

Heat Kernels on Covering Manifolds and Further Topics

Noncommutative Floquet–Bloch theory for nilpotent groups and a corrected polycyclic outlook

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Abstract

This note is based on the author’s lecture at the RIMS workshop *Interactions of Representation Theory and Related Topics* in June 2026. We first recall how the long-time behavior of the heat kernel on an abelian covering is obtained from the ordinary Floquet–Bloch decomposition and perturbation of the lowest twisted eigenvalue. For a nonabelian nilpotent deck group, the full unitary dual is non-type I and cannot be used as an ordinary Brillouin zone. The replacement developed in [11] begins on the lattice side: the regular representation is first decomposed into factor representations, the relevant central-induced factors are lifted to monomial representations of the Malcev completion, and Fujiwara’s theorem decomposes those monomial representations into Kirillov representations π_l . One then applies an exact restriction theorem to rational π_l , a finite-dimensional rational refinement governed by Howe’s classification, and a separate positive finitely additive Pytlik-type trace formula. The analytic companion [12] uses these exact rational fibers before passing to smooth Kirillov or Schrödinger normal forms and obtains local long-time expansions for nilpotent coverings. The Heisenberg group and the Harper operator illustrate the mechanism explicitly.

The final section corrects several statements about polycyclic groups that were too strong in the lecture slides. A general polycyclic group need not be a single semidirect product of an abelian and a nilpotent group, and every discrete group is unimodular. For symmetric finite-range random walks the known polycyclic dichotomy is polynomial decay in the virtually nilpotent case and cube-root stretched-exponential decay, in the comparison sense, in the exponentially growing case. For the special abelian-by-cyclic groups $\mathbb{Z}^d \rtimes_M \mathbb{Z}$ there is an exact Mackey–Floquet reduction to one-dimensional orbit Hamiltonians and exact finite periodic matrices. However, an explicit local/off-diagonal all-order expansion and a sharp Lifshitz constant remain separate analytic problems; they do not follow from a cubic Taylor term of a lowest band.

1 The heat-kernel problem on a covering

Let M be a compact connected Riemannian manifold and let

$$\pi : X \longrightarrow M$$

be a normal covering with deck transformation group Γ . The same formalism applies, with minor modifications, to a covering of a finite connected non-bipartite graph. We write $k_X(t, p, q)$ for the heat kernel of the nonnegative Laplacian Δ_X ; thus

$$u(t, p) = \int_X k_X(t, p, q) f(q) dq$$

solves $(\partial_t + \Delta_X)u = 0$ with $u(0, \cdot) = f$.

If Γ is finite, then X is compact and

$$k_X(t, p, q) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \varphi_j(p) \overline{\varphi_j(q)}, \quad 0 = \lambda_0 < \lambda_1 \leq \dots$$

Since $\varphi_0 = \text{vol}(X)^{-1/2}$,

$$\lim_{t \rightarrow \infty} k_X(t, p, q) = \frac{1}{\text{vol}(X)}.$$

For an infinite covering the spectrum reaches zero continuously and the problem becomes one of describing the low-energy density of states together with the corresponding local amplitudes.

The model example is $X = \mathbb{R}^b$ with deck group $\mathbb{Z}^b$:

$$k_{\mathbb{R}^b}(t, x, y) = (4\pi t)^{-b/2} \exp\left(-\frac{|x - y|^2}{4t}\right).$$

The exponent $b/2$ is the first indication that the large-time heat decay is controlled by the volume-growth geometry of the deck group. For a nilpotent group the Euclidean dimension b is replaced by the Bass–Guivarc’h homogeneous dimension.

2 Abelian coverings and classical Floquet–Bloch theory

Assume first that $\Gamma \simeq \mathbb{Z}^b$. Its unitary dual is the torus $\hat{\Gamma} \simeq \mathbb{T}^b$. For $\chi \in \hat{\Gamma}$, let E_χ be the flat line bundle over M associated with χ and let Δ_χ be the corresponding twisted Laplacian. The regular representation and the space of square-integrable functions decompose as

$$R_\Gamma \simeq \int_{\hat{\Gamma}}^{\oplus} \chi d\chi, \quad L^2(X) \simeq \int_{\hat{\Gamma}}^{\oplus} L^2(M, E_\chi) d\chi.$$

Consequently,

$$k_X(t, p, q) = \int_{\hat{\Gamma}} k_\chi(t, p, q) d\chi, \quad k_\chi(t, p, q) = \sum_{j \geq 0} e^{-\lambda_j(\chi)t} \varphi_{j,\chi}(p) \overline{\varphi_{j,\chi}(q)}. \quad (1)$$

The trivial character is the unique point at which the lowest eigenvalue vanishes. The long-time asymptotics are therefore localized near $\chi = \mathbf{1}$.

A useful geometric coordinate system near $\mathbf{1}$ is obtained from harmonic one-forms. If ω is harmonic and

$$\chi_{\tau\omega}([c]) = \exp\left(2\pi i \tau \int_c \omega\right),$$

then a gauge transformation identifies $\Delta_{\chi_{\tau\omega}}$ with a family on the fixed space $L^2(M)$:

$$L_{\tau\omega}f = \Delta_M f + 4\pi i\tau \langle \omega, df \rangle + 4\pi^2 \tau^2 |\omega|^2 f.$$

Differentiating the lowest eigenvalue equation at $\tau = 0$ and using $\delta\omega = 0$ gives

$$\lambda'_0(0) = 0, \quad \lambda''_0(0) = \frac{8\pi^2}{\text{vol}(M)} \int_M |\omega|^2 dv_g > 0 \quad (\omega \neq 0). \quad (2)$$

Thus the Hessian at the trivial character is positive definite. The Morse lemma and the Laplace method applied to (1) give the classical result of Kotani–Sunada and its predecessors [15, 18].

Theorem 2.1 (Abelian heat asymptotics). *Let $\pi : X \rightarrow M$ be a normal $\mathbb{Z}^b$ -covering. For fixed $p, q \in X$,*

$$k_X(t, p, q) \sim C_{\text{ab}}(p, q) t^{-b/2} \quad (t \rightarrow \infty),$$

and, under the standard smoothness assumptions, there is a complete expansion in integral powers of t^{-1} . The leading coefficient is determined by the Hessian (2), equivalently by the Albanese metric of the covering.

The point to retain is the two-step structure:

1. a representation-theoretic decomposition over a compact parameter space;
2. perturbation of the lowest fiber eigenvalue at the trivial representation.

The nilpotent theory preserves this structure, but the first step must be replaced because the full unitary dual is no longer an ordinary torus.

3 The non-type I obstruction and the nilpotent replacement

Let Γ be a torsion-free finitely generated nilpotent group and let G be its Malcev completion. Then G is a connected simply connected nilpotent Lie group containing Γ as a lattice. The lattice determines a rational structure $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g} = \text{Lie}(G)$.

For countable discrete groups, Thoma's theorem says that the type I property is equivalent to the existence of an abelian subgroup of finite index [19]. Hence a nonabelian torsion-free nilpotent lattice is non-type I. By the Glimm theory, its irreducible dual cannot be treated as a standard Borel parameter space comparable with $\mathbb{T}^b$ [7]. This is not a technical inconvenience that can be removed by choosing better coordinates; it is the basic obstruction.

3.1 The lattice-first route and the origin of the Kirillov fibers

It is important not to begin by taking the ordinary Plancherel decomposition of G and simply restricting every Lie-group representation to Γ . Restriction can identify distinct Lie-group parameters. Already for $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, the nontrivial character

$$x \longmapsto e^{2\pi i \xi x} \quad (\xi \in \mathbb{Z} \setminus \{0\})$$

is trivial on Γ . More generally, the restriction of a character of $\mathbb{R}$ depends only on ξ modulo $\mathbb{Z}$. Thus a Lie-group representation may be nontrivial while its restriction is trivial, and a direct restriction of the Lie-group Plancherel parameters would not by itself encode the correct lattice multiplicities or provide the desired lattice inversion formula.

The construction therefore starts with the right regular representation ρ_R^Γ of the lattice. Bekka's factor-Plancherel theorem gives the ordinary direct-integral decomposition

$$\rho_R^\Gamma \simeq \int_{\text{Ch}(\Gamma)}^\oplus \pi_t d\nu(t), \quad (3)$$

where $\text{Ch}(\Gamma)$ is Thoma's character space and π_t is a finite-type factor representation for ν -almost every t [3, 19]. For the nilpotent Plancherel layer used here, the relevant quasi-equivalence classes have a central-induced monomial model: up to quasi-equivalence, they are represented by

$$\pi_t \sim \text{Ind}_{Z(\Gamma)}^\Gamma \chi_t^\Gamma, \quad (4)$$

with a unitary character χ_t^Γ of $Z(\Gamma)$ [3, 10].

Choose a real lift of this central character to a character χ_t^G of $Z(G)$, so that $\chi_t^G|_{Z(\Gamma)} = \chi_t^\Gamma$. The corresponding central-induced model is lifted to the monomial representation

$$\tilde{\pi}_t := \text{Ind}_{Z(G)}^G \chi_t^G. \quad (5)$$

The lift is unique only modulo an integral central character, exactly as in the elementary pair $\mathbb{Z} \subset \mathbb{R}$; different such lifts have the same restriction to $Z(\Gamma)$. The observation that this extension mechanism can be implemented through the Baker–Campbell–Hausdorff coordinates was communicated to the author by Takashi Iwamoto.

Let $\mathfrak{z} = \text{Lie}(Z(G))$ and choose $l_t^0 \in \mathfrak{g}^*$ extending the differential of the central character, so that $d\chi_t^G(X) = 2\pi i l_t^0(X)$ for $X \in \mathfrak{z}$. Set

$$\Lambda_t = l_t^0 + \mathfrak{z}^\perp = \{l \in \mathfrak{g}^* : l|_{\mathfrak{z}} = l_t^0|_{\mathfrak{z}}\}.$$

Fujiwara's Plancherel theorem for monomial representations gives

$$\tilde{\pi}_t \simeq \int_{\Lambda_t}^\oplus \pi_l d\nu_t(l), \quad (6)$$

with the usual stratification or cross-section interpretation of the orbit parameters, where π_l is the irreducible Kirillov representation of G associated with the coadjoint orbit of l [5, 6]. This is the step at which the representations π_l used below are produced: they are not inserted independently of the lattice decomposition. One then restricts these π_l back to Γ ; on the rational locus the exact restriction theorem and the arithmetic finite-dimensional refinement apply. The final finitely additive trace formula is assembled from those rational finite-dimensional fibers, rather than by restricting the ordinary Fujiwara measure itself.

Thus the complete representation-theoretic route is

$$\rho_R^\Gamma \xrightarrow{\text{factor Plancherel}} \pi_t \xrightarrow{\text{central monomial lift}} \tilde{\pi}_t \xrightarrow{\text{Fujiwara}} \pi_l \xrightarrow{\text{restriction}} \pi_l|_\Gamma. \quad (7)$$

The lattice-first arrow records from the outset all identifications of Lie-group parameters that become invisible after restriction.

Once the Kirillov family has been produced by (3)–(6), the construction in [11] keeps the following three levels separate.

1. An ordinary Hilbert direct integral obtained by restricting rational Kirillov representations of G to Γ .
2. A further finite-dimensional refinement on an arithmetic rational locus, using Howe's classification of finite-dimensional representations of nilpotent lattices [9].
3. A positive finitely additive normalized-trace functional supported on those finite-dimensional rational representations, generalizing Pytlik's formula for the discrete Heisenberg group [17].

Only the first item is an ordinary direct integral. The third is a trace identity, not a countably additive Plancherel decomposition of the regular representation.

For the restriction step, fix a Kirillov parameter $l \in \mathfrak{g}^*$ occurring in a family of the form (6); the theorem below applies, in particular, to every rational $l \in \mathfrak{g}_{\mathbb{Q}}^*$. Write

$$B_l(X, Y) = l([X, Y]), \quad \mathfrak{r}_l = \{X \in \mathfrak{g} : B_l(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

A polarization for l is a maximal subordinate subalgebra $\mathfrak{m} \subset \mathfrak{g}$, namely $l([\mathfrak{m}, \mathfrak{m}]) = 0$. For rational $l \in \mathfrak{g}_{\mathbb{Q}}^*$ one may choose $\mathfrak{m}$ rational. Put $M = \exp \mathfrak{m}$ and $\chi_l(\exp X) = e^{2\pi i l(X)}$. The orbit method associates to l the irreducible representation

$$\pi_l = \text{Ind}_M^G \chi_l.$$

Theorem 3.1 (Exact restriction theorem [11]). *Let $l \in \mathfrak{g}_{\mathbb{Q}}^*$ and let π_l be the corresponding irreducible unitary representation of G . One can choose a rational polarization and a lattice-adapted weak Malcev system $X_1, \dots, X_n$ such that*

$$\pi_l|_{\Gamma} \simeq \int_{[0,1]^{n-m}}^{\oplus} \text{Ind}_{M \cap \Gamma}^{\Gamma} (\chi_l|_{M \cap \Gamma}) dt, \quad (8)$$

where $m = \dim M$ and

$$l_t = \text{Ad}^*(e^{t_{m+1}X_{m+1}}) \dots \text{Ad}^*(e^{t_n X_n})l.$$

In the non-ideal case the notation includes the standard transport of the polarization along the codimension-one induction.

The fibers in (8) need not be irreducible or finite-dimensional. The finite-dimensional statement is a second, arithmetic step. Let

$$\mathbb{Q}^{\text{odd}} = \{p/q \in \mathbb{Q} : (p, q) = 1, q \text{ odd}\}.$$

Theorem 3.2 (Finite-dimensional rational refinement [11]). *Suppose that $t \in ([0, 1] \cap \mathbb{Q}^{\text{odd}})^{n-m}$ satisfies the corresponding odd-period condition in Howe's classification. Then there is a transported lattice subgroup $M_t \cap \Gamma$, a discrete polarization $P_t \subset \Gamma$, and an extension χ_t^P of the inducing character such that*

$$\text{Ind}_{M_t \cap \Gamma}^{\Gamma} (\chi_t) \simeq \int_{P_t / \widehat{(M_t \cap \Gamma)}}^{\oplus} \text{Ind}_{P_t}^{\Gamma} (\chi_t^P \otimes \eta) d\eta. \quad (9)$$

Every representation in the integrand is finite-dimensional and irreducible.

Applying Theorems 3.1 and 3.2 to the rational Kirillov constituents produced by (6), and then assembling them over the original factor decomposition (3), gives the following normalized-trace inversion formula by Howe's theorem and the finite-quotient construction [6, 11].

Theorem 3.3 (Generalized Pytlik formula). *There exists a positive finitely additive probability measure μ on $\widehat{\Gamma}$, supported on the finite-dimensional irreducible representations $\widehat{\Gamma}_{\text{fin}}$, such that for $f \in \ell^1(\Gamma)$ and $\sigma \in \Gamma$,*

$$f(\sigma) = \int_{\widehat{\Gamma}_{\text{fin}}} \frac{1}{\dim \rho} \text{Tr}(\rho(\sigma^{-1})\rho(f)) d\mu(\rho). \quad (10)$$

The associated Plancherel identity also holds. Formula (10) is a finitely additive trace identity; it is not an ordinary Hilbert-space direct integral over the full unitary dual.

This separation resolves a common ambiguity. Infinite-dimensional Kirillov representations are indispensable as smooth analytic models, but the exact lattice-side trace in (10) is taken only through normalized finite-dimensional traces.

4 From rational fibers to nilpotent heat asymptotics

Let

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots, \quad \Gamma_{j+1} = [\Gamma, \Gamma_j],$$

be the lower central series. The polynomial growth degree is

$$D = \sum_{j \geq 1} j \text{rank}(\Gamma_j / \Gamma_{j+1}), \quad (11)$$

by the Bass–Guivarc’h formula [2]. After passing to the associated stratified nilpotent model, the first layer $\mathfrak{g}^{(1)}$ generates the Lie algebra and the natural representation-theoretic model operator is

$$\mathcal{H}_l = - \sum_{r=1}^k d\pi_l(X_r)^2, \quad X_1, \dots, X_k \text{ a basis of } \mathfrak{g}^{(1)}. \quad (12)$$

It is a positive hypoelliptic, more precisely Rockland-type, operator in the Kirillov model.

The analytic argument in [12] is organized in the following order.

1. At each rational parameter, use the exact finite-dimensional fibers of Theorems 3.1–3.3.
2. Compare a fixed finite-order observable with the corresponding smooth Kirillov model. The error is controlled by the rational denominator or height, uniformly in the numerator and in the compact Bloch variables.
3. Bounded-height subsets have zero mass for the finitely additive trace functional. Hence the large-height comparison passes to the exact trace.
4. On the model side, positive Rockland estimates and a Plancherel–Mellin argument justify the coefficient-weighted spectral sums at every fixed order.

Irrational parameters are useful for writing a smooth ambient normal form, but they make no direct contribution to the exact Pytlik functional.

Theorem 4.1 (Local nilpotent heat expansion [12]). *Let $\pi : X \rightarrow M$ be a normal covering with torsion-free finitely generated nilpotent deck group Γ , and let D be given by (11). For fixed $p, q \in X$ and every $N \geq 0$,*

$$k_X(t, p, q) = t^{-D/2} \sum_{j=0}^N C_j(p, q) t^{-j/2} + o(t^{-D/2-N/2}). \quad (13)$$

If the conjugation/parity symmetry of the localized model eliminates the odd half-orders, only integral powers of t^{-1} occur. The coefficients are local and are expressed through the geometric perturbation data on M and spectral invariants of the model operators (12).

The leading power $t^{-D/2}$ agrees with the earlier local limit theory of Alexopoulos and others [1]. The additional point of (13) is the local coefficient expansion: it is not merely an integrated Edgeworth expansion. In concrete groups the leading constant naturally involves a spectral zeta value $\zeta_{\mathcal{H}}(D/2)$. For the Heisenberg group the model is a harmonic oscillator; for the Engel group it is a quartic oscillator, explaining the appearance of Bessel/resolvent formulae in the latter case.

5 The Heisenberg prototype and the Harper model

The discrete Heisenberg group is

$$\Gamma = \text{Heis}_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

and its Malcev completion is $G = \text{Heis}_3(\mathbb{R})$. If X, Y, Z are the standard Lie-algebra generators, then $[X, Y] = Z$ and the dilations have weights 1, 1, 2. Hence $D = 1+1+2 = 4$.

For $h \neq 0$, the Schrödinger representation can be written on $L^2(\mathbb{R})$ as

$$(\rho_h(x, y, z)f)(u) = e^{2\pi i h(z+uy)} f(u+x). \quad (14)$$

When $h = p/q$ is rational in lowest terms, its restriction to Γ admits an exact decomposition into q -dimensional representations:

$$\rho_h|_{\Gamma} \simeq \int_0^1 \oplus \int_0^1 \oplus \rho_{\text{fin},(h, \{qhx_2\}, x_3)} dx_2 dx_3. \quad (15)$$

This is the model for the general restriction theorem. It also explains why the dimension of the relevant finite-dimensional fibers tends to infinity when the central character approaches the trivial one.

Let ω_1, ω_2 be the harmonic one-forms associated with the two horizontal directions and put $A = \|\omega_1\|_{L^2(M)}$, $B = \|\omega_2\|_{L^2(M)}$. The regular model operator is

$$\mathcal{H}_{\text{Heis}} = -A^2 \frac{d^2}{ds^2} + 4\pi^2 B^2 s^2,$$

with eigenvalues $2\pi AB(2n+1)$. Therefore

$$\zeta_{\mathcal{H}_{\text{Heis}}}(2) = \sum_{n \geq 0} [2\pi AB(2n+1)]^{-2} = \frac{1}{32A^2B^2}.$$

With the model-density normalization of [12], the leading diagonal coefficient is

$$C_0(p, p) = \frac{\text{vol}(M)^2}{16\|\omega_1\|_{L^2(M)}^2 \|\omega_2\|_{L^2(M)}^2}.$$

Higher terms are obtained from the perturbation of the oscillator and the geometric coefficients of the twisted Laplacian.

The Harper operator provides a finite-dimensional toy model for the same logic. In the phase convention $\theta \in \mathbb{R}$ it is

$$(H_\theta u)(m, n) = u(m+1, n) + u(m-1, n) + e^{im\theta} u(m, n+1) + e^{-im\theta} u(m, n-1).$$

For $\theta = 2\pi p/q$, the operator decomposes into exact $q \times q$ magnetic matrices arising from the representations in (15). Only after this exact rational finite-dimensionalization does one pass to the smooth Schrödinger model. Near the bottom of the spectrum,

$$h_\theta = -2 \cos\left(\sqrt{\theta} \frac{1}{i} \frac{d}{ds}\right) - 2 \cos(\sqrt{\theta} s) = -4 + \theta \left(-\frac{d^2}{ds^2} + s^2\right) + O(\theta^2), \quad (16)$$

which formally gives Wilkinson's expansion

$$E_n = -4 + (2n+1)\theta + O(\theta^2).$$

The comparison between the bounded operator on the left of (16) and the unbounded harmonic oscillator is not an operator-norm Taylor expansion; its rigorous semiclassical justification uses localization and tunneling estimates of Helffer–Sjöstrand [8, 20]. In the present program the Harper example serves to illustrate the order of operations: exact finite rational fibers first, smooth semiclassical normal form second.

6 Polycyclic groups: corrections and a cyclic outlook

Definition 6.1. A group Γ is polycyclic if it has a finite subnormal series

$$\{e\} = \Gamma_0 \triangleleft \Gamma_1 \triangleleft \cdots \triangleleft \Gamma_r = \Gamma$$

with cyclic quotients Γ_{j+1}/Γ_j , finite or infinite.

Every finitely generated nilpotent group is polycyclic, and every polycyclic group is solvable. The converses fail. More importantly, a general polycyclic group need not have a preferred global decomposition $A \rtimes N$ or $N \rtimes A$ with A abelian and N nilpotent. Such semidirect products are important subclasses, not a definition or a normal form for all polycyclic groups.

There are three corrections to the preliminary slides.

1. Every discrete group is unimodular. Thus a dichotomy between “unimodular” and “non-unimodular” *deck groups* is inapplicable. Non-unimodular solvable Lie groups form a different continuous setting.
2. A nonnegative self-adjoint lowest band has a local minimum at the trivial unitary twist. If it is smooth, the first nonzero homogeneous term cannot be odd. In particular, a nonzero linear or cubic leading term is not the source of a positive heat-kernel asymptotic.
3. If a real phase satisfies $F(x) \asymp |x|^m$ near an isolated minimum, then m is even and the Laplace integral scales polynomially like $t^{-k/m}$. It does not produce $e^{-ct^{1/3}}$. Airy functions arise from oscillatory or complex cubic normal forms, not from a positive cubic Laplace minimum.

The correct coarse statement for polycyclic groups is a growth/return probability dichotomy. For positive functions on the integers, write $f \simeq g$ if they are comparable after multiplication by positive constants and a constant rescaling of the time variable.

Theorem 6.2 (Polycyclic return-probability dichotomy). *Let Γ be a finitely generated polycyclic group and let $p_{2n}(e, e)$ be the return probability of a symmetric finitely supported generating random walk.*

1. *If Γ is virtually nilpotent of homogeneous growth degree D , then*

$$p_{2n}(e, e) \simeq n^{-D/2}.$$

2. *If Γ is not virtually nilpotent, then it has exponential growth and*

$$p_{2n}(e, e) \simeq \exp(-n^{1/3}).$$

The growth dichotomy is the Milnor–Wolf theorem [21]. In the second case the lower bound is due to Pittet–Saloff-Coste for solvable groups of finite Prüfer rank, a class containing the polycyclic groups [16]; the matching upper comparison follows from the standard volume-growth/isoperimetric heat-kernel bound. The theorem is a coarse on-diagonal comparison. It does not provide an exact exponential constant, an off-diagonal rate function, or an all-order local expansion. It also must not be extended to arbitrary amenable groups of exponential growth, where many other return-probability scales occur.

A useful exact Floquet–Mackey theory is nevertheless available for the special abelian-by-cyclic group

$$\Gamma_M = \mathbb{Z}^d \rtimes_M \mathbb{Z}, \quad M \in GL_d(\mathbb{Z}),$$

with multiplication $(a, n)(b, m) = (a + M^n b, n + m)$. Put $T = M^{-T}$ and

$$V(\xi) = 2 \sum_{r=1}^d c_r (1 - \cos(2\pi \xi_r)), \quad \xi \in \mathbb{T}^d.$$

For the natural right-convolution Laplacian with horizontal rates c_r and vertical rate κ , Fourier transform in the normal subgroup $\mathbb{Z}^d$ gives the one-dimensional orbit Hamiltonian

$$(H_\xi v)_j = \kappa(2v_j - v_{j+1} - v_{j-1}) + V(T^{-j}\xi)v_j. \quad (17)$$

Proposition 6.3 (Exact cyclic Mackey–Floquet formula). *For the return heat kernel of the preceding Laplacian,*

$$p_t(e, e) = \int_{\mathbb{T}^d} \langle \delta_0, e^{-tH_\xi} \delta_0 \rangle d\xi. \quad (18)$$

If ξ is periodic of exact period m under T , then Mackey theory gives an m -dimensional representation with boundary phase $z \in \mathbb{T}$, and the image of the Laplacian is the cyclic Jacobi matrix

$$(H_{\xi,z}^{(m)} v)_j = \kappa(2v_j - v_{j+1} - v_{j-1}) + V(T^{-j}\xi)v_j, \quad j \in \mathbb{Z}/m\mathbb{Z},$$

with $v_{j+m} = zv_j$. Torsion grids in $\mathbb{T}^d$ approximate Haar measure and yield an exact normalized-trace formula supported on these finite periodic Mackey fibers.

Formula (18) is a genuine countably additive torus integral, whereas the periodic finite-dimensional formula is a separate tracial skeleton. This distinction is the cyclic analogue of the separation made in Section 3.

For hyperbolic M , the cube-root scale is better interpreted through a local spectral distribution

$$N(E) = \int_{\mathbb{T}^d} \langle \delta_0, \mathbf{1}_{[0,E]}(H_\xi) \delta_0 \rangle d\xi, \quad p_t(e, e) = \int_0^\infty e^{-tE} dN(E).$$

If one could prove the sharp local Lifshitz tail

$$-\log N(E) \sim aE^{-1/2},$$

then the Laplace principle would give

$$-\log p_t(e, e) \sim 3 \left(\frac{a^2 t}{4} \right)^{1/3}.$$

The cyclic-extension draft [13] proves the exact fiber formulas and a Bowen-tube/Feynman–Kac lower bound on the cube-root scale, but it identifies the matching local observability/Faber–Krahn upper bound and the exact Lifshitz constant as open. Thus the coarse Theorem 6.2 is known, while the refined local spectral mechanism and all-order compact-base covering asymptotics remain a research program.

7 Concluding perspective

The passage from abelian to nilpotent coverings does not consist in replacing one torus by a more complicated full unitary dual. The correct structure is

$\begin{array}{c} \rho_R^\Gamma \xrightarrow{\text{factor Plancherel}} \pi_t \xrightarrow{\text{central lift}} \tilde{\pi}_t \xrightarrow{\text{Fujiwara}} \pi_l \\ \Downarrow \text{restriction to } \Gamma \text{ and rational refinement} \\ \text{finite-dimensional rational fibers} \xrightarrow{\text{normalized traces}} \text{Pytlík-type identity} \\ \Downarrow \text{large-height comparison} \\ \text{smooth Kirillov model and local heat-kernel expansion} \end{array}$
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The Heisenberg/Harper example makes every arrow explicit. The cyclic extension $\mathbb{Z}^d \rtimes_M \mathbb{Z}$ shows how a related finite periodic Mackey skeleton survives in an exponentially growing polycyclic setting, but the analytic mechanism changes: rare orbit segments and local spectral scarcity replace a finite-order Taylor expansion of a single band. Keeping exact representation theory, asymptotic model comparison, and coarse random-walk estimates logically separate is essential in both settings.

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