

A Pieri rule for H -equivariant K -group of semi-infinite flag manifolds

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Abstract

We study the Pieri problem in the H -equivariant K -group $K_H(\mathbf{Q}_G)$ of the semi-infinite flag manifold. Motivated by the equivariant Pieri–Chevalley formula of Kato, Naito, and Sagaki and the Pieri-type multiplication formula for quantum Grothendieck polynomials, we formulate a conjectural Pieri rule describing the tensor product of an arbitrary semi-infinite Schubert class with a special Schubert class. The conjectural formula is indexed by k -Pieri chains, which are directed paths in the quantum Bruhat graph satisfying suitable Pieri conditions.

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1 Introduction

Equivariant quantum K -theory of flag varieties is a central topic in Schubert calculus. A fundamental problem is to obtain explicit multiplication formulas for Schubert classes. Among them, Pieri rules describe multiplication by distinguished Schubert classes and often admit elegant combinatorial descriptions in terms of the quantum Bruhat graph.

Let $G = SL_{n+1}(\mathbb{C})$, and let B and H be the Borel subgroup of upper triangular matrices and the maximal torus of diagonal matrices, respectively. Denote by $Fl_{n+1} = G/B$ the complete flag variety of type A_n . Quantum Grothendieck polynomials and quantum double Grothendieck polynomials provide polynomial representatives for Schubert classes in quantum K -theory and equivariant quantum K -theory of Fl_{n+1} , respectively; see [LM06, MNS25a, MNS25b]. These polynomial models reveal a close relationship between the multiplicative structure of quantum K -theory and the combinatorics of the quantum Bruhat graph.

The semi-infinite flag manifold $\mathbf{Q}_G$ provides a natural framework for studying multiplication formulas in quantum K -theory. Its H -equivariant K -group $K_H(\mathbf{Q}_G)$ has a basis consisting of the structure sheaves of semi-infinite Schubert varieties, and the K -theoretic Peterson isomorphism identifies $K_H(\mathbf{Q}_G)$ with the equivariant quantum K -theory of Fl_{n+1} ; see [Kat23, Kat25]. Under this correspondence, the quantum parameters are realized as translation operators.

A fundamental multiplication formula in $K_H(\mathbf{Q}_G)$ was established by Kato, Naito, and Sagaki, who proved an equivariant Pieri–Chevalley formula describing the tensor product of a semi-infinite Schubert class with a line bundle class; see [KNS20]. Their formula is expressed in terms of semi-infinite Lakshmibai–Seshadri paths and serves as the starting point for the present work.

The purpose of this paper is to establish a *Pieri rule* in $K_H(\mathbf{Q}_G)$. We obtain a conjectural explicit expansion of the tensor product of an arbitrary semi-infinite Schubert class

with a distinguished Schubert class in the basis of semi-infinite Schubert classes. The expansion is indexed by k -Pieri chains, namely directed paths in the quantum Bruhat graph satisfying suitable Pieri conditions.

Our approach combines the equivariant Chevalley formula in $K_H(\mathbf{Q}_G)$ with a combinatorial analysis of the quantum Bruhat graph. This enables us to organize the contributing terms into k -Pieri chains and derive the desired Pieri rule, yielding an explicit combinatorial description of multiplication in $K_H(\mathbf{Q}_G)$.

2 Preliminary

2.1 Algebraic groups

Let $G = SL_{n+1}(\mathbb{C})$ be the connected, simply connected simple algebraic group of type A_n over $\mathbb{C}$. Let B be the Borel subgroup of upper triangular matrices and H the maximal torus of diagonal matrices. We denote by $R(H)$ the representation ring of H .

Let $Q^\vee = \sum_{i=1}^n \mathbb{Z} \alpha_i^\vee$ be the coroot lattice of G , where $\alpha_1^\vee, \dots, \alpha_n^\vee$ are the simple coroots, and let $Q^{\vee,+} = \sum_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i^\vee$ be the dominant coroot lattice. We write $P = \sum_{i=1}^n \mathbb{Z} \varpi_i$ for the weight lattice of G , where $\varpi_1, \dots, \varpi_n$ are the fundamental weights, and denote by $\mathbb{Z}[P] = \sum_{\lambda \in P} \mathbb{Z} \mathbf{e}^\lambda$ its group algebra.

The Weyl group of G is $W = S_{n+1} = \langle s_1, \dots, s_n \rangle$, where $s_i = (i, i+1)$ is the simple reflection corresponding to α_i . The affine Weyl group is $W_{\text{af}} \cong W \ltimes Q^\vee$, and we write $W_{\text{af}}^{\geq 0} \cong W \ltimes Q^{\vee,+}$ for the subset consisting of affine Weyl group elements whose translation part is dominant.

2.2 H -equivariant K -group of the semi-infinite flag manifold

Following [KNS20], the semi-infinite flag manifold associated with G is $\mathbf{Q}_G = G(\mathbb{C}((t)))/B(\mathbb{C}[[t]])$. Its H -equivariant K -group is $K_H(\mathbf{Q}_G) = \prod_{x \in W_{\text{af}}^{\geq 0}} \mathbb{Z}[P][\mathcal{O}_{\mathbf{Q}_G}(x)]$, where $[\mathcal{O}_{\mathbf{Q}_G}(x)]$ denotes the semi-infinite Schubert class indexed by $x \in W_{\text{af}}^{\geq 0}$.

2.3 H -equivariant quantum K -theory of the full flag manifold

Following [Lee04], let $Fl_{n+1} = G/B$ be the complete flag manifold of type A_n . Its H -equivariant K -group is $K_H(Fl_{n+1}) = \bigoplus_{w \in W} \mathbb{Z}[P][\mathcal{O}^w]$, where $[\mathcal{O}^w]$ denotes the Schubert class indexed by $w \in W$. The corresponding H -equivariant quantum K -theory ring is $QK_H(Fl_{n+1}) = K_H(Fl_{n+1}) \otimes_{R(H)} \mathbb{Z}[P][[Q]]$, where $Q = (Q_1, \dots, Q_n)$ is the family of Novikov variables with $Q_i = Q^{\alpha_i^\vee}$ for $1 \leq i \leq n$.

Theorem 2.1 (see [Kat25]). *We have an $\mathbb{Z}[P]$ -isomorphism of commutative algebras*

$$\Phi : QK_H(Fl_{n+1}) \rightarrow K_H(\mathbf{Q}_G)$$

such that $\mathbf{e}^\mu[\mathcal{O}^w][Q^\xi] \mapsto \mathbf{e}^{-\mu}[\mathcal{O}_{\mathbf{Q}_G}(wt_\xi)]$ for $\mu \in P$, $w \in W$, $\xi \in Q^{\vee,+}$.

Definition 2.2 (Demazure operators, see [LM06]). *For $1 \leq i \leq n-1$, let*

$$\pi_i^{(y)} = \text{id} + \frac{1 - y_i}{y_i - y_{i+1}} (\text{id} - s_i)$$

acting on $\text{Frac}(\mathbb{Z}[y_1^{\pm 1}, \dots, y_n^{\pm 1}])$, where s_i exchanges y_i and y_{i+1} .

If $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced decomposition, define

$$\pi_w^{(y)} = \pi_{i_1}^{(y)} \cdots \pi_{i_\ell}^{(y)}.$$

This operator is well-defined, i.e., independent of the choice of reduced decomposition.

Definition 2.3 (Quantum double Grothendieck polynomials, see [LM06]). *Let w_0 be the longest element of $W = S_{n+1}$. Define*

$$\mathfrak{G}_{w_0}^Q(x, y) := \prod_{k=1}^n \left(\sum_{l=0}^k (-1)^l (1 - y_{n+1-l})^l F_l^k(x_1, \dots, x_{n+1}) \right),$$

where

$$F_l^k(x_1, \dots, x_{n+1}) = \sum_{\substack{J \subset \{1, \dots, k\} \\ |J|=l}} \prod_{\substack{j \in J \\ j+1 \notin J}} (1 - Q_j) \prod_{j \in J} (1 - x_j).$$

For $w \in W$, the quantum double Grothendieck polynomial is defined by

$$\mathfrak{G}_w^Q(x, y) := \pi_{ww_0}^{(y)} \mathfrak{G}_{w_0}^Q(x, y).$$

Remark 2.4.

- $\mathfrak{G}_w(x, y) = \mathfrak{G}_w^{Q=0}(x, y)$, the double Grothendieck polynomial;
- $\mathfrak{G}_w^Q(x, y) = \mathcal{Q}(\mathfrak{G}_w(x, y))$, where $\mathcal{Q}$ denotes the quantization map;
- $\mathfrak{G}_w^Q(x) = \mathfrak{G}_w^Q(x, y=0)$, the quantum Grothendieck polynomial.

Setting

$$1 - y_j = 1 - \mathbf{e}^{-\epsilon_j}, \quad 1 \leq j \leq n,$$

we identify

$$R(H) \cong \mathbb{Z}[\mathbf{e}^{\pm \epsilon_1}, \dots, \mathbf{e}^{\pm \epsilon_n}] \cong \mathbb{Z}[(1 - y_1)^{\pm 1}, \dots, (1 - y_n)^{\pm 1}], \quad (\text{ID1})$$

$$R(H)[[Q]][x_1, \dots, x_{n+1}] \cong \mathbb{Z}[[Q]][x_1, \dots, x_{n+1}][(1 - y_1)^{\pm 1}, \dots, (1 - y_n)^{\pm 1}]. \quad (\text{ID2})$$

Theorem 2.5 (see [MNS25b]). *Under the identification (ID2), there exists an $R(H)[[Q]]$ -algebra isomorphism*

$$\Psi^Q : R(H)[[Q]][x_1, \dots, x_{n+1}] / I^Q \longrightarrow QK_H(Fl_{n+1})$$

such that

$$\Psi^Q(\mathfrak{G}_w^Q(x, y) \bmod I^Q) = [\mathcal{O}^w],$$

where I^Q is the ideal of $R(H)[[Q]][x_1, \dots, x_{n+1}]$ generated by the elements

$$\sum_{\substack{J \subset \overline{1, n+1} \\ |J|=l}} \prod_{\substack{j \in J \\ j+1 \notin J}} \overline{1 - Q_j} \prod_{j \in J} (1 - x_j) - \sum_{\substack{J \subset \overline{1, n+1} \\ |J|=l}} \mathbf{e}^{\epsilon_J},$$

where

$$\overline{1 - Q_j} = \begin{cases} 1 - Q_j, & 1 \leq j \leq n, \\ 1, & j = n+1. \end{cases}$$

Motivated by the Pieri-type multiplication formula for quantum Grothendieck polynomials $\mathfrak{G}_w^Q(x)$ established in [NS25], we study the following problem.

Problem 2.6. *Find a Pieri rule in $K_H(\mathbf{Q}_G)$: For $w \in S_{n+1}$, $1 \leq k \leq n+1$,*

$$[\mathcal{O}_{\mathbf{Q}_G}(s_1 \dots s_k)] \otimes [\mathcal{O}_{\mathbf{Q}_G}(w)] = \sum_{v \in W_{af}^{\geq 0}} (\text{coefficient} = ?) [\mathcal{O}_{\mathbf{Q}_G}(v)].$$

3 Result

The equivariant Pieri–Chevalley formula describes multiplication by line bundle classes. To obtain a Pieri rule, one has to iterate this formula. The resulting terms are naturally encoded by directed paths in the quantum Bruhat graph. This motivates the following notion of k -Pieri chains.

Definition 3.1 (quantum Bruhat graph, see [NS25]). *The quantum Bruhat graph $\text{QBG}(S_{n+1})$ is the directed graph whose vertices are the elements of S_{n+1} and whose directed edges are of the form $x \xrightarrow{(a,b)} y$, with $x, y \in S_{n+1}$ and $1 \leq a < b \leq n+1$, such that $y = x \cdot (a, b)$ and either of the following holds:*

$$(B) \ell(y) = \ell(x) + 1,$$

$$(Q) \ell(y) = \ell(x) - 2(b - a) + 1.$$

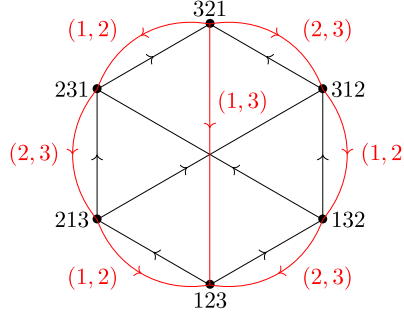
An edge satisfying condition (B) (resp., (Q)) is called a Bruhat edge (resp., a quantum edge). For a path

$$\mathbf{p} : x_0 \xrightarrow{(a_1, b_1)} x_1 \xrightarrow{(a_2, b_2)} \dots \xrightarrow{(a_r, b_r)} x_r$$

in $\text{QBG}(S_{n+1})$, we set $\ell(\mathbf{p}) := r$, $\text{end}(\mathbf{p}) := x_r$, and

$$\text{wt}(\mathbf{p}) := \sum_{x_{i-1} \xrightarrow{(a_i, b_i)} x_i \text{ is a quantum edge}} (a_i, b_i).$$

Example 3.2. $\text{QBG}(S_3)$



Definition 3.3 (k -Pieri chain, see [NS25]). *Let $w \in S_{n+1}$ and $k \geq 1$. A directed path*

$$\mathbf{p} : x_0 \xrightarrow{(a_1, b_1)} x_1 \xrightarrow{(a_2, b_2)} \dots \xrightarrow{(a_r, b_r)} x_r$$

in $\text{QBG}(S_{n+1})$ is called a k -Pieri chain if it satisfies the following conditions:

(P0) $a_s \leq k < b_s$, and (a_s, b_s) is not repeated for all $1 \leq s \leq r$;

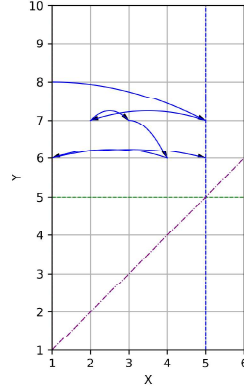
(P1) $b_1 \geq b_2 \geq \dots \geq b_r$;

(P2) If $r \geq 3$, and if $a_t = a_s$ for some $1 \leq t < s \leq r - 1$, then $(a_s, b_s) \prec (a_{s+1}, b_{s+1})$,

where $(a, b) \prec (c, d)$ means $(b > d)$ or $(b = d \text{ and } a < c)$.

Let $\mathbf{P}^k(w)$ denote the set of all k -Pieri chains starting at w .

Example 3.4. *The points in the figure below represent the labels of a 5-Pieri chain starting from $w = (1, 2, 7)(3, 6, 5, 8)$:*



Points: (1, 8), (5, 7), (2, 7), (3, 7), (4, 6), (1, 6), (5, 6)

Theorem 3.5 (Naito, Nguyen, Sagaki). *For $w, v \in S_{n+1}$, there exists at most one k -Pieri chain $\mathbf{p}$ from w to v .*

Fix $1 \leq k \leq n$. Let $J = \{j_1, j_2, \dots, j_p\} \subset [k] = \{1, 2, \dots, k\}$, with $1 \leq j_1 < j_2 < \dots < j_p \leq k$. Let $(\beta_a, \dots, \beta_1)$ be the sequence

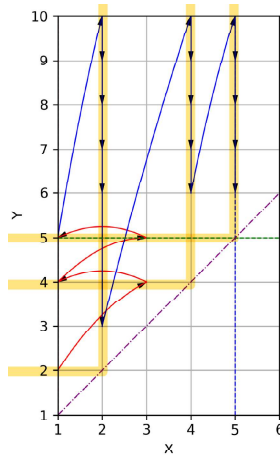
$$\begin{aligned} & (j_1, n+1), (j_1, n), \dots, \cancel{(j_1, j_p)}, \dots, \cancel{(j_1, j_2)}, \dots, (j_1, j_1+2), (j_1, j_1+1), \\ & (j_2, n+1), (j_2, n), \dots, \cancel{(j_2, j_p)}, \dots, \cancel{(j_2, j_3)}, \dots, (j_2, j_2+2), (j_2, j_2+1), \\ & \dots \dots \dots \\ & (j_{p-1}, n+1), (j_{p-1}, n), \dots, \cancel{(j_{p-1}, j_p)}, \dots, (j_{p-1}, j_{p-1}+2), (j_{p-1}, j_{p-1}+1), \\ & (j_p, n+1), (j_p, n), \dots, (j_p, j_p+2), (j_p, j_p+1), \end{aligned}$$

Let $(\gamma_1, \dots, \gamma_b)$ be the sequence

$$\begin{aligned} & (j_1, n+1), (j_1, n), \dots, \cancel{(j_1, j_p)}, \dots, \cancel{(j_1, j_2)}, \dots, (j_1, j_1+2), (j_1, j_1+1), \\ & (j_2, n+1), (j_2, n), \dots, \cancel{(j_2, j_p)}, \dots, \cancel{(j_2, j_3)}, \dots, (j_2, j_2+2), (j_2, j_2+1), \\ & \dots \dots \dots \\ & (j_{p-1}, n+1), (j_{p-1}, n), \dots, \cancel{(j_{p-1}, j_p)}, \dots, (j_{p-1}, j_{p-1}+2), (j_{p-1}, j_{p-1}+1), \\ & (j_p, n+1), (j_p, n), \dots, (j_p, j_p+2), (j_p, j_p+1), \end{aligned}$$

where (j_r, j_s) does not appear in the sequence above for any $1 \leq r < s \leq p$.

Example 3.6. *For $n = 9$, $k = 5$, and $J = 2, 4, 5$, the red arrows represent the β -sequence, while the blue arrows represent the γ -sequence. The labels of the chain are represented by points in $\mathbb{N}^2$.*

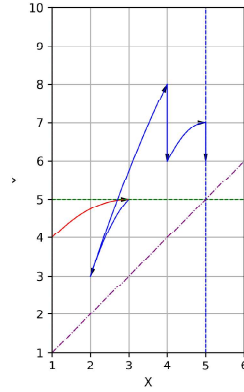


Definition 3.7 ($\beta\gamma$ -chain, see [MNS25b]). Let $w \in S_{n+1}$, $J \subset [k]$. A $\beta\gamma$ -chain of type J starting from w is a directed path in $\text{QBG}(S_{n+1})$ of the form:

$$\mathbf{d} : w = \underbrace{x_s \xrightarrow{\beta_{a_s}} \cdots \xrightarrow{\beta_{a_1}} x_0}_{=: \mathbf{d}_\beta, \text{ the } \beta\text{-part of } \mathbf{d}} = \underbrace{y_0 \xrightarrow{\gamma_{b_1}} \cdots \xrightarrow{\gamma_{b_t}} y_t}_{=: \mathbf{d}_\gamma, \text{ the } \gamma\text{-part of } \mathbf{d}} =: \text{end}(\mathbf{d}),$$

with $s \geq 0$ and $a \geq a_s > \cdots > a_1 \geq 1$,
with $t \geq 0$ and $1 \leq b_1 < b_2 < \cdots < b_t \leq b$.

Example 3.8. A $\beta\gamma$ -chain $\mathbf{d}$ of type $J = \{2, 4, 5\}$ starting from $w = (1, 2, 7)(3, 6, 5, 8)$:

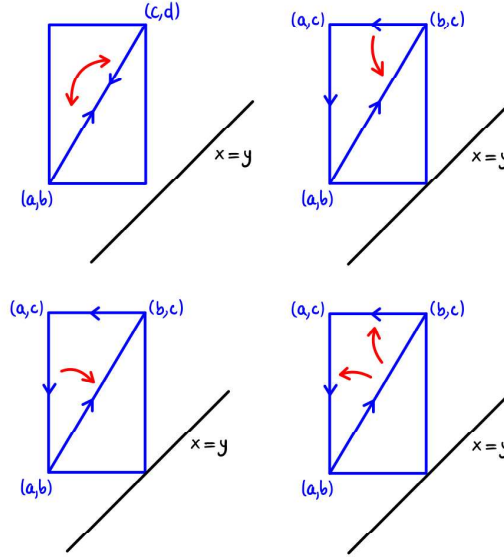


Points: (1, 4), (3, 5), (2, 3), (4, 8), (4, 6), (5, 7), (5, 6)

Here, $\mathbf{d}_\beta$ has edges (1, 4), (3, 5) and $\mathbf{d}_\gamma$ has edges (2, 3), (4, 8), (4, 6), (5, 7), (5, 6).

Lemma 3.9 (see [NS25]). Let $v \in S_{n+1}$, and $a, b, c, d \in \mathbb{Z}_+$.

- Assume that $a < b$, $c < d$, and $\{a, b\} \cap \{c, d\} = \emptyset$. If $(v; (a, b), (c, d))$ is a directed path, then so is $(v; (c, d), (a, b))$.
- Assume that $a < b < c$. If $(v; (a, c), (b, c))$ is a directed path, then so is $(v; (b, c), (a, b))$. Also, if $(v; (b, c), (a, c))$ is a directed path, then so is $(v; (a, b), (b, c))$.
- Assume that $a < b < c$. If $(v; (a, b), (a, c))$ is a directed path, then so is $(v; (b, c), (a, b))$. Also, if $(v; (a, c), (a, b))$ is a directed path, then so is $(v; (a, b), (b, c))$.
- Assume that $a < b < c$. If $(v; (a, b), (b, c))$ is a directed path, then either $(v; (b, c), (a, c))$ or $(v; (a, c), (a, b))$ is a directed path. Also, if $(v; (b, c), (a, b))$ is a directed path in the quantum Bruhat graph, then either $(v; (a, c), (b, c))$ or $(v; (a, b), (a, c))$ is a directed path.



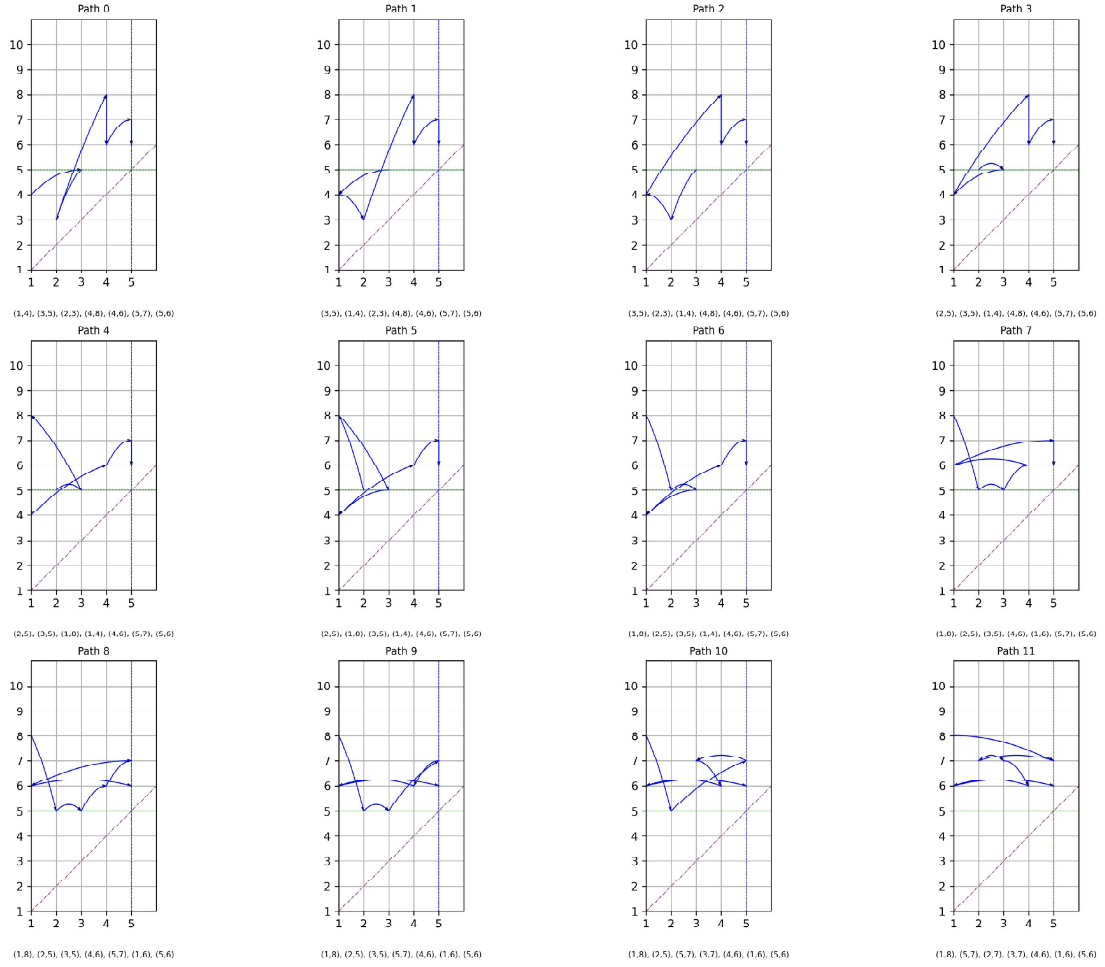
We say that two paths in $\text{QBG}(S_{n+1})$ are equivalent if they can be transformed to each other by a series of rules in the lemma.

Theorem 3.10 (Naito, Nguyen, Sagaki). *For each k -Pieri chain $\mathbf{p}$ starting from $w \in S_{n+1}$, there exists a unique $\beta\gamma$ -chain $\mathbf{d}$ starting from w such that $\mathbf{p}$ and $\mathbf{d}$ are equivalent.*

Example 3.11. *The $\beta\gamma$ -chain and 5-Pieri chain in the previous Examples are equivalent by the following transformation:*

$(1, 4), (3, 5), (2, 3), (4, 8), (4, 6), (5, 7), (5, 6)$
 $(3, 5), (1, 4), (2, 3), (4, 8), (4, 6), (5, 7), (5, 6)$
 $(3, 5), (2, 3), (1, 4), (4, 8), (4, 6), (5, 7), (5, 6)$
 $(2, 5), (3, 5), (1, 4), (4, 8), (4, 6), (5, 7), (5, 6)$
 $(2, 5), (3, 5), (1, 8), (1, 4), (4, 6), (5, 7), (5, 6)$
 $(2, 5), (1, 8), (3, 5), (1, 4), (4, 6), (5, 7), (5, 6)$
 $(1, 8), (2, 5), (3, 5), (1, 4), (4, 6), (5, 7), (5, 6)$
 $(1, 8), (2, 5), (3, 5), (4, 6), (1, 6), (5, 7), (5, 6)$
 $(1, 8), (2, 5), (3, 5), (4, 6), (5, 7), (1, 6), (5, 6)$
 $(1, 8), (2, 5), (3, 5), (5, 7), (4, 6), (1, 6), (5, 6)$
 $(1, 8), (2, 5), (5, 7), (3, 7), (4, 6), (1, 6), (5, 6)$
 $(1, 8), (5, 7), (2, 7), (3, 7), (4, 6), (1, 6), (5, 6)$

The corresponding chains are illustrated below.



For $J \subset [k]$, set $\epsilon_J := \sum_{j \in J} \epsilon_j$, where $\epsilon_j = \varpi_j - \varpi_{j+1}$ for $1 \leq j \leq n+1$.

The correspondence established in Theorem 3.10 suggests that the equivariant Pieri–Chevalley formula in $K_H(\mathbf{Q}_G)$ admits a description in terms of k -Pieri chains. This leads to the following conjectural Pieri rule.

Conjecture 3.12 (Naito, Nguyen, Sagaki). *It holds that $[\mathcal{O}_{\mathbf{Q}_G}(s_1 \dots s_k)] \otimes [\mathcal{O}_{\mathbf{Q}_G}(w)]$*

$$= \sum_{J \subset [k]} \sum_{\mathbf{p} \in \mathbf{P}^k(w)} \mathbf{e}^{|J| \epsilon_1 - \text{end}(\mathbf{p}_\beta) \epsilon_J} (-1)^{|J| + \ell(\mathbf{p}_\gamma)} [\mathcal{O}_{\mathbf{Q}_G}(\text{end}(\mathbf{p}) t_{\text{wt}(\mathbf{p})})],$$

where $\mathbf{p}_\beta := \mathbf{d}_\beta$ and $\mathbf{p}_\gamma := \mathbf{d}_\gamma$, with $\mathbf{d}$ is the unique $\beta\gamma$ -chain that is equivalent to $\mathbf{p}$.

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