

q -Deformation of Graphic Arrangements and Graphic Monomial Arrangements

Tongyu Nian

The University of Osaka

Abstract

We introduce two new classes of hyperplane arrangements inspired by graphic arrangements. Building on the analogy between the braid arrangement and the arrangement of all hyperplanes over a finite field, we define q -deformation of graphic arrangements and extend the notion from graphs to simplicial complexes by replacing cliques with faces. We also study a special setting called graphic monomial arrangements, which is defined over any field containing primitive r -th roots. We also discuss its defining polynomial, characteristic polynomial and freeness.

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1 Introduction

The theory of hyperplane arrangements lies at the intersection of combinatorics, algebra, and geometry. A central object of study is the *characteristic polynomial*,

which encodes combinatorial information about the arrangement, and properties such as *freeness* and *supersolvability*, which reflect deeper algebraic structure.

A classical family of examples is provided by *graphic arrangements*, associated to graphs, whose characteristic polynomial coincides with the chromatic polynomial of the underlying graph. A landmark theorem due to Stanley characterizes, in purely graph-theoretic terms, precisely when a graphic arrangement is free or supersolvable: the answer is that the graph must be *chordal*.

Recent work of Nian–Tsuje–Uchiumi–Yoshinaga [1], and of Nian [2], introduces two natural generalizations of graphic arrangements:

- (i) the q -deformation $\mathcal{A}_G^q$, an arrangement over a finite field $\mathbb{F}_q^\ell$ that recovers the graphic arrangement in the limit $q \rightarrow 1$; and
- (ii) the *graphic monomial arrangement* $\mathcal{M}(G, r)$, which incorporates r -th roots of unity and whose characteristic polynomial is an explicit rescaling of the chromatic polynomial.

Both families are shown to enjoy the same property as the classical case: chordality of the graph is equivalent to supersolvability, which is equivalent to freeness. The proof strategy relies on the deletion–restriction recursion and on an explicit construction of a basis for the module of derivations.

In a third direction, Berget [4] showed that the *internal zonotopal algebra* of the Gale dual of $\mathcal{M}(K_\ell, r)$ obtain a representation of the monomial group $W = G(r, 1, \ell)$ isomorphic to a specific induced representation.

This report is organized as follows. Section 2 reviews the necessary background on hyperplane arrangements. Section 3 introduces q -deformations and graphic monomial arrangements, and states the main equivalence theorem. Section 4 discusses the representation-theoretic consequences via zonotopal algebras. Section 5 collects open questions and directions for future research.

2 Background on Hyperplane Arrangements

2.1 Hyperplane Arrangements and the Characteristic Polynomial

Let $\mathbb{K}$ be a field and $V = \mathbb{K}^\ell$ an ℓ -dimensional vector space.

Definition 2.1 (Hyperplane arrangement). A *hyperplane* $H \subset V$ is an affine subspace of codimension one, i.e., the zero locus of a linear polynomial $\alpha_H \in \mathbb{K}[x_1, \dots, x_\ell]$. A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of hyperplanes in V . The *defining polynomial* of $\mathcal{A}$ is

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H.$$

The combinatorics of an arrangement is largely governed by its *intersection poset* $L(\mathcal{A})$, consisting of all nonempty intersections of subsets of hyperplanes, ordered by reverse inclusion, with the whole space V as minimum element.

Definition 2.2 (Characteristic polynomial). For an arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$, the *characteristic polynomial* is

$$\chi(\mathcal{A}, t) = \sum_{I \subseteq [n]} (-1)^{|I|} t^{\dim H_I}, \quad H_I = \bigcap_{i \in I} H_i.$$

Equivalently, one can define $\chi(\mathcal{A}, t)$ via the Möbius function of $L(\mathcal{A})$.

A fundamental tool for computing characteristic polynomials is the *deletion–restriction* relation.

Proposition 2.3 (Deletion–restriction). *For any $H \in \mathcal{A}$, define the restriction $\mathcal{A}^H := \{X \cap H \mid X \in \mathcal{A} \setminus H, X \cap H \neq \emptyset\}$. Then*

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A} \setminus H, t) - \chi(\mathcal{A}^H, t).$$

2.2 Coxeter Arrangements and Graphic Arrangements

Definition 2.4 (Coxeter arrangement). A *Coxeter group* is a group of the form $\langle r_1, \dots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle$ with $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. The *Coxeter arrangement* associated to a finite Coxeter group W is the arrangement whose hyperplanes are the reflecting hyperplanes of W .

Definition 2.5 (Graphic arrangement). Let $G = ([\ell], E)$ be a simple graph on ℓ vertices. The *graphic arrangement* of G is

$$\mathcal{A}_G := \{\ker(x_i - x_j) \mid (i, j) \in E\} \subset \mathbb{K}^\ell.$$

Graphic arrangements are subarrangements of the type- $A_{\ell-1}$ Coxeter arrangement. A classical result states that $\chi(\mathcal{A}_G, t) = \chi(G, t)$, the chromatic polynomial of G .

Example 2.6. Let P_3 be the path graph on vertices $\{1, 2, 3\}$ with edges $(1, 2)$ and $(2, 3)$. Then $\mathcal{A}_{P_3}$ consists of the two hyperplanes $x_1 - x_2 = 0$ and $x_2 - x_3 = 0$ in $\mathbb{K}^3$. The chromatic polynomial is $\chi(P_3, t) = t(t-1)^2$, and the characteristic polynomial is $\chi(\mathcal{A}_{P_3}, t) = t^3 - 2t^2 + t$.

2.3 Supersolvability and Freeness

Definition 2.7 (Supersolvable arrangement). An arrangement $\mathcal{A} \subset \mathbb{K}^\ell$ is *supersolvable* if its intersection lattice $L(\mathcal{A})$ is supersolvable, i.e., there exists a chain of subarrangements

$$\emptyset = \mathcal{A}_0 \subsetneq \mathcal{A}_1 \subsetneq \dots \subsetneq \mathcal{A}_\ell = \mathcal{A}$$

such that $\text{rank}(\mathcal{A}_i) = i$ and for any two distinct hyperplanes $H, H' \in \mathcal{A}_i \setminus \mathcal{A}_{i-1}$ there exists $H'' \in \mathcal{A}_{i-1}$ with $H \cap H' \subseteq H''$. Such a chain is called a *filtration*.

For a supersolvable arrangement with filtration as above,

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - |\mathcal{A}_i \setminus \mathcal{A}_{i-1}|).$$

Definition 2.8 (Free arrangement). Let $S = \mathbb{K}[x_1, \dots, x_\ell]$ and Der_V be the S -module of $\mathbb{K}$ -linear derivations $\theta : S \rightarrow S$ satisfying the Leibniz rule. Define the *module of $\mathcal{A}$ -derivations*:

$$D(\mathcal{A}) := \{\theta \in \text{Der}_V \mid \theta(Q(\mathcal{A})) \in (Q(\mathcal{A}))\}.$$

The arrangement $\mathcal{A}$ is *free* with *exponents* $(e_1, \dots, e_\ell)$ if $D(\mathcal{A})$ is a free S -module with a basis $\theta_1, \dots, \theta_\ell$ where $\deg \theta_i = e_i$.

Theorem 2.9 (Terao's factorization theorem, [3, Ch. 4]). *If $\mathcal{A}$ is free with exponents $(e_1, \dots, e_\ell)$, then*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - e_i).$$

The implication supersolvable $\Rightarrow$ free holds in general; the converse fails. For graphic arrangements, however, a complete picture is available.

Theorem 2.10 (Stanley). *For a graph G and its graphic arrangement $\mathcal{A}_G$, the following conditions are equivalent:*

- (1) G is chordal (every cycle of length ≥ 4 has a chord).
- (2) $\mathcal{A}_G$ is supersolvable.
- (3) $\mathcal{A}_G$ is free.

Example 2.11. For the path graph P_3 , the derivations $\theta_1 = \partial_1 + \partial_2 + \partial_3$, $\theta_2 = (x_1 - x_2)\partial_1$, $\theta_3 = (x_2 - x_3)\partial_3$ form a basis of $D(\mathcal{A}_{P_3})$, so $\mathcal{A}_{P_3}$ is free with exponents $(0, 1, 1)$. The characteristic polynomial $t(t-1)^2$ factors accordingly.

3 q -Deformations and Graphic Monomial Arrangements

3.1 q -Deformation of Graphic Arrangements

The q -deformation introduced in [1] replaces the single hyperplane associated to each edge by a family of hyperplanes parametrized by elements of the multiplicative group $\mathbb{F}_q^\times$.

Definition 3.1 (q -deformation [1]). Let $G = ([\ell], E)$ be a graph and q a prime power. Denote by $\mathcal{C}(G)$ the set of cliques of G :

$$\mathcal{C}(G) := \{C \subseteq V(G) \mid (x, y) \in E(G) \text{ for all } x, y \in C\}.$$

The q -deformation of the graphic arrangement is the arrangement $\mathcal{A}_G^q$ in $\mathbb{F}_q^\ell$ defined by

$$\mathcal{A}_G^q := \{ \{a_{i_1}x_{i_1} + \dots + a_{i_r}x_{i_r} = 0\} \mid \{i_1, \dots, i_r\} \in \mathcal{C}(G), a_{i_j} \in \mathbb{F}_q^\times \}.$$

When $G = K_\ell$ is the complete graph, every subset of vertices is a clique and $\mathcal{A}_{K_\ell}^q$ contains all hyperplanes in $\mathbb{F}_q^\ell$. In this extremal case,

$$\chi(\mathcal{A}_{K_\ell}^q, t) = (t-1)(t-q)(t-q^2) \cdots (t-q^{\ell-1}).$$

The relationship to the chromatic polynomial is made precise by the following limit result.

Proposition 3.2 (Nian-Tsujie-Uchiumi-Yoshinaga [1]). *Let $\chi(\mathcal{A}_G^q, t)$ be the characteristic polynomial of $\mathcal{A}_G^q$. Then*

$$\lim_{q \rightarrow 1} \frac{\chi(\mathcal{A}_G^q, q^t)}{(q-1)^\ell} \equiv \chi(G, t) \pmod{q-1}.$$

Furthermore, equality holds (not just modulo $q-1$) when G is chordal or triangle-free.

3.2 Graphic Monomial Arrangements

A complementary generalization incorporates roots of unity.

Definition 3.3 (Graphic monomial arrangement [2]). Let $G = ([\ell], E)$ be a graph and r a positive integer. Suppose the field $\mathbb{K}$ contains a primitive r -th root of unity z (i.e., $z^r = 1_{\mathbb{K}}$, $z^k \neq 1_{\mathbb{K}}$ for $0 < k < r$). The *graphic monomial arrangement* is

$$\mathcal{M}(G, r) := \{ \{x_i - z^k x_j = 0\} \mid (i, j) \in E, 1 \leq k \leq r \} \cup \{ \{x_i = 0\} \mid i = 1, \dots, \ell \}.$$

Proposition 3.4 (Nian [2]). *The defining polynomial and characteristic polynomial of $\mathcal{M}(G, r)$ are:*

$$Q(\mathcal{M}(G, r)) = \prod_{i=1}^{\ell} x_i \cdot \prod_{(i,j) \in E} (x_i^r - x_j^r),$$

$$\chi(\mathcal{M}(G, r), t) = r^{\ell} \chi\left(G, \frac{t-1}{r}\right).$$

An analogue of Proposition 3.2 holds in this setting.

Proposition 3.5 (Nian [2]).

$$\lim_{r \rightarrow 0} \frac{\chi(\mathcal{M}(G, r), (r+1)^t)}{r^{\ell}} = \chi(G, t).$$

3.3 Freeness

A central result of [1, 2] is that chordality remains the key condition for freeness and supersolvability, in both new families.

Theorem 3.6 (Nian-Tsujie-Uchiumi-Yoshinaga [1], Nian [2]). *Let $G = ([\ell], E)$ be a graph. The following five conditions are equivalent:*

- (1) G is chordal.
- (2) $\mathcal{A}_G^q$ is supersolvable.
- (3) $\mathcal{A}_G^q$ is free.
- (4) $\mathcal{M}(G, r)$ is supersolvable.
- (5) $\mathcal{M}(G, r)$ is free.

This theorem extends Theorem 2.10 to both the q -deformed and monomial settings, and shows that the chordal condition is a robust combinatorial characterization of the algebraic properties of these arrangement families.

Remark 3.7. The proof of Theorem 3.6 uses the deletion–restriction recursion (Proposition 2.3) together with an explicit construction of a free basis for $D(\mathcal{A}_G^q)$ and $D(\mathcal{M}(G, r))$ when G is chordal. The basis is built inductively using a perfect elimination ordering of the chordal graph. The supersolvability direction uses a filtration compatible with the clique structure.

Furthermore, [2] extends the framework to simplicial complexes, replacing the role of cliques by simplices and obtaining analogous combinatorial characterizations in that broader setting.

4 A Representation from the Graphic Monomial Arrangement

4.1 Internal Zonotopal Algebras

The third part of the theory connects $\mathcal{M}(K_\ell, r)$ to representation theory via the internal zonotopal algebra introduced by Holtz and Ron.

Definition 4.1 (Internal zonotopal algebras [6]). Let $\mathcal{A}$ be a central arrangement in $V = \mathbb{C}^\ell$. For $h \in V$, define

$$\rho_{\mathcal{A}}(h) := \#\{H \in \mathcal{A} \mid h \notin H\}.$$

For $k \in \{-2, -1, 0\}$, set

$$I_{\mathcal{A},k} := \langle h^{\max\{\rho_{\mathcal{A}}(h)+k+1, 0\}} \mid h \in V \rangle \subset \text{Sym}(V).$$

The quotient algebras $S_{\mathcal{A},k} := \text{Sym}(V)/I_{\mathcal{A},k}$ for $k = 0$, $k = -1$, and $k = -2$ are called the *external*, *central*, and *internal zonotopal algebras* of $\mathcal{A}$, respectively.

Theorem 4.2 (Ardila-Postnikov [5]). *For a central essential arrangement $\mathcal{A}$, the ideal $I_{\mathcal{A},k}$ is generated by $\{h^{\rho_{\mathcal{A}}(h)+k+1} \mid h \text{ spans a line of } \mathcal{A}\}$ for $k \in \{-2, -1, 0\}$. Moreover, the Hilbert series of $S_{\mathcal{A},k}$ is closely related to the Tutte polynomial of the matroid $M(\mathcal{A})$.*

4.2 Gale Duality

Definition 4.3 (Gale dual). Given a central arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ with n hyperplanes, the Gale dual of $\mathcal{A}$ is the hyperplane arrangement corresponding to the dual of the vector matroid $M(\mathcal{A})$.

Proposition 4.4. *The graphic monomial arrangement $\mathcal{M}(G, r)$ contains the coordinate hyperplanes $\{x_i = 0\}$ for $i = 1, \dots, \ell$, so its normal vectors can be written in the form $[I_\ell \mid A]$. The normal vectors of $\mathcal{M}(G, r)^\perp$ are then $[-A^t \mid I_{n-\ell}]$.*

Example 4.5. Let $\mathcal{A} = \mathcal{M}(K_2, 2)$, the arrangement in $\mathbb{C}^2$ defined by $xy(x-y)(x+y)$. The normal vector matrix of $\mathcal{A}$ is

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

The Gale dual has normal vectors

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix},$$

so $\mathcal{A}$ is self-dual. One computes $\langle x^2, y^2, (x \pm y)^2 \rangle \subset I_{\mathcal{A},-2}$ and the Hilbert series of $S_{\mathcal{A},-2}$ is $1 + 2q$.

4.3 The Representation of the Monomial Group

Definition 4.6 (Monomial reflection group). The *monomial group* $W = G(r, 1, \ell)$ is the group of all $\ell \times \ell$ matrices whose nonzero entries are r -th roots of unity and which have exactly one nonzero entry in each row and column. It is generated by the symmetric group S_ℓ (permutation matrices) and the diagonal matrices g_i obtained from I_ℓ by replacing the i -th diagonal entry by a primitive r -th root of unity z .

Remark 4.7. The reflection arrangement of $W = G(r, 1, \ell)$ is precisely the graphic monomial arrangement $\mathcal{M}(K_\ell, r)$.

Theorem 4.8 (Berget 2018, [4]). *Let $\mathcal{A} = \mathcal{M}(K_\ell, r)$ be the reflection arrangement of $W = G(r, 1, \ell)$, and let $\mathcal{A}^\perp$ be its Gale dual. Then the internal zonotopal algebra $S_{\mathcal{A}^\perp, -2}$ carries a natural action of W , making it a representation of W .*

Moreover, let S denote the degree- $(\ell - 1)$ component of $S_{\mathcal{A}^\perp, -2}$, and let $C \leq W$ be the subgroup generated by an ℓ -cycle $c \in S_\ell \subset W$ and the scalar matrix $zI \in W$. Define the one-dimensional character χ of C by $\chi(c) = z^{r/\ell}$ and $\chi(zI) = z^{\ell-1}$. Then $S \cong \text{Ind}_C^W(\chi)$ as representations of W .

4.4 Explicit Group Action on $\mathcal{M}(K_\ell, r)$

The proof of Theorem 4.8 requires an explicit description of the W -action on the hyperplanes of $\mathcal{M}(K_\ell, r)$, which Nian works out in [2] as follows.

Label the hyperplanes: $H_i := \{x_i = 0\}$ and $H_{ij}^k := \{x_i - z^k x_j = 0\}$. For a permutation $\pi \in S_\ell$:

$$\pi(H_i) = H_{\pi(i)}, \quad \pi(H_{ij}^k) = H_{\pi(i)\pi(j)}^k.$$

For the diagonal generator g_i (which multiplies x_i by z):

$$g_i(H_i) = zH_i, \quad g_i(H_{ji}^k) = zH_{ji}^{k-1}, \quad g_i(H_{ij}^k) = H_{ij}^{k+1},$$

and $g_m(H_{ij}^k) = H_{ij}^k$ for $m \neq i, j$.

Setting $y_{ij}^k = (H_i - z^k H_j) - H_{ij}^k$, the kernel K of the map $H_i \mapsto e_i$, $H_{ij}^k \mapsto e_i - z^k e_j$ contains $\mathcal{M}^\perp(K_\ell, r)$ and has an explicit W -action:

$$\pi(y_{ij}^k) = y_{\pi(i)\pi(j)}^k, \quad g_i(y_{ij}^k) = z y_{ij}^{k-1}, \quad g_j(y_{ij}^k) = y_{ij}^{k+1}, \quad g_m(y_{ij}^k) = y_{ij}^k \quad (m \neq i, j).$$

5 Discussion and Future Directions

The combinatorial condition of chordality governs the algebraic properties of freeness and supersolvability uniformly across three families of arrangements introduced in this report. The proofs exploit the deletion-restriction recursion and clique-based inductive structures in a way that is robust across the different algebraic settings.

Several natural directions for future investigation remain open.

Representations from General Graphs

Theorem 4.8 and its proof in [2] apply to the special case $G = K_\ell$, where $\mathcal{M}(K_\ell, r)$ is the full reflection arrangement of $W = G(r, 1, \ell)$. A natural question is:

For a general graph G , does the internal zonotopal algebra of the Gale dual of $\mathcal{M}(G, r)$ carry an action of some group, and can the resulting representation be identified combinatorially?

When G is a proper subgraph of K_ℓ , the arrangement $\mathcal{M}(G, r)$ is no longer a reflection arrangement, so the group action is less transparent. New techniques may be needed.

Monodromy Representations

Another source of representations from hyperplane arrangements is the *monodromy* of the complement $\mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$. The fundamental group of this complement (the *arrangement group*) acts on the cohomology of fibrations associated to the arrangement. It would be interesting to investigate whether the q -deformation or graphic monomial arrangements give rise to monodromy representations with combinatorial descriptions analogous to Theorem 4.8.

Extension to Simplicial Complexes

As announced in [2], the q -deformation framework extends from graphs to simplicial complexes, replacing cliques by simplices. The full analogue of Theorem 3.6 in that generality, and the identification of the appropriate combinatorial condition replacing chordality, is an ongoing research direction.

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