is invertible and E is a brick. By brick-splitting, $E \in \mathcal{T} \cup \mathcal{F}$; but $E \in \mathcal{T}$ forces the quotient map to S_2 to vanish, and $E \in \mathcal{F}$ forces the inclusion of S_1 to vanish, both contradictions. $\square$

For a torsion class $\mathcal{T}$ in $\text{mod } A$, let $e_{\mathcal{T}}$ denote the idempotent supported on the simples lying in $\mathcal{T}$. Then, Proposition 3.4 shows that $e_{\mathcal{T}}A(1 - e_{\mathcal{T}}) = 0$, so A admits a triangular decomposition along $\mathcal{T}$. We say A is *fully cyclic* if $eA(1 - e) = 0$ forces $e \in \{0, 1\}$ for every idempotent e . Then, we obtain the following result.

Corollary 3.5 *If A is fully cyclic, then the only brick-splitting torsion pairs in $\text{mod } A$ are the trivial ones. In particular, this holds whenever A is a connected self-injective algebra.*

In the preceding corollary, the assertion on the self-injective case follows because a non-trivial idempotent decomposition $eA(1 - e) = 0$ would force the projective module $A(1 - e)$, which is then also injective, to be supported only on the vertices corresponding to $1 - e$. This contradicts connectedness of A via $D((1 - e)A) \cong A(1 - e)$.

In comparison between the classical notion of splitting pairs and our new notion of brick-splitting pairs, two natural questions are when brick-splitting torsion pairs actually coincide with splitting ones, and how large the gap between the two notions can be. For hereditary algebras, being functorially finite closes this gap.

Proposition 3.6 *If A is a tame hereditary algebra, then every torsion pair in $\text{mod } A$ is splitting if and only if it is brick-splitting. More generally, for arbitrary hereditary algebra A , every functorially finite brick-splitting torsion pair is splitting.*

The above proposition implies that to find brick-splitting torsion pairs which are not splitting, such as the one discussed in Example 3.2, one should consider those algebras which are not tame hereditary nor locally representation-directed algebras (those algebras for which $\text{ind } A = \text{brick } A$), because for either of these families the two notions of splitting always coincide. In particular, for tame hereditary case, the important point is the structure of the Auslander-Reiten quiver of such algebras. More specifically, for a fix length $d \in \mathbb{Z}_{>0}$, almost all non-brick indecomposables of length d lie in tubes whose quasi-simple modules are bricks. If two Hom-orthogonal quasi-simples of the same tube were split between $\mathcal{T}$ and $\mathcal{F}$, then a non-split extension between them would produce a brick outside $\mathcal{T} \cup \mathcal{F}$, contradicting brick-splitting, which implies an entire tube lies on one side; for more details, see [AIMP1, Proposition 3.11].

4. BRICK-DIRECTED ALGEBRAS

We start by recalling that $X \in \text{ind } A$ is *directing* if it lies on no cycle $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_m = X_0$ of non-zero non-isomorphisms in $\text{mod } A$. Then, A is said to be a *representation-directed* algebra if every indecomposable is directing. Over representation-directed, every indecomposable is known to be rigid (i.e., has no self-extension), and such algebras are always representation-finite; for more details, see [ASS, IX]. We now weaken this classical directedness notion by specializing to the cycle of bricks, and introduce a brick-analogue of representation-directed algebras.

Definition 4.1 A *brick-cycle* is a cycle $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_m = X_0$ of non-zero non-invertible morphisms in $\text{mod } A$ with every $X_i \in \text{brick } A$. We say A is a *brick-directed* algebra if $\text{mod } A$ contains no brick-cycle.

Every representation-directed algebra is evidently brick-directed, but the converse is far from true in general. For instance, the algebra A of Example 3.2 is brick-directed without being representation-directed. In fact, every brick of A lies on a cycle of indecomposables that are not all bricks, while the three bricks themselves admit no cyclic sequence of non-zero non-isomorphisms among themselves. Thus brick-directedness genuinely enlarges the classical notion. As recalled below, the new notion of brick-directed permits representation-infinite examples, and even examples of brick-infinite algebras.

For our study of brick-directedness, it is convenient to combinatorially encode Hom-orthogonality of bricks. In that regard, we introduce the *brick quiver* of A , denoted by $Q^b(A)$, as following: the vertex set is $\text{brick } A$, with an arrow $X \rightarrow Y$ whenever $X \neq Y$ and $\text{Hom}_A(X, Y) \neq 0$. Then, from the definition it immediately follows that A is brick-directed exactly when $Q^b(A)$ is acyclic.

Proposition 4.2 *For a connected algebra A , the brick quiver $Q^b(A)$ is connected and has no loops. It has finitely many arrows if and only if it has finitely many vertices, if and only if A is brick-finite. Each simple module labels a vertex of $Q^b(A)$ with an arrow to or from any non-simple brick sharing a composition factor with it.*

Connectedness of $Q^b(A)$ follows because every non-simple brick X has an arrow to some simple summand of $\text{top}(X)$ and an arrow from some simple summand of $\text{soc}(X)$, and any arrow $\alpha : i \rightarrow j$ of Ω_A that is not a loop produces a length-two brick connecting S_i and S_j . The remaining statements of the above proposition follow directly from the definitions. Together with Theorem 3.3 one obtains a useful bijective reformulation as follows: the brick-splitting torsion classes of A correspond exactly to the successor-closed subsets of $Q^b(A)$, via $\mathcal{T} \mapsto \mathcal{T} \cap \text{brick } A$. See [AIMP1, Proposition 3.5].

Combining Proposition 4.2 with Demonet’s correspondence between chains of bricks and chains of torsion classes [KD, Appendix], one obtains the following characterization, which is the source of many subsequent results.

Theorem 4.3 *For an algebra A , the following are equivalent:*

- (1) A is brick-directed;
- (2) $Q^b(A)$ is acyclic;
- (3) there is a maximal chain $\{\mathcal{T}_i\}_{i \in I}$ in $\text{tors } A$ all of whose members are brick-splitting.

Moreover, such a maximal chain corresponds bijectively to a total order on $\text{brick } A = \{B_j\}_{j \in J}$ with $\text{Hom}_A(B_i, B_j) = 0$ whenever $i < j$.

Example 4.4 For $n = 2$, let $\Lambda_2 = kQ_2/I_2$ where Q_2 has vertices $1, 2$, a loop α at 1 , an arrow $\beta : 1 \rightarrow 2$, a loop γ at 2 , and $I_2 = \langle \alpha^2, \gamma^2, \gamma\beta\alpha \rangle$. This is a representation-infinite string algebra, known as “wind-wheel” algebra of rank 2, and it admits a unique band $\beta^{-1}\gamma^{-1}\beta\alpha$. It is known that Λ_2 is brick-finite [Mo], and we have

$$\text{brick } \Lambda_2 = \left\{ 1, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\}.$$

Its brick quiver $Q^b(\Lambda_2)$ has arrows $2 \rightarrow \frac{1}{2} \rightarrow 1$, together with $\frac{1}{2} \rightarrow 2, \frac{1}{2} \rightarrow \frac{1}{2}, \frac{1}{2} \rightarrow \frac{1}{2}, 2 \rightarrow \frac{1}{2}$, and $\frac{1}{2} \rightarrow \frac{1}{2}$. This quiver is acyclic, so Λ_2 is brick-directed even though it is representation-infinite and, therefore, not representation-directed. Observe that if take the quotient of Λ_2 by the relation that kills γ , we recover the algebra A of Example 3.2.

We recall that each representation-directed algebra is triangular, that is, the Ext-quiver of the algebra has no oriented cycle; see [ASS, IX]. As a consequence of our analysis, we conclude that every brick-directed algebra is in fact *weakly triangular*, that is, the only oriented cycles in its Ext-quiver are powers of loops; see [AIMP1, Corollary 4.2]. This difference can be seen in Examples 3.2 and 4.4.

As shown in the next propositions, brick-directedness is inherited by many natural operations.

Proposition 4.5 *Let A be brick-directed. Then, we have the following:*

- (1) A/J is brick-directed for every two-sided ideal J .
- (2) eAe is brick-directed for every idempotent $e \in A$.
- (3) every τ -tilting reduction of A is brick-directed.

In the previous proposition, we remark that in each case $\text{mod}(-)$ embeds as a full subcategory of $\text{mod } A$, thus, a brick-cycle downstairs would produce one upstairs, contradicting brick-directedness of A . In particular, strictly wild algebras are never brick-directed (however, an arbitrary wild algebra can be brick-directed). For connected hereditary algebras, we have that kQ is brick-directed exactly when Q is Dynkin or the Kronecker quiver.

Finally, a gluing construction lets one manufacture brick-directed algebras of arbitrary rank and representation type. Given connected bound quivers (Q, I) and (Q', I') with x_n a sink of Q and y_1 a source of Q' , identifying x_n with y_1 and killing all length-two paths through the new vertex produces an algebra Λ of rank $|Q_0| + |Q'_0| - 1$ in which every indecomposable is entirely supported on the Q -side or the Q' -side, and Λ is brick-directed if and only if both pieces are. Iterating this construction yields the following result, allowing us to construct explicit examples of brick-directed algebras which have various representation-types and brick-types; see [AIMP1, Corollary 4.13].

Corollary 4.6 *For every positive integer n , there exists a connected brick-directed algebra of rank n of each of the following types: (I) representation-finite; (II) brick-finite tame; (III) brick-infinite tame; (IV) brick-finite wild; (V) brick-infinite wild.*

We note that Case (II) is realized by the “wind-wheel” algebras of Ringel [Ri], which are known to be minimal representation-infinite special biserial algebras that are nonetheless brick-finite [Mo]. On the other hand, Cases (III)–(V) glue Kronecker-type or local wild pieces onto Dynkin quivers of appropriate rank. For details, see [AIMP1, Section 4.3]. Notably, and in sharp contrast with representation-directed algebras, arbitrary brick-directed algebras need not admit a preprojective component in their Auslander–Reiten quiver.

5. THE BRICK-FINITE CASE: LEFT MODULARITY, EXTREMALITY, AND TRIMNESS

We now specialize to brick-finite algebras, where $\text{tors } A$ is a finite (semidistributive) lattice and Theorem 2.2 applies. In particular, combining Theorems 3.3 and 4.3 with Theorem 2.1 yields the main result of this manuscript.

Theorem 5.1 *Let A be a brick-finite algebra. The following are equivalent:*

- (1) A is brick-directed;
- (2) $\text{tors } A$ is a left modular lattice;
- (3) $\text{tors } A$ is an extremal lattice;
- (4) $\text{tors } A$ is a trim lattice.

Moreover, if these conditions hold, then the spine of $\text{tors } A$ consists precisely of the brick-splitting torsion classes, and it forms a distributive sublattice of $\text{tors } A$.

Sketch. (1) $\Leftrightarrow$ (2) is immediate from Theorems 3.3 and 4.3. More precisely, A is brick-directed if and only if $\text{tors } A$ has a maximal chain of brick-splitting torsion classes, if and only if (by Theorem 3.3) it has a maximal chain of left modular elements, which is the definition of left modularity for $\text{tors } A$.

For (1) $\Leftrightarrow$ (3), note that by Theorem 2.1, $|\text{JI}(\text{tors } A)| = |\text{MI}(\text{tors } A)| = |\text{brick } A|$, so $\text{tors } A$ is extremal exactly when it admits a maximal chain of length $|\text{brick } A|$. If A is brick-directed, the chain from Theorem 4.3 realizes every brick as a label (Theorem 3.3(2)), hence has length $|\text{brick } A|$. Conversely, a maximal chain of length $|\text{brick } A|$ has all its labels distinct, hence exhausts $\text{brick } A$,

giving (via Demonet's correspondence [KD, Appendix]) a total order on brick A with vanishing backward Homs, so A is brick-directed.

The equivalence (1) $\Leftrightarrow$ (4) now follows since $\text{tors } A$ is always semidistributive, so Theorem 2.2 identifies trimness with the conjunction of (2) and (3), which are already equivalent to (1). The description of the spine follows from the identification of left modular elements with brick-splitting torsion classes, together with the known description of the spine of a trim lattice [TW, Thm. 3.7]. $\square$

Below, we consider several consequences of Theorem 5.1 which are of significance and provide further insights into the classical and modern notions treated in our work.

Corollary 5.2 *Let A be brick-finite with $\text{tors } A$ extremal (equivalently, left modular). Then $\text{tors}(A/J)$, $\text{tors}(eAe)$, and $\text{tors } B$ for B a τ -tilting reduction of A are all extremal.*

Corollary 5.3 *A brick-finite algebra A is representation-directed if and only if it is representation-finite and the length of $\text{tors } A$ equals $|\text{ind } A|$.*

Indeed, if A is representation-directed then $\text{ind } A = \text{brick } A$, so A is brick-directed and Theorem 5.1 gives a chain of length $|\text{brick } A| = |\text{ind } A|$. Conversely the length of $\text{tors } A$ is always at most $|\text{brick } A| \leq |\text{ind } A|$, so equality forces $\text{brick } A = \text{ind } A$ and a chain realizing all of $\text{ind } A$, which by Demonet's correspondence yields a total order on $\text{ind } A$ with vanishing backward Homs.

Then next corollary strengthens the classical fact that indecomposables over a representation-directed algebra are determined by their dimension vector. In particular, we show a stronger result for brick-finite brick-directed algebras, which need not be representation-finite.

Corollary 5.4 *Let A be a brick-finite and brick-directed algebra. Then, distinct bricks of A have distinct dimension vectors.*

The preceding result is proved by choosing, along the ordering $X_1, \dots, X_r$ of brick A from Theorem 4.3, a compatible sequence of stability parameters θ_i separating each X_i from its predecessors and successors; if $[X_l] = [X_m]$ for $l < m$ one gets $\theta_l([X_m]) < 0$ from $X_m \in \mathcal{F}_{\theta_l}$ and $\theta_l([X_l]) \geq 0$ from $X_l \in \overline{\mathcal{T}}_{\theta_l}$, a contradiction. We suppress the precise definitions of these stability parameters here and refer to [AIMP1, Corollary 5.7] for details.

Remark 5.5 The converse of Corollary 5.4 fails. Let $A = kQ/I$ with Q the triangle quiver on vertices $1, 2, 3$ with arrows $\alpha : 1 \rightarrow 2$, $\beta : 2 \rightarrow 3$, $\gamma : 3 \rightarrow 1$, and $I = \langle \beta\alpha, \gamma\beta, \alpha\gamma \rangle$. Then $\text{ind } A = \text{brick } A$ consists of the three simples and the three length-two modules $\frac{1}{2}, \frac{2}{3}, \frac{3}{1}$, all of pairwise distinct dimension vectors, yet $\frac{1}{2} \rightarrow \frac{3}{1} \rightarrow \frac{2}{3} \rightarrow \frac{1}{2}$ is a brick-cycle, so A is not brick-directed, and by Theorem 5.1 the lattice $\text{tors } A$, although finite and semidistributive, is neither left modular nor extremal.

Example 5.6 Returning to the wind-wheel algebra Λ_2 of Example 4.4, note that since $Q^b(\Lambda_2)$ is acyclic, Λ_2 is brick-directed, and Theorem 5.1 shows that $\text{tors } \Lambda_2$, a finite lattice of length equal to the number of indecomposable τ -rigid modules, is simultaneously left modular, extremal, and trim, with spine consisting of the five brick-splitting torsion classes obtained by successively adjoining $2, \frac{1}{2}, \frac{1}{2}$ or $\frac{1}{2}$, and finally 1 , to the brick quiver of Example 4.4.

6. BEYOND BRICK-FINITENESS

Theorem 5.1 is a statement about *finite* lattices, since extremality and trimness were classically only defined in that setting. It is natural to ask whether the equivalence between brick-directedness and left modularity also holds for brick-infinite algebras, that is, where $\text{tors } A$ is an infinite lattice. We remark that several of the constructions in Corollary 4.6 already produce brick-directed algebras

which are brick-infinite, so the question is not vacuous. This question is answered affirmatively in our recent work [AIMP2], which we summarize briefly.

The key observation is that $\text{tors } A$ is always *completely semidistributive* and weakly atomic. Moreover, it is a *well-separated κ -lattice*, meaning that it is bispatial (namely, every element is both a join of completely join-irreducible elements and a meet of completely meet-irreducible elements), and there exists the canonical bijection $\kappa : \text{JI}^c(\text{tors } A) \rightarrow \text{MI}^c(\text{tors } A)$ of Theorem 2.1 that satisfies a strong separation property; for details, see [DI+, RST]. For such lattices, left modularity of an element x admits the following combinatorial reformulation, echoing the earlier characterization of brick-splitting torsion classes given in Theorem 3.3.

Theorem 6.1 ([AIMP2, Thm. 2.6]) *Let L be a weakly atomic completely semidistributive lattice. For $x \in L$, the following are equivalent: x is left modular; for every completely join-irreducible j , either $j \leq x$ or $x \leq \kappa(j)$. Moreover the set of left modular elements forms a completely distributive complete sublattice of L .*

Observe that if Theorem 6.1 is applied to $\text{tors } A$, we recover Theorem 3.3, that is, a torsion class is left modular exactly when every brick is comparable to it in the appropriate sense (i.e., lies in $\mathcal{T}$ or in $\mathcal{T}^\perp$).

To extend extremality, one needs a version that does not presuppose finiteness of chains. As in [AIMP2], an arbitrary lattice L is called *extremal* if there is a bijection $\lambda : \text{JI}^c(L) \rightarrow \text{MI}^c(L)$ and a maximal chain C such that each $x \in C$ satisfies $x \leq \lambda(j)$ whenever $j \not\leq x$, and each cover relation in C picks out a matched pair $(j, \lambda(j))$ under λ . This recovers the classical notion of extremality for finite lattices, and the following generalization of Theorem 2.2 holds.

Theorem 6.2 ([AIMP2, Thm. 2.9]) *Let L be a well-separated κ -lattice. Then, L is left modular if and only if L is extremal.*

Since $\text{tors } A$ is always a well-separated κ -lattice, Theorem 6.2 applies verbatim. In fact, the same argument as in Theorem 5.1 (without the finiteness hypothesis) shows that a (possibly brick-infinite) algebra A is brick-directed if and only if $\text{tors } A$ is left modular, if and only if $\text{tors } A$ is extremal in the generalized sense above. In particular, the brick-infinite tame and wild algebras of type (III) and (V) in Corollary 4.6, obtained by gluing, provide explicit infinite left modular and extremal lattices of torsion classes, of arbitrarily large “rank” n . This shows that the phenomenon described by Theorem 5.1 is not an artifact of brick-finiteness, but in fact a structural feature of the interaction between bricks and the lattice of torsion classes that holds in full generality.

CONCLUDING REMARKS

The results surveyed here identify left modularity, extremality, and trimness of $\text{tors } A$ with the purely representation-theoretic condition of brick-directedness, at least once A is brick-finite, and this identification extends, via the theory of well-separated κ -lattices, to arbitrary finite dimensional algebras. Although some further progress in these results will appear in our future work, several natural questions remain open. In particular, it is interesting to ask whether the module variety of a brick-directed algebra always has at most one brick component in each dimension vector. We refer the interested reader to [AIMP1] and [AIMP2] for full proofs and further results, including a treatment via wall-and-chamber structures and Newton polytopes that we did not discuss here.

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