

Reciprocal characters and the quantum affine Schur–Weyl duality (a survey)

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Abstract

This note is a survey of results obtained in joint work with Maxim Gurevich [GV26] and presented in the author’s talk at the RIMS Symposium *Interactions of representation theory and related topics* in June 2026. We construct the reciprocal character, an invariant of finite-dimensional modules over affine Hecke algebras of type A , and show that it mirrors the dominant part of q -characters on the other side of the quantum affine Schur–Weyl duality. We explain the classical picture that motivates the construction, give a combinatorial formula for reciprocal multiplicities, and connect the theory to Lusztig’s canonical bases.

1 Introduction

Quantum affine Schur–Weyl duality [CP96, GRV94] links the representation theory of the affine Hecke algebras $H_{n,v}$ of type A with the representation theory of the quantum affine algebras $U_v(\widehat{\mathfrak{sl}}_{N+1})$. Here $v \in \mathbb{C}^\times$ is a fixed deformation parameter that is not a root of unity. The duality is realized by a family of exact functors

$$\mathcal{F}_{N,n}: \text{Rep}(H_{n,v}) \longrightarrow \mathcal{C}_N$$

from finite-dimensional $H_{n,v}$ -modules to the category $\mathcal{C}_N$ of finite-dimensional type 1 $U_v(\widehat{\mathfrak{sl}}_{N+1})$ -modules. The sum $\mathcal{F}_N = \bigoplus_{n \geq 0} \mathcal{F}_{N,n}$ is a monoidal functor that sends simple modules to simple modules or to zero, which ties the two theories together. It is useful to note that the link extends to the theory of p -adic groups. A classical theorem of Borel [Bor76] identifies finite-dimensional $H_{n,v}$ -modules, when $v^2 = p$ is prime, with those admissible $\text{GL}_n(\mathbb{Q}_p)$ -representations generated by their Iwahori-fixed vectors.

Our focus is the notion of a character in these two settings, and how the notions are intertwined by quantum affine Schur–Weyl duality. This note surveys results of the author’s joint work with Maxim Gurevich [GV26], where complete proofs and a detailed treatment can be found. Here we present the classical analogy behind the results, work out small examples, and give the minimum of detail needed to state the main theorems precisely.

Let us set up the language. A *segment* is an integer interval $\Delta = [x, y]$, with *length* $\ell(\Delta) = y - x + 1$, and Seg denotes the set of segments. A *multisegment* is a finite formal sum $\mathbf{m} = \Delta_1 + \cdots + \Delta_r$ of segments, with *total length* $n_{\mathbf{m}} = \ell(\Delta_1) + \cdots + \ell(\Delta_r)$. We restrict attention throughout to the integral part¹ of the representation theory on both sides of quantum affine Schur–Weyl duality. The general case reduces to this one by a standard decomposition into blocks, see [GV26, Sections 3.3.5 and 5.2.2].

On the Hecke side, we study the ring

$$\mathbf{R} = \bigoplus_{n \geq 0} G(H_{n,v}) ,$$

¹On the Hecke side, the integral part is the full subcategory of modules in which every eigenvalue of the commuting generators is an integer power of v^2 . On the quantum affine side, it is the full subcategory of modules whose composition factors have all roots of their i -th Drinfeld polynomial inside $v^{2\mathbb{Z}+i+1}$.

where $G(H_{n,v})$ is the Grothendieck group of the integral part of the category of finite-dimensional $H_{n,v}$ -modules and the multiplication comes from parabolic induction. To a multisegment $\mathbf{m}$ one attaches a *standard module* $\zeta(\mathbf{m})$, induced from one-dimensional segment modules, together with its unique simple quotient $Z(\mathbf{m})$. The Zelevinsky classification [Zel80] identifies $\mathbf{R}$ with the polynomial ring $\mathbb{Z}[\text{Seg}]$ on the variable set Seg , in such a way that the classes $M(\mathbf{m}) = [\zeta(\mathbf{m})]$ of standard modules correspond to the monomials $\mathbf{m}$.

On the quantum affine side, we study the Grothendieck ring $G[N]$ of the integral part of the monoidal category $\mathcal{C}_N$. The classification by Drinfeld polynomials [CP94] identifies $G[N]$ with the polynomial ring $\mathbb{Z}[\mathcal{D}_N]$ on the variables

$$\mathcal{D}_N = \{Y_{i,p} : 1 \leq i \leq N, p \in \mathbb{Z}, p+i \text{ odd}\},$$

which correspond to the fundamental modules. The parity condition expresses the integral normalization on this side. The classes $V(\mathbf{m})$ of standard modules, tensor products of fundamental modules, again correspond to the monomials.²

The two polynomial rings are related by the surjections

$$P_N: \mathbb{Z}[\text{Seg}] \rightarrow \mathbb{Z}[\mathcal{D}_N], \quad P_N([x, y]) = \begin{cases} Y_{y-x+1, x+y} & y-x+1 \leq N, \\ 1 & y-x+1 = N+1, \\ 0 & y-x+1 > N+1. \end{cases}$$

Since $(y-x+1) + (x+y) = 2y+1$ is odd, P_N lands in $\mathbb{Z}[\mathcal{D}_N]$. Note that segments and the variables $Y_{i,p}$ are two notations for the same objects, matched by $[x, y] \rightarrow Y_{y-x+1, x+y}$. Segments are natural in the Hecke theory and $Y_{i,p}$ in the theory of q -characters, and we therefore use the same letter $\mathbf{m}$ for a monomial in either notation.

The relevant quantum affine invariant is the q -character of Frenkel and Reshetikhin [FR99]. The q -character is an injective ring homomorphism χ^N from $G[N]$ into a Laurent polynomial ring in the variables $Y_{i,p}$, recording dimensions of ℓ -weight spaces. Truncating to the monomials with no negative powers gives the *dominant q -character*

$$\chi_+^N: G[N] \longrightarrow \mathbb{Z}[\mathcal{D}_N].$$

By a theorem of Nakajima [Nak04], χ_+^N is an additive bijection, although it is no longer multiplicative.

The main construction of [GV26] is a Hecke-algebraic counterpart of χ_+^N . Let $\mathbf{m} = \Delta_1 + \cdots + \Delta_r$ be a multisegment with $\Delta_j = [x_j, y_j]$, and let $E(\mathbf{m}) = \{y_j\}_j = \{d_1 < \cdots < d_l\}$ be the set of right ends of its segments. For $d \in E(\mathbf{m})$, set $\mathbf{m}[d] = \sum_{j: y_j=d} \Delta_j$, so that $\mathbf{m} = \mathbf{m}[d_1] + \cdots + \mathbf{m}[d_l]$. The standard modules $\zeta(\mathbf{m}[d_i])$ are simple, and the *indicator module*

$$\zeta(\mathbf{m})^{\otimes} = \zeta(\mathbf{m}[d_1]) \boxtimes \cdots \boxtimes \zeta(\mathbf{m}[d_l])$$

is a simple module over the parabolic subalgebra $H_{n_{\mathbf{m}},v} < H_{n_{\mathbf{m}},v}$ attached to the composition $n_{\mathbf{m}} = (n_{\mathbf{m}[d_1]}, \dots, n_{\mathbf{m}[d_l]})$. Parabolic induction of $\zeta(\mathbf{m})^{\otimes}$ recovers the standard module $\zeta(\mathbf{m})$. The *reciprocal character* of a finite-dimensional $H_{n,v}$ -module V is

$$\text{ch}^{\otimes}(V) = \sum_{\mathbf{m}: n_{\mathbf{m}}=n} [\text{Res}_{n_{\mathbf{m}}}(V) : \zeta(\mathbf{m})^{\otimes}] \mathbf{m} \in \mathbb{Z}[\text{Seg}],$$

²For ease of presentation we write a monomial in the variables $\mathcal{D}_N$ as $\mathbf{m}$, where [GV26] writes $\exp(\mathbf{m})$ for the monomial and $\mathbf{m}$ for its exponent function. In the same way a multisegment $\Delta_1 + \cdots + \Delta_r$ stands for the monomial $\Delta_1 \cdots \Delta_r$ in $\mathbb{Z}[\text{Seg}]$.

where the coefficients are Jordan–Hölder multiplicities of indicator modules inside the parabolic restrictions of V . The invariant is additive on short exact sequences, and it defines an additive isomorphism $\text{ch}^\otimes: \mathbf{R} \rightarrow \mathbb{Z}[\text{Seg}]$. The name refers to Frobenius reciprocity, as explained in Section 4.2.

The main theorem exhibits the reciprocal character as the mirror image of the dominant q -character through the quantum affine Schur–Weyl duality. In the diagram, $[\mathcal{F}_N]: \mathbf{R} \rightarrow G[N]$ denotes the map induced on Grothendieck groups by the exact functor $\mathcal{F}_N$.

Theorem A ([GV26]). *For every $N \geq 1$, the diagram*

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{[\mathcal{F}_N]} & G[N] \\ \text{ch}^\otimes \downarrow & & \downarrow \chi_+^N \\ \mathbb{Z}[\text{Seg}] & \xrightarrow{P_N} & \mathbb{Z}[\mathcal{D}_N] \end{array}$$

commutes. In other words, $\chi_+^N(\mathcal{F}_{N,n}(V)) = P_N(\text{ch}^\otimes(V))$ holds for every finite-dimensional $H_{n,v}$ -module V .

The proof of Theorem A rests on a closed combinatorial formula for all reciprocal multiplicities of standard modules, in terms of a count of dominant bi-tableaux (Theorem 5.2).

The second circle of results places the reciprocal character inside Lusztig’s theory of canonical bases [Lus90]. The representation theory of the algebras $H_{n,v}$ categorifies Lusztig’s algebra $\mathbf{f}$ of type A_∞ , by a theorem of Ariki [Ari96]. Under this categorification, simple modules match the dual canonical basis and standard modules match a dual PBW basis. The algebra $\mathbf{f}$ also carries a third distinguished basis, a *monomial basis* $\{m_{\mathbf{m}}\}$ indexed by multisegments, constructed through the general method of Lusztig [Lus90, Section 7.8], as formalized by Reineke [Rei01]. We prove that the reciprocal character is precisely the coordinate map with respect to the dual monomial basis, specialized at $q = 1$. As a consequence, the dominant q -character computes the specializations at $q = 1$ of the transition matrices between the monomial basis and the canonical and PBW bases. These statements appear as Theorem B and Corollary 6.2 in Section 6.

The note is organized as follows. Section 2 presents the classical analogy for symmetric groups and general linear groups. Section 3 recalls the two Grothendieck rings and the quantum affine Schur–Weyl duality functor. Section 4 presents the two character invariants and the Frobenius reciprocity motivation. Section 5 develops the computation of reciprocal multiplicities, their role in the proof of Theorem A, and a worked example of Theorem A. Section 6 states the quantum group results. Section 7 collects remarks on further directions.

2 The classical picture

This section presents the classical analogy behind the main results. We recall the relation between representations of symmetric groups and of general linear groups, and how the two notions of character are tied together.

2.1 Characters of symmetric groups

For a representation (π, V) of the symmetric group S_n , the character $\text{ch}(\pi): g \mapsto \text{Tr } \pi(g)$ is a class function, giving a map

$$\text{ch}: \text{Rep}(S_n) \longrightarrow C(S_n)$$

into the space of class functions. Since S_n -representations are semisimple and the characters of simple modules are linearly independent, the character determines a representation up to isomorphism.

For $n = n_1 + n_2$, the embedding $S_{n_1} \times S_{n_2} \hookrightarrow S_n$ gives the induction product $(\pi_1, \pi_2) \mapsto \text{Ind}(\pi_1 \boxtimes \pi_2)$. This operation turns

$$\Lambda = \bigoplus_{n \geq 0} C(S_n)$$

into a commutative ring, which is identified with the ring of symmetric functions in infinitely many variables [Mac95, Ch. I, §7]. Under this identification the trivial representation $\mathbf{1}_n$ goes to the complete homogeneous symmetric function

$$\text{ch}(\mathbf{1}_n) = h_n = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \cdots x_{i_n} ,$$

and the h_n are free polynomial generators, so that $\Lambda = \mathbb{C}[h_1, h_2, \dots]$ is a polynomial ring.

For a partition $\lambda = (n_1 \geq \dots \geq n_k)$ of n , with Young subgroup $S_\lambda = S_{n_1} \times \dots \times S_{n_k}$, the corresponding monomial in the generators is again a character,

$$h_\lambda = h_{n_1} \cdots h_{n_k} = \text{ch}(M_\lambda) , \quad M_\lambda = \text{Ind}(\mathbf{1}_{n_1} \boxtimes \dots \boxtimes \mathbf{1}_{n_k}) ,$$

and the monomials h_λ , over partitions λ of n , form a basis of $C(S_n)$. Every representation π of S_n therefore expands as

$$\text{ch}(\pi) = \sum_{\lambda} c_{\pi, \lambda} h_\lambda ,$$

and the integers $c_{\pi, \lambda}$ determine the character. Inverting the matrix of coefficients means expanding the h_λ back in the irreducible characters, indexed by the set $\text{Irr}(S_n)$ of irreducible representations of S_n up to isomorphism,

$$h_\lambda = \sum_{\pi \in \text{Irr}(S_n)} k_{\lambda, \pi} \text{ch}(\pi) ,$$

$$k_{\lambda, \pi} = \dim \text{Hom}_{S_n}(\pi, M_\lambda) = [\text{Res}_{S_\lambda}(\pi) : \mathbf{1}_{n_1} \boxtimes \dots \boxtimes \mathbf{1}_{n_k}] = \dim \pi^{S_\lambda} ,$$

where the second equality is Frobenius reciprocity. The $k_{\lambda, \pi}$ are the Kostka numbers [Mac95, Ch. I, §6]. Counting dimension of the invariants is another way to record characters. The reciprocal character surveyed in this note is a version of this count for a setting where semisimplicity fails, see Section 4.2.

Remark 2.1. The same count for the modules M_μ themselves gives the symmetric matrix

$$[\text{Res}_{S_\lambda}(M_\mu) : \mathbf{1}_{n_1} \boxtimes \dots \boxtimes \mathbf{1}_{n_k}] = \dim (M_\mu)^{S_\lambda} = \sum_{\pi \in \text{Irr}(S_n)} k_{\mu, \pi} k_{\lambda, \pi} .$$

Write π_ν for the irreducible representation labeled by a partition ν , so that the numbers $K_{\nu\lambda} = k_{\lambda, \pi_\nu}$ form the Kostka matrix K . The sum above is then the entry $(K^\top K)_{\mu\lambda}$ of the transition matrix expressing the functions h_μ in the basis of monomial symmetric functions m_λ [Mac95, Ch. I, §6]. The affine Hecke analogs of these multiplicities are central to this survey. We later call them reciprocal multiplicities, and Section 6 casts them again as entries of a transition matrix.

2.2 Polynomial representations of GL_N

A representation (π, W) of $\mathrm{GL}_N = \mathrm{GL}_N(\mathbb{C})$ is polynomial when the matrix entries of $\pi(g)$ are polynomial in the entries of g . A polynomial representation decomposes into weight spaces

$$W = \bigoplus_{\mu \in \mathbb{Z}_{\geq 0}^N} W_\mu, \quad W_\mu = \{ w \in W : \pi(\mathrm{diag}(x_1, \dots, x_N)) w = x_1^{\mu_1} \cdots x_N^{\mu_N} w, \forall x_i \in \mathbb{C}^\times \}.$$

The character

$$\chi(W) = \sum_{\mu} \dim(W_\mu) x_1^{\mu_1} \cdots x_N^{\mu_N} \in \Lambda_N := \mathbb{C}[x_1, \dots, x_N]^{S_N}$$

is a symmetric polynomial, by conjugating with permutation matrices, and it again determines a polynomial representation up to isomorphism.

2.3 Characters under Schur–Weyl duality

The group GL_N acts diagonally on $(\mathbb{C}^N)^{\otimes n}$, by $g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$, and the group S_n permutes the tensor factors, by $\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. The two actions commute. This is the setting of the classical Schur–Weyl duality, see [EGH⁺11, Section 4.18]. For $\pi \in \mathrm{Rep}(S_n)$, viewed as a right $\mathbb{C}[S_n]$ -module set

$$F_N(\pi) = \pi \otimes_{\mathbb{C}[S_n]} (\mathbb{C}^N)^{\otimes n} \in \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_N),$$

and assemble these functors into $F_N: \bigoplus_n \mathrm{Rep}(S_n) \rightarrow \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_N)$. The functor satisfies $F_N(\mathrm{Ind}(\pi_1 \boxtimes \pi_2)) \cong F_N(\pi_1) \otimes F_N(\pi_2)$. Finally, let

$$p_N: \Lambda \rightarrow \Lambda_N$$

be the projection setting $x_i = 0$ for $i > N$.

Theorem 2.2 (see [Mac95, Ch. I, Appendix A]). *The diagram*

$$\begin{array}{ccc} \bigoplus_{n \geq 0} \mathrm{Rep}(S_n) & \xrightarrow{F_N} & \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_N) \\ \downarrow \mathrm{ch} & & \downarrow \chi \\ \Lambda & \xrightarrow{p_N} & \Lambda_N \end{array}$$

commutes, that is, $\chi(F_N(\pi)) = p_N(\mathrm{ch}(\pi))$.

Example 2.3. Take $\pi = \mathbf{1}_n$. The lower path gives $p_N(h_n) = h_n(x_1, \dots, x_N)$. On the other hand, tensoring with the trivial module takes coinvariants, so $F_N(\mathbf{1}_n) = \mathbf{1}_n \otimes_{\mathbb{C}[S_n]} (\mathbb{C}^N)^{\otimes n} \cong S^n(\mathbb{C}^N)$ is the symmetric power. In the standard basis $e_1, \dots, e_N$ of $\mathbb{C}^N = \bigoplus_{i=1}^N \mathbb{C}e_i$, its weight spaces are the lines spanned by the monomials $e_{i_1} \cdots e_{i_n}$ with $i_1 \leq \cdots \leq i_n \leq N$, and therefore $\chi(S^n \mathbb{C}^N) = h_n(x_1, \dots, x_N)$. The two paths agree.

On the level of Grothendieck rings, after tensoring the classical Grothendieck rings with $\mathbb{C}$, both vertical maps of the diagram in Theorem 2.2 are ring isomorphisms. The results surveyed here reproduce this diagram in the quantum affine setting, where semisimplicity fails on both

sides. There the vertical maps $\text{ch}^\otimes$ and χ_+^N are additive bijections, but they are not multiplicative. The following dictionary sets up the analogy.

classical		quantum affine
S_n	$\rightsquigarrow$	$H_{n,v}$
GL_N	$\rightsquigarrow$	$U_v(\widehat{\mathfrak{sl}}_{N+1})$
$\Lambda = \bigoplus_n C(S_n)$	$\rightsquigarrow$	$\mathbb{Z}[\text{Seg}] \cong \mathbf{R}$
Λ_N	$\rightsquigarrow$	$\mathbb{Z}[\mathcal{D}_N] \cong G[N]$
F_N	$\rightsquigarrow$	$\mathcal{F}_N$
ch	$\rightsquigarrow$	$\text{ch}^\otimes$ (reciprocal character)
χ	$\rightsquigarrow$	χ_+^N (dominant q -character)

3 The two sides of the quantum affine Schur–Weyl duality

Throughout, fix $v \in \mathbb{C}^\times$ that is not a root of unity. All algebras and categories below depend on this parameter.

3.1 Affine Hecke algebras

The affine Hecke algebra $H_n = H_{n,v}$ of type A is the associative $\mathbb{C}$ -algebra generated by elements $T_1, \dots, T_{n-1}$ and invertible commuting elements $y_1, \dots, y_n$, subject to

$$\begin{aligned} (T_i - v^2)(T_i + 1) &= 0, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad (|i - j| > 1), \\ T_i y_i T_i &= v^2 y_{i+1}, \quad T_i y_j = y_j T_i \quad (j \neq i, i+1). \end{aligned}$$

It is a deformation of the group algebra of $S_n \ltimes \mathbb{Z}^n$.

A composition $\underline{n} = (n_1, \dots, n_r)$ of n gives a parabolic embedding $H_{\underline{n}} = H_{n_1} \otimes \dots \otimes H_{n_r} \hookrightarrow H_n$, and with it the restriction functor $\text{Res}_{\underline{n}}$ and the induction functor

$$\text{Ind}_{\underline{n}}(V) = H_n \otimes_{H_{\underline{n}}} V,$$

which is exact and left adjoint to $\text{Res}_{\underline{n}}$. For $V_i \in \text{Rep}(H_{n_i})$ we write $V_1 \times \dots \times V_r := \text{Ind}_{\underline{n}}(V_1 \boxtimes \dots \boxtimes V_r)$. The class $[V_1 \times V_2]$ in the Grothendieck group does not depend on the order of the factors. Induction thus makes

$$\mathbf{R} = \bigoplus_{n \geq 0} G(H_n)$$

into a commutative ring. Here $G(H_n)$ denotes the Grothendieck group of the integral part of the category, namely of the modules in which all eigenvalues of the elements y_i are integer powers of v^2 .

For a segment $\Delta = [x, y]$, the *segment module* $Z(\Delta)$ is the one-dimensional module over H_{y-x+1} given by $T_i \mapsto -1$ and $y_i \mapsto v^{2(y-i+1)}$. For a multisegment $\mathbf{m} = \Delta_1 + \dots + \Delta_r$, fix an ordering of its segments with non-decreasing right ends, and set

$$\zeta(\mathbf{m}) = Z(\Delta_1) \times \dots \times Z(\Delta_r), \quad M(\mathbf{m}) := [\zeta(\mathbf{m})] \in \mathbf{R}.$$

The isomorphism class of the *standard module* $\zeta(\mathbf{m})$ does not depend on the choice of such an ordering.

Theorem 3.1 ([Zel80]). *Each standard module $\zeta(\mathbf{m})$ admits a unique simple quotient $Z(\mathbf{m})$, and the map $\mathbf{m} \mapsto Z(\mathbf{m})$ is a bijection between multisegments and isomorphism classes of simple modules. Moreover, the classes $\{M(\mathbf{m})\}_{\mathbf{m}}$ form a $\mathbb{Z}$ -basis of $\mathbf{R}$, and the assignment $\mathbf{m} \mapsto M(\mathbf{m})$ defines a ring isomorphism*

$$\mathbb{Z}[\text{Seg}] \xrightarrow{\sim} \mathbf{R}.$$

The multiplicities $[\zeta(\mathbf{m}) : Z(\mathbf{n})]$ of simple modules inside standard modules are known to be values at 1 of Kazhdan–Lusztig polynomials of finite type A [KL87].

3.2 The quantum affine side

We refer to [CP94] for the definition of the quantum affine algebra $U_v(\widehat{\mathfrak{sl}}_{N+1})$ and for the basic theory of its finite-dimensional type 1 modules, which form the category $\mathcal{C}_N$. The category is monoidal.

For $1 \leq i \leq N$ and $p \in \mathbb{Z}$, the *fundamental module* $L(Y_{i,p})$ is the evaluation module attached to the i -th fundamental representation $\bigwedge^i \mathbb{C}^{N+1}$ of $U_v(\mathfrak{sl}_{N+1})$, with spectral parameter v^p . For a monomial $\mathbf{m} = Y_{i_1,p_1} \cdots Y_{i_r,p_r}$ in the variables $\mathcal{D}_N$ of Section 1, the *standard module* is the tensor product

$$\vartheta(\mathbf{m}) = L(Y_{i_1,p_1}) \otimes \cdots \otimes L(Y_{i_r,p_r}), \quad V(\mathbf{m}) := [\vartheta(\mathbf{m})],$$

whose class in the Grothendieck group does not depend on the order of the factors. When the factors are ordered suitably, $\vartheta(\mathbf{m})$ admits a unique simple quotient $L(\mathbf{m})$.

In line with our integral convention, we write $G[N]$ for the Grothendieck ring of the *integral part* of $\mathcal{C}_N$, the full subcategory of modules whose simple factors are all of the form $L(\mathbf{m})$, for monomials $\mathbf{m}$ in the variables $\mathcal{D}_N$. Every simple module of $\mathcal{C}_N$ is a tensor product of spectral twists of simple modules of the integral part [HL10, Section 3.7], so the restriction loses no information.

Theorem 3.2 (see [CP94, FR99]). *The map $\mathbf{m} \mapsto L(\mathbf{m})$ is a bijection between monomials in the variables $\mathcal{D}_N$ and isomorphism classes of simple modules in the integral part of $\mathcal{C}_N$. This is a form of the classification by Drinfeld polynomials. Moreover, the classes $\{V(\mathbf{m})\}_{\mathbf{m}}$ form a $\mathbb{Z}$ -basis of $G[N]$, and $\mathbf{m} \mapsto V(\mathbf{m})$ defines a ring isomorphism*

$$\mathbb{Z}[\mathcal{D}_N] \xrightarrow{\sim} G[N].$$

3.3 The quantum affine Schur–Weyl duality functor

See Chari and Pressley [CP96] for the construction of exact functors

$$\mathcal{F}_{N,n}: \text{Rep}(H_n) \longrightarrow \mathcal{C}_N$$

as a quantum affine analog of the Schur–Weyl construction of Section 2. The combined functor $\mathcal{F}_N = \bigoplus_{n \geq 0} \mathcal{F}_{N,n}$ satisfies $\mathcal{F}_N(V_1 \times V_2) \cong \mathcal{F}_N(V_1) \otimes \mathcal{F}_N(V_2)$, and each simple module is sent to a simple module or to zero. For $n \leq N$, the functor $\mathcal{F}_{N,n}$ is an equivalence onto the full subcategory of modules of level n , those whose restriction to $U_v(\mathfrak{sl}_{N+1})$ has all irreducible constituents in $(\mathbb{C}^{N+1})^{\otimes n}$ [CP96, Theorem 4.2].

On Grothendieck rings the duality becomes purely combinatorial. Recall the surjection $P_N: \mathbb{Z}[\text{Seg}] \rightarrow \mathbb{Z}[\mathcal{D}_N]$ of Section 1, sending a segment of length at most N to a variable, a segment of length $N+1$ to 1, and longer segments to 0.

Theorem 3.3 ([CP96, Theorem 7.6], [GV26]). *The induced map $[\mathcal{F}_N]: \mathbf{R} \rightarrow G[N]$ coincides with P_N , when both rings are presented through the isomorphisms of Theorems 3.1 and 3.2. In particular*

$$[\mathcal{F}_N](M(\mathbf{m})) = V(P_N(\mathbf{m})), \quad [\mathcal{F}_N](Z(\mathbf{m})) = [L(P_N(\mathbf{m}))],$$

where the right sides are interpreted as 0 when $P_N(\mathbf{m}) = 0$.

4 The two character invariants

4.1 q -characters

A finite-dimensional module W in the integral part of $\mathcal{C}_N$ decomposes into the ℓ -weight spaces $W_{\mathbf{m}}$, the generalized eigenspaces of a distinguished commutative subalgebra of $U_v(\widehat{\mathfrak{sl}}_{N+1})$, indexed by Laurent monomials $\mathbf{m}$ in the variables $\mathcal{D}_N$. The q -character of Frenkel and Reshetikhin [FR99] is the resulting generating function

$$\chi^N(W) = \sum_{\mathbf{m}} \dim(W_{\mathbf{m}}) \mathbf{m} \in \mathbb{Z}[Y_{i,p}^{\pm 1}]_{1 \leq i \leq N, p+i \text{ odd}}.$$

It is an injective ring homomorphism on $G[N]$, but it is not surjective. A monomial is *dominant* when all its variables appear with nonnegative powers. Composing χ^N with the $\mathbb{Z}$ -linear projection that keeps dominant monomials and kills all others, for example $Y_{1,0}^2 + Y_{1,0}Y_{2,1}^{-1} \mapsto Y_{1,0}^2$, yields the *dominant q -character*

$$\chi_+^N: G[N] \longrightarrow \mathbb{Z}[\mathcal{D}_N].$$

Theorem 4.1 ([Nak04, Theorem 3.5(3)]). *The map χ_+^N is an additive bijection.*

The map is not multiplicative. For simple modules one often finds $\chi_+^N(L(\mathbf{m})) = \mathbf{m}$, but not always, a phenomenon quantified by the theory surveyed here.

4.2 The reciprocal character

We now construct the Hecke-side counterpart of χ_+^N announced in Section 1. Call two segments *linked* when neither contains the other and their union is again a segment. Call a multisegment *right-aligned* when all of its segments have the same right end. Standard modules of right-aligned multisegments are simple, since segments with a common right end are pairwise unlinked, and a product of segment modules of pairwise unlinked segments is irreducible [Zel80].

Recall from Section 1 the right ends $E(\mathbf{m}) = \{d_1 < \dots < d_l\}$ of a multisegment $\mathbf{m}$, its right-aligned parts $\mathbf{m}[d_i]$, the composition $\underline{n}_{\mathbf{m}} = (n_{\mathbf{m}[d_1]}, \dots, n_{\mathbf{m}[d_l]})$, and the indicator module

$$\zeta(\mathbf{m})^{\otimes} = \zeta(\mathbf{m}[d_1]) \boxtimes \dots \boxtimes \zeta(\mathbf{m}[d_l]) \in \text{Rep}(H_{\underline{n}_{\mathbf{m}}}).$$

It is simple, since each $\mathbf{m}[d_i]$ is right-aligned, and $\zeta(\mathbf{m}) = \text{Ind}_{\underline{n}_{\mathbf{m}}}(\zeta(\mathbf{m})^{\otimes})$ holds by the construction of standard modules.

Example 4.2. For $\mathbf{m} = [0, 0] + [0, 1] + [1, 1]$ the right ends are $d_1 = 0$ and $d_2 = 1$, with $\mathbf{m}[0] = [0, 0]$ and $\mathbf{m}[1] = [0, 1] + [1, 1]$. Thus $\underline{n}_{\mathbf{m}} = (1, 3)$ and $\zeta(\mathbf{m})^{\otimes} = Z([0, 0]) \boxtimes \zeta([0, 1] + [1, 1])$. The second factor is a simple module of dimension three, since H_3 is a free module of rank 3 over its parabolic subalgebra $H_{(2,1)}$.

Definition 4.3. The *reciprocal character* of a finite-dimensional H_n -module V is

$$\text{ch}^{\otimes}(V) = \sum_{\mathbf{m}: \underline{n}_{\mathbf{m}}=n} [\text{Res}_{\underline{n}_{\mathbf{m}}}(V) : \zeta(\mathbf{m})^{\otimes}] \mathbf{m} \in \mathbb{Z}[\text{Seg}].$$

Only finitely many summands are nonzero, since there are finitely many compositions of n and each parabolic restriction of V has finite length. Jordan–Hölder multiplicities are additive on short exact sequences, so $\text{ch}^{\otimes}$ descends to an additive map $\text{ch}^{\otimes}: \mathbf{R} \rightarrow \mathbb{Z}[\text{Seg}]$.

The map $\text{ch}^{\otimes}$ is in fact an additive isomorphism. This will become clear in Section 6.

Let us explain the name. By Frobenius reciprocity, for every simple H_n -module Z ,

$$\text{Hom}_{H_{\underline{n}_{\mathbf{m}}}}(\zeta(\mathbf{m})^{\otimes}, \text{Res}_{\underline{n}_{\mathbf{m}}}(Z)) \cong \text{Hom}_{H_n}(\zeta(\mathbf{m}), Z),$$

and since $\zeta(\mathfrak{m})$ has the unique simple quotient $Z(\mathfrak{m})$, both spaces have dimension 1 when $Z \cong Z(\mathfrak{m})$ and 0 otherwise. Passing from Hom spaces to Jordan–Hölder multiplicities, which may be larger, breaks the symmetry:

$$\begin{aligned} \dim \operatorname{Hom}_{H_n}(\zeta(\mathfrak{m}), Z) &\leq [\zeta(\mathfrak{m}) : Z] , \\ \dim \operatorname{Hom}_{H_{\underline{n}\mathfrak{m}}}(\zeta(\mathfrak{m})^{\otimes}, \operatorname{Res}_{\underline{n}\mathfrak{m}}(Z)) &\leq [\operatorname{Res}_{\underline{n}\mathfrak{m}}(Z) : \zeta(\mathfrak{m})^{\otimes}] . \end{aligned}$$

In a semisimple situation both inequalities are equalities and the two counts agree. For affine Hecke algebras they differ. The right side of the first inequality is controlled by Kazhdan–Lusztig polynomials evaluated at 1. The right side of the second gives the coefficients of Definition 4.3, a new invariant sitting on the other side of Frobenius reciprocity.

For later use we record the reciprocal multiplicities of standard modules,

$$A(\mathfrak{m}, \mathfrak{n}) = [\operatorname{Res}_{\underline{n}\mathfrak{m}}(\zeta(\mathfrak{m})) : \zeta(\mathfrak{n})^{\otimes}] ,$$

so that $\operatorname{ch}^{\otimes}(M(\mathfrak{m})) = \sum_{\mathfrak{n}} A(\mathfrak{m}, \mathfrak{n}) \mathfrak{n}$. In the classical picture, the counterpart of the matrix $A(\mathfrak{m}, \mathfrak{n})$ is the transition matrix of Remark 2.1.

5 Combinatorics of reciprocal multiplicities

This section explains the computation of the reciprocal multiplicities $A(\mathfrak{m}, \mathfrak{n})$, which is the core of the proof of Theorem A.

We think of parabolic restriction as “splitting” the segments of a standard module. To organize this, fix a multisegment $\mathfrak{m} = \Delta_1 + \cdots + \Delta_r$ with $\Delta_i = [x_i, y_i]$ and $E(\mathfrak{m}) = \{d_1 < \cdots < d_l\}$. We define the finite set $\mathcal{K}(\mathfrak{m})$ of *bi-tableaux* of $\mathfrak{m}$, so that each $\mathcal{Q} = (a_j^i, b_j^i)_{1 \leq i \leq r, 1 \leq j \leq k_i} \in \mathcal{K}(\mathfrak{m})$ is an array of pairs of integers that assigns to each segment Δ_i the sequence $(a_1^i, b_1^i), \dots, (a_{k_i}^i, b_{k_i}^i)$, subject to

$$x_i = a_{k_i}^i < \cdots < a_1^i \leq y_i , \quad b_1^i < \cdots < b_{k_i}^i , \quad b_j^i \in E(\mathfrak{m}) .$$

The a -entries split the segment Δ_i into the nonempty pieces $[a_j^i, a_{j-1}^i - 1]$, with the convention $a_0^i = y_i + 1$, and the entry b_j^i assigns the j -th piece to the right end $b_j^i \in E(\mathfrak{m})$. The pieces assigned to a common right end d form the multisegments

$$\mathfrak{m}_{\mathcal{Q}, d} = \sum_{(i,j): b_j^i = d} [a_j^i, a_{j-1}^i - 1] , \quad d \in E(\mathfrak{m}) ,$$

the composition $\underline{n}_{\mathcal{Q}} = (n_{\mathfrak{m}_{\mathcal{Q}, d_1}}, \dots, n_{\mathfrak{m}_{\mathcal{Q}, d_l}})$ and the class

$$M(\mathcal{Q}) = M(\mathfrak{m}_{\mathcal{Q}, d_1}) \boxtimes \cdots \boxtimes M(\mathfrak{m}_{\mathcal{Q}, d_l}) \in G(H_{\underline{n}_{\mathcal{Q}}}) ,$$

where $M(0)$ is understood as the class of the trivial module of $H_0 = \mathbb{C}$.

The next proposition is the geometric lemma of Bernstein and Zelevinsky [BZ77].

Proposition 5.1. *For every composition $\underline{n} = (n_1, \dots, n_l)$ of $n_{\mathfrak{m}}$ into l parts $n_i \geq 0$, in the Grothendieck group of $H_{\underline{n}}$,*

$$\operatorname{Res}_{\underline{n}} M(\mathfrak{m}) = \sum M(\mathcal{Q}) ,$$

the sum running over the bi-tableaux $\mathcal{Q} \in \mathcal{K}(\mathfrak{m})$ with $\underline{n}_{\mathcal{Q}} = \underline{n}$.

For every multisegment $\mathbf{n}$, Proposition 5.1 and [GV26, Lemma 3.4.1] give

$$A(\mathbf{m}, \mathbf{n}) = \sum_{\mathcal{Q}} [M(\mathcal{Q}) : \zeta(\mathbf{n})^{\otimes}].$$

The sum is empty if $n_{\mathbf{m}} \neq n_{\mathbf{n}}$ or $E(\mathbf{n}) \not\subseteq E(\mathbf{m})$. Otherwise, it runs over $\mathcal{Q} \in \mathcal{K}(\mathbf{m})$ with $n_{\mathbf{m}_{\mathcal{Q},d}} = n_{\mathbf{n}[d]}$ for $d \in E(\mathbf{n})$ and $\mathbf{m}_{\mathcal{Q},d} = 0$ for $d \in E(\mathbf{m}) \setminus E(\mathbf{n})$. At these missing endpoints, we insert zero parts into $\underline{n}_{\mathbf{n}}$ and trivial H_0 factors into $\zeta(\mathbf{n})^{\otimes}$. A summand contributes precisely when each $\zeta(\mathbf{n}[d])$ occurs as a composition factor of $\zeta(\mathbf{m}_{\mathcal{Q},d})$.

The multiplicity $[M(\mathcal{Q}) : \zeta(\mathbf{n})^{\otimes}]$ is 0 or 1, see [GV26, Lemma 3.3.1]. Hence $A(\mathbf{m}, \mathbf{n})$ can be interpreted as a count of the bi-tableaux that contribute multiplicity 1.

Let $\mathcal{A}(\mathbf{m}) \subseteq \mathcal{K}(\mathbf{m})$ denote the set of bi-tableaux with $y_i \leq b_1^i$ for every i and with all entries a_j^i in $\{x_1, \dots, x_r\}$. Only bi-tableaux from $\mathcal{A}(\mathbf{m})$ can contribute, see [GV26, Section 1.4]. A bi-tableau contributes to $A(\mathbf{m}, \mathbf{n})$ exactly when, for each d , the pieces of $\mathbf{m}_{\mathcal{Q},d}$ can be concatenated into the segments of $\mathbf{n}[d]$, which all end at d . Informally, a piece $[a_j^i, a_{j-1}^i - 1]$ that ends before b_j^i must connect to the piece of a pair (i', j') with $(a_{j'}^{i'}, b_{j'}^{i'}) = (a_{j-1}^i, b_j^i)$, and distinct pieces must connect to distinct pieces. For a segment $\Delta = [x, y]$, set

$$t(\mathcal{Q}, \Delta) = \#\{(i, j) : (a_j^i, b_j^i) = (x, y)\}, \quad \overleftarrow{t}(\mathcal{Q}, \Delta) = \#\{(i, j) : (a_{j-1}^i, b_j^i) = (x, y), y_i < y\}.$$

Thus $t(\mathcal{Q}, \Delta)$ counts the pairs with entries (x, y) , while $\overleftarrow{t}(\mathcal{Q}, \Delta)$ counts the pairs that must connect to a pair with entries (x, y) . The bi-tableau is *dominant* when $t(\mathcal{Q}, \Delta) \geq \overleftarrow{t}(\mathcal{Q}, \Delta)$ for every segment Δ , which holds exactly when the connections can be chosen injectively. A dominant bi-tableau then determines the multisegment

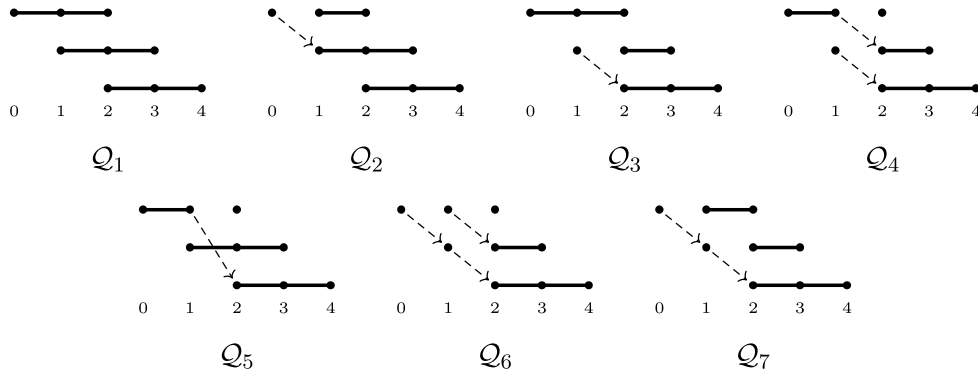
$$\theta(\mathcal{Q}) = \sum_{\Delta \in \text{Seg}} (t(\mathcal{Q}, \Delta) - \overleftarrow{t}(\mathcal{Q}, \Delta)) \Delta.$$

Let $\mathcal{A}(\mathbf{m})_+$ denote the set of dominant bi-tableaux in $\mathcal{A}(\mathbf{m})$. For $\mathcal{Q} \in \mathcal{A}(\mathbf{m})_+$, exactly one indicator module occurs in $M(\mathcal{Q})$, namely $\zeta(\theta(\mathcal{Q}))^{\otimes}$, and it occurs with multiplicity 1 [GV26, Lemma 3.4.2].

Theorem 5.2 ([GV26]). *For all multisegments $\mathbf{m}, \mathbf{n}$,*

$$A(\mathbf{m}, \mathbf{n}) = \#\{\mathcal{Q} \in \mathcal{A}(\mathbf{m})_+ : \theta(\mathcal{Q}) = \mathbf{n}\}.$$

Example 5.3. Take $\mathbf{m} = [0, 2] + [1, 3] + [2, 4]$, so that $E(\mathbf{m}) = \{2, 3, 4\}$. The set $\mathcal{A}(\mathbf{m})_+$ consists of the seven bi-tableaux $\mathcal{Q}_1, \dots, \mathcal{Q}_7$ drawn below. Each picture shows the segments $[0, 2]$, $[1, 3]$, $[2, 4]$ of $\mathbf{m}$ from top to bottom, split into pieces by the a -entries. A piece consisting of a single point appears as a bare dot. A dashed arrow points from the end of a piece to the piece it connects to in the reassembly. The maximal chains of pieces are then the segments of $\theta(\mathcal{Q}_k)$.



Reading off the chains gives

$$\begin{aligned}\theta(\mathcal{Q}_1) &= [0, 2] + [1, 3] + [2, 4] , & \theta(\mathcal{Q}_2) &= [0, 3] + [1, 2] + [2, 4] , \\ \theta(\mathcal{Q}_3) &= [0, 2] + [1, 4] + [2, 3] , & \theta(\mathcal{Q}_4) &= [0, 3] + [1, 4] + [2, 2] , \\ \theta(\mathcal{Q}_5) &= \theta(\mathcal{Q}_6) = [0, 4] + [1, 3] + [2, 2] , & \theta(\mathcal{Q}_7) &= [0, 4] + [1, 2] + [2, 3] .\end{aligned}$$

Since $\theta(\mathcal{Q}_5) = \theta(\mathcal{Q}_6)$ while the remaining values are pairwise distinct, Theorem 5.2 gives $A(\mathbf{m}, \mathbf{n}) = 2$ for $\mathbf{n} = [0, 4] + [1, 3] + [2, 2]$, $A(\mathbf{m}, \mathbf{n}) = 1$ for the other five values of θ , and $A(\mathbf{m}, \mathbf{n}) = 0$ for every other $\mathbf{n}$.

Theorem A follows once the identity

$$\chi_+^N(\mathcal{F}_{N,n_m}(\zeta(\mathbf{m}))) = \sum_{\mathbf{n}} A(\mathbf{m}, \mathbf{n}) P_N(\mathbf{n})$$

is proved for every multisegment $\mathbf{m}$ and every $N \geq 1$. Indeed, both sides of Theorem A are additive in V , and the classes of standard modules span $\mathbf{R}$. The identity is proved by expanding the q -character of a standard module and matching the dominant monomials of the product with the count of Theorem 5.2. The expansion uses the formula

$$\chi^N(L(Y_{i,p})) = \sum_{1 \leq j_1 < \dots < j_i \leq N+1} \prod_{k=1}^i Y_{j_k, p+i+j_k-2k} Y_{j_k-1, p+i+j_k-2k+1}^{-1} ,$$

in which $Y_{0,s}$ and $Y_{N+1,s}$ are read as 1 [FR99, Sections 5.4.1 and 8], [Nak03, Section 4].

The following example verifies Theorem A in a small case where both invariants are nontrivial.

Example 5.4. Let $N = 1$ and $\mathbf{m} = 2[0, 0] + [1, 1]$, and set $\mathbf{n} = [0, 0] + [0, 1]$.

Hecke side. Zelevinsky's theory gives the decomposition

$$M(\mathbf{m}) = [Z(\mathbf{m})] + [Z(\mathbf{n})] .$$

The count of Theorem 5.2 gives $A(\mathbf{m}, \mathbf{m}) = 1$ and $A(\mathbf{m}, \mathbf{n}) = 2$. All other reciprocal multiplicities of $\zeta(\mathbf{m})$ vanish, so

$$\text{ch}^\otimes(M(\mathbf{m})) = \mathbf{m} + 2\mathbf{n} .$$

The segments of $\mathbf{n}$ are unlinked, so $\zeta(\mathbf{n})$ is simple, and a similar count gives $\text{ch}^\otimes([Z(\mathbf{n})]) = \mathbf{n}$. Subtracting, we obtain

$$\text{ch}^\otimes([Z(\mathbf{m})]) = \mathbf{m} + \mathbf{n} .$$

Quantum affine side. We work in $U_v(\widehat{\mathfrak{sl}}_2)$. Mirroring the Hecke side through Theorem 3.3, the standard module decomposes as

$$V(Y_{1,0}^2 Y_{1,2}) = [L(Y_{1,0}^2 Y_{1,2})] + [L(Y_{1,0})] .$$

For a fundamental module, $\chi^1(L(Y_{1,p})) = Y_{1,p} + Y_{1,p+2}^{-1}$, so its dominant q -character is the single monomial $Y_{1,p}$. Since χ^1 is a ring homomorphism,

$$\begin{aligned}\chi^1(L(Y_{1,0}^2 Y_{1,2})) &= \chi^1(V(Y_{1,0}^2 Y_{1,2})) - \chi^1(V(Y_{1,0})) \\ &= (Y_{1,0} + Y_{1,2}^{-1})^2 (Y_{1,2} + Y_{1,4}^{-1}) - (Y_{1,0} + Y_{1,2}^{-1}) ,\end{aligned}$$

with dominant part

$$\chi_+^1(L(Y_{1,0}^2 Y_{1,2})) = Y_{1,0}^2 Y_{1,2} + Y_{1,0} . \quad (1)$$

Matching. Since $P_1([0, 0]) = Y_{1,0}$, $P_1([1, 1]) = Y_{1,2}$ and $P_1([0, 1]) = 1$, we have $P_1(\mathbf{m}) = Y_{1,0}^2 Y_{1,2}$, $P_1(\mathbf{n}) = Y_{1,0}$.

$$P_1(\text{ch}^\otimes([Z(\mathbf{m})])) = Y_{1,0}^2 Y_{1,2} + Y_{1,0} = \chi_+^1(L(P_1(\mathbf{m}))) ,$$

which is Theorem A for the simple module $Z(\mathbf{m})$. Both invariants are nontrivial here, since neither $\text{ch}^\otimes([Z(\mathbf{m})])$ nor $\chi_+^1(L(P_1(\mathbf{m})))$ is a single monomial.

6 Monomial bases

In the introduction of [Nak04] it is suggested that, in the quantum affine setting, the dominant part of the q -character plays the role of the monomial bases from Lusztig's theory of canonical bases. This section makes the suggestion precise.

Let $R(q)$ be the free $\mathbb{Q}(q)$ -algebra on a countable alphabet $\{\alpha_d\}_{d \in \mathbb{Z}}$ of simple roots of type A_∞ , with words as basis and concatenation as product. For $c \geq 1$ write $[c]_q! = \prod_{s=1}^c \frac{q^s - q^{-s}}{q - q^{-1}}$ and $\alpha^{(c)} = \alpha^c / [c]_q!$ for the divided powers. The quotient of $R(q)$ by the quantum Serre relations

$$\alpha_i \alpha_j = \alpha_j \alpha_i \quad (|i - j| > 1), \quad \alpha_i^{(2)} \alpha_j - \alpha_i \alpha_j \alpha_i + \alpha_j \alpha_i^{(2)} = 0 \quad (|i - j| = 1), \quad (2)$$

is *Lusztig's algebra* $\mathbf{f}$, the positive part of the quantized enveloping algebra of $\mathfrak{sl}_\infty$ [Lus90]. Writing $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$, the $\mathbb{A}$ -subalgebra $\mathbf{f}_\mathbb{A} \subset \mathbf{f}$ generated by the divided powers $\alpha_d^{(c)}$ is an integral form.

The bridge to affine Hecke algebras is a theorem of Ariki [Ari96], in the formulation of [Lec04]. It provides a group isomorphism

$$\Phi: \mathbf{f}_\mathbb{A}^*|_{q=1} \xrightarrow{\sim} \mathbf{R}$$

from the specialization at $q = 1$ of the graded dual of $\mathbf{f}_\mathbb{A}$. We write $x|_{q=1}$ for the specialization of $x \in \mathbf{f}_\mathbb{A}^*$. Under Φ , the dual bases of two distinguished bases of $\mathbf{f}_\mathbb{A}$, both indexed by multisegments, match classes in $\mathbf{R}$:

$$\begin{aligned} \text{the canonical basis } \mathcal{B} = \{b_\mathbf{m}\} : \quad & \Phi(b_\mathbf{m}^*|_{q=1}) = [Z(\mathbf{m})] \quad (\text{simple modules}), \\ \text{a PBW basis } \mathcal{C} = \{e_\mathbf{m}\} : \quad & \Phi(e_\mathbf{m}^*|_{q=1}) = M(\mathbf{m}) \quad (\text{standard modules}). \end{aligned}$$

The first line is Ariki's theorem, for Lusztig's basis [Lus90]. The second holds for a suitable PBW basis [LNT03].

The algebra $\mathbf{f}_\mathbb{A}$ carries a third distinguished basis, a monomial basis modeled on the string bases of Berenstein and Zelevinsky [BZ93]. Let $\mathbf{m}$ be a multisegment, with right ends $E(\mathbf{m}) = \{d_1 < \dots < d_l\}$ and right-aligned parts $\mathbf{m}[d]$ as in Section 4.2. For $d \in E(\mathbf{m})$ set

$$m_{\mathbf{m}[d]} = \alpha_d^{(c_d)} \alpha_{d-1}^{(c_{d-1})} \alpha_{d-2}^{(c_{d-2})} \dots, \quad c_e = \#\{\text{segments of } \mathbf{m}[d] \text{ containing point } e\},$$

a descending product of divided powers, and following the indicator decomposition $\mathbf{m} = \mathbf{m}[d_1] + \dots + \mathbf{m}[d_l]$, set

$$m_\mathbf{m} = m_{\mathbf{m}[d_1]} m_{\mathbf{m}[d_2]} \dots m_{\mathbf{m}[d_l]}.$$

The elements $\{m_\mathbf{m}\}$ form an $\mathbb{A}$ -basis $\mathcal{M}$ of $\mathbf{f}_\mathbb{A}$ [Rei01].

Write the transition matrices between the three bases as

$$m_\mathbf{m} = \sum_{b \in \mathcal{B}} h_{\mathbf{m},b} b = \sum_{\mathbf{n}} s_{\mathbf{m},\mathbf{n}} e_\mathbf{n} \in \mathbf{f}_\mathbb{A}, \quad h_{\mathbf{m},b}, s_{\mathbf{m},\mathbf{n}} \in \mathbb{A}. \quad (3)$$

Theorem B ([GV26]). *Under Φ , the reciprocal character $\text{ch}^\otimes$ is the coordinate map with respect to the dual monomial basis at $q = 1$, that is, $\text{ch}^\otimes(\Phi(m_\mathbf{m}^*|_{q=1})) = \mathbf{m}$ for every multisegment $\mathbf{m}$. In terms of the transition coefficients (3), for every multisegment $\mathbf{n}$,*

$$\text{ch}^\otimes([Z(\mathbf{n})]) = \sum_{\mathbf{m}} h_{\mathbf{m},\mathbf{n}}(1) \cdot \mathbf{m}, \quad \text{ch}^\otimes(M(\mathbf{n})) = \sum_{\mathbf{m}} s_{\mathbf{m},\mathbf{n}}(1) \cdot \mathbf{m},$$

with sums over all multisegments $\mathbf{m}$. Equivalently, for standard modules, the reciprocal multiplicities are $A(\mathbf{m}, \mathbf{n}) = s_{\mathbf{n},\mathbf{m}}(1) = e_\mathbf{m}^*(m_\mathbf{n})|_{q=1}$.

The two expansions follow from the first statement through the dual basis rule, which transposes (3) into $e_n^* = \sum_m s_{m,n} m_m^*$ and $b_n^* = \sum_m h_{m,b_n} m_m^*$.

The monomial basis is unitriangular with respect to the PBW basis [Lus90, Section 7.8], [Rei01, Theorem 4.2], so by Theorem B the matrix $A(m, n) = s_{n,m}(1)$ of reciprocal multiplicities is unitriangular as well. As $\text{ch}^\otimes(M(m)) = \sum_n A(m, n) n$, this proves the following.

Corollary 6.1. *The map $\text{ch}^\otimes: \mathbf{R} \rightarrow \mathbb{Z}[\text{Seg}]$ is an additive isomorphism.*

Applying P_N and then Theorem A turns Theorem B into a statement about $U_v(\widehat{\mathfrak{sl}}_{N+1})$ alone, the precise form of the suggestion of Nakajima [Nak04].

Corollary 6.2. *Let W be a simple or standard module in $\mathcal{C}_N$, parametrized by the dominant monomial $P_N(m_0)$ for a multisegment m_0 . Then*

$$\chi_+^N(W) = \begin{cases} \sum_m h_{m,b_{m_0}}(1) \cdot P_N(m) & \text{for simple } W = L(P_N(m_0)) , \\ \sum_m s_{m,m_0}(1) \cdot P_N(m) & \text{for standard } W = \vartheta(P_N(m_0)) , \end{cases}$$

with sums over all multisegments m .

Example 6.3. Take $m = 2[0, 0] + [1, 1]$ and $n = [0, 0] + [0, 1]$, the pair of Example 5.4. The construction gives

$$m_m = \alpha_0^{(2)} \alpha_1 , \quad m_n = \alpha_0 \cdot \alpha_1 \alpha_0 = m_m + \alpha_1 \alpha_0^{(2)} ,$$

with the second word shaped by the decomposition $n = n[0] + n[1]$ and the last equality by the quantum Serre relation. Both summands lie in the canonical basis. Using unitriangularity (see [Ant23, Section 2]), we get $m_m = b_m$ and $m_n = b_m + b_n$. The matrix h is often the identity, and this pair is a minimal case with a nontrivial entry, here $h_{n,b_m} = 1$. It is natural to ask what $\Phi(m_m^*|_{q=1})$ is. In general it need not be the class of a module, since its coefficients in the basis of simple classes can be negative, as happens here. From $b_m = m_m$ and $b_n = m_n - m_m$ one gets $m_n^* = b_n^*$ and $m_m^* = b_m^* - b_n^*$, and applying Φ gives

$$\Phi(m_n^*|_{q=1}) = [Z(n)] , \quad \Phi(m_m^*|_{q=1}) = [Z(m)] - [Z(n)] .$$

Since $\text{ch}^\otimes$ is additive, Example 5.4 gives $\text{ch}^\otimes([Z(n)]) = n$ and $\text{ch}^\otimes([Z(m)] - [Z(n)]) = (m+n) - n = m$, so both classes have their indexing multisegment as reciprocal character, as Theorem B asserts. Applying P_1 to $\text{ch}^\otimes([Z(m)]) = m + n$, with Theorem A turning the left side into $\chi_+^1(L(P_1(m)))$, recovers the identity (1) of Example 5.4, which is Corollary 6.2 for the simple module $L(P_1(m))$.

The transition matrix from the monomial basis to the PBW basis also occurs as an intermediate step in the algorithms of [Ant23, SZ25] for computing canonical bases. For the bases used here, Theorem B identifies its transpose at $q = 1$ with the matrix of reciprocal multiplicities of standard modules.

7 Outlook

The identity of Theorem B and its quantum affine corollary admit natural deformations on both sides. On the quantum affine side, the q -character is refined by Nakajima's t -analog [Nak04], whose dominant part lies in $\mathbb{Z}[\mathcal{D}_N][t, t^{-1}]$. On the quantum group side, the transition coefficients $h_{m,b}$ and $s_{m,n}$ are Laurent polynomials in q before specialization. It is natural to conjecture that

the two deformations match, with Nakajima's t playing the role of Lusztig's q , and that Theorem B is the shadow at $t = 1$ of this richer compatibility. A promising framework for such questions is provided by quiver Hecke algebras and their generalized Schur–Weyl duality functors [KKK18].

A different perspective on the same Grothendieck rings comes from the cluster algebra categorification of Hernandez and Leclerc [HL10]. The relation between the cluster-theoretic picture and the monomial-basis coordinates studied here remains to be explored.

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