

CHROMATIC SYMMETRIC FUNCTIONS OF GRAPHS WITH HESSENBERG FUNCTIONS

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ABSTRACT. The goal of this note is to share the author's ongoing attempts to understand relations between graph colorings and e -positivity of chromatic symmetric functions in terms of new classes of symmetric functions, *palette* and *mosaic* functions. We also review the history on Stanley-Stembridge conjecture since 1993.

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1. INTRODUCTION

The goal of this note is to share the author's ongoing attempts to understand relations between graph colorings and new classes of symmetric functions, *palette*

Date: July 24, 2026.

2020 Mathematics Subject Classification. Primary:05E05; Secondary:05C15.

Key words and phrases. chromatic symmetric function, elementary symmetric function, graph coloring, Hessenberg function, integer partition, monomial symmetric function, set partition.

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This is based on the author's talk at the workshop “Interactions of representation theory and related topics”, Kyoto RIMS, 11:00–11:45 am, on June 4, 2026.

and *mosaic* functions. Let us briefly review the history on this topic. Stanley (1995) [8] introduced the chromatic symmetric functions for a graph. He (and Stanley–Stembridge (1993) [9] before) stated a conjecture on *e-positivity* of such functions. Afterward, Gasharov (1996) [3] proved *s-positivity* (Schur-positivity) in $(3+1)$ -free cases to support the conjecture. Apart from these, Tymoczko (2008) [10] showed that the cohomology ring of some Hessenberg varieties admits an S_n -action. Guay-Paquet (2013) [5] reduced the conjecture to $(3+1)$ - & $(2+2)$ -free graphs; in fact, a graph of this class originates from geometry of Hessenberg varieties. Shareshian–Wachs (2016) [7] combined Stanley–Stembridge’s and Tymoczko’s ideas with introducing the chromatic quasi-symmetric function for a graph. Recently, Hikita (2024) [6] presented a proof of the conjecture. Indeed, he used heavy machinery in the probability theory. We nonetheless aim to better understand his argument only with classical combinatorics of graphs and tableaux. Our main results are Theorems 3.5, 3.18, 3.20 and 3.25. Some of our discussions are straightforward and we sometimes omit a proof.

Remark 1.1. We assume basic knowledge on Young tableaux as in [2].

1.1. Chromatic symmetric functions. Throughout we deal with a finite simple graph $G = (V, \text{Edge}(G))$, $V = [n] = \{1, \dots, n\}$. For $i, j \in [n]$, $i - j$ means $\{i, j\} \in \text{Edge}(G)$. A coloring on G is any map $\kappa : [n] \rightarrow \mathbf{N}$. It is *proper* if

$$i - j \implies \kappa(i) \neq \kappa(j).$$

Let $K(G)$ denote the set of all proper colorings on G . It is convenient to express a vertex i colored by a as i_a .

Example 1.2. If $G = 1-2-3$, $a, b, c \in \mathbf{N}$ and $a < b < c$, then $1_a-2_b-3_c, 1_a-2_b-3_a$ are both proper colorings on G .

Definition 1.3 (Stanley [8]). Let $x = (x_1, x_2, \dots)$ be commuting variables. The *weight* of $\kappa \in K(G)$ is $x_\kappa = x_{\kappa(1)} \cdots x_{\kappa(n)}$. The *chromatic symmetric function* of G is

$$X_G(x) = \sum_{\kappa \in K(G)} x_\kappa.$$

Remark 1.4. This is a homogeneous and symmetric formal power series.

Let $\Lambda_n = \mathbf{Z}[[x]]_n^{S_\infty}$ be the homogeneous part of the ring of symmetric functions. Recall two of the distinguished bases. For $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$, define *elementary symmetric function*

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_l}$$

where

$$e_k = \sum_{\substack{a_1 < \cdots < a_k \\ 2}} x_{a_1} \cdots x_{a_k},$$

and *monomial symmetric function*

$$m_\lambda = \sum_{\delta \in \text{Perm}(\lambda), a_k < a_{k+1}} x_{a_1}^{\delta_1} \cdots x_{a_l}^{\delta_l}$$

where $\delta \in \text{Perm}(\lambda)$ means $\delta = (\delta_1, \dots, \delta_l)$ and there exists some $w \in S_l$ such that $\delta_k = \lambda_{w(k)}$.

Definition 1.5. Say $F(x) \in \Lambda_n$ is *e-positive* if

$$F(x) = \sum_{\lambda \vdash n} a_\lambda e_\lambda, \quad a_\lambda \in \mathbf{Z}_{\geq 0}.$$

Define *m-positivity* similarly.

Proposition 1.6. If $F(x) \in \Lambda_n$ is *e-positive*, then it is *m-positive*.

Proof. It follows from expansions $e_\mu = \sum_\lambda K_{\lambda'\mu} s_\lambda$ and $s_\lambda = \sum_\nu K_{\lambda\nu} m_\nu$ with Schur functions and Kostka numbers which are all nonnegative. $\square$

Example 1.7. In particular, a single e_λ is *m-positive*. See Tables 2 and 3.

We will give a proof of that $X_G(x)$ is *m-positive* (Theorem 3.5). It is then natural to ask when it is *e-positive*. To explain some details, we need more definitions. Let P be a poset on $[n]$. Define a graph $\text{inc}(P)$ with vertices $[n]$ and edges:

$$i - j \iff i \text{ is incomparable to } j \text{ in } P.$$

Say P is *(3+1)-free* if it avoids an induced subposet isomorphic to $\{1 < 2 < 3\} + \{1\}$ (+ is the direct sum of posets). Say G is *(3+1)-free* if $G = \text{inc}(P)$ with P *(3+1)-free*.

Conjecture 1.8 (Stanley–Stembridge [8, Conjecture 5.1], [9, Conjecture 5.5]). If G is *(3+1)-free*, then $X_G(x)$ is *e-positive*.

For the last thirty years, researchers verified its correctness for many classes of G . Among these, Guay-Paquet made essential progress by narrowing down G to graphs of a smaller class. Say P is *(2+2)-free* if P avoids an induced subposet isomorphic to $\{1 < 2\} + \{1 < 2\}$. Usage of “ G is *(2+2)-free*” is similar.

Theorem 1.9 ([5, Theorem 5.1]). To prove Stanley–Stembridge conjecture, it is enough to show that if G is *(3+1)- and (2+2)-free*, then $X_G(x)$ is *e-positive*.

Recently, Hikita (2024) [6] presented a proof of Stanley–Stembridge conjecture. Indeed, he uses heavy machinery in the probability theory. However, since the conjecture is just about the sum of elementary symmetric functions, it should be possible to understand his argument with only classical combinatorics of graphs and tableaux. We will provide some ideas in this direction after this section. For now, let us review a bit on Hessenberg functions.

1.2. Hessenberg graphs.

Definition 1.10. A map $h : [n] \rightarrow [n]$ is a *Hessenberg function* if:

- $h(i) \geq i$ for $i \in [n]$,
- $h(i) \leq h(i+1)$ for $i \in [n-1]$.

Denote by $H(n)$ the set of all such functions.

From $h \in H(n)$, construct a graph $G(h)$ as follows. Its vertex set is $[n]$ and define an edge $i-j$ if $h(i) \geq j$. Say a graph G is *Hessenberg* if $G = G(h)$ for some $h \in H(n)$.

Fact 1.11 ([7, p.510-513]). The following are equivalent:

- (1) G is Hessenberg.
- (2) G is both $(3+1)$ - and $(2+2)$ -free.

1.3. Cores of this enumeration problem. Here are the cores of this enumeration problem.

- From m -positive X_G , how to “assemble” weights of proper colorings which sum up to a single e_λ ?
- How to count “copies” of those e_λ ’s?
- How to connect graph colorings and a pair of Young tableaux through a Knuth array with *dual* RSK?
- How is Hessenberg-ness of a graph related to e -positivity of X_G ?

For this purpose, we have been planning to develop a little new theory with various objects on m - and e -side as in Table 1. Such a naming is for the sake of our formalism. Palette m - and e -functions are nothing but commuting Gebhard–Sagan chromatic functions [4], though.

TABLE 1. objects on m - and e -side

m -side	e -side
m_λ	e_λ
m -factorial	e -factorial
palette m -function m_π	palette e -function e_σ
mosaic	ensemble
mosaic function m_μ	ensemble function e_ε

2. PALETTE SYMMETRIC FUNCTIONS

2.1. Palette (set partition). Following Gebhard–Sagan [4], we recall basic ideas on set partitions.

Definition 2.1. Say π is a *palette* (set partition) of $[n]$ if $\pi = \{\pi_1, \dots, \pi_N\}$ with $\pi_k \subseteq [n]$, $\pi_k \neq \emptyset$, $\pi_k \cap \pi_l = \emptyset$ for $k \neq l$ and $\bigsqcup \pi_k = [n]$. Let $\Pi(n)$ denote the set of all such π 's.

Each π_k is a *block* of π . Call $N = \ell(\pi)$ the length of π . For ease of notation, write $\pi = \pi_1 / \dots / \pi_N$ with omitting commas and parenthesis as

$$\pi = 136/47/58/2.$$

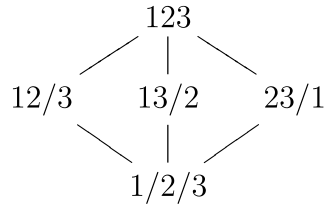
In other words, a palette is an *unordered* list of disjoint subsets of $[n]$. Customarily, we list blocks of π in a way that $(|\pi_k|)$ is weakly decreasing. There is a choice of a total order on blocks of the same size, though. It does not matter for now; we will take care of this later on. By $i =_\pi j$, we mean i, j belong to the same block of π . In such a case, we intuitively say i, j have the same color and call a block a *color block* instead.

Define a partial order on $\Pi(n)$ by refinement as $\pi \leq \sigma \iff$ each block of π is contained in some block in σ . This makes $\Pi(n)$ a lattice with meet $\wedge$ and join $\vee$. Notice that

$$\min \Pi(n) = \widehat{0} = 1/2/\dots/n, \quad \max \Pi(n) = \widehat{1} = 12 \dots n.$$

In particular, $r(\pi) = n - \ell(\pi)$ defines a rank function with $r(\widehat{0}) = 0, r(\widehat{1}) = n - 1$.

Example 2.2. This is the Hasse diagram of $\Pi(3)$.



$\Pi(4)$ consists of 15 set partitions as 1234, 123/4, 124/3, 134/2, 234/1, 12/34, 13/24, 14/23, 12/3/4, 13/2/4, 14/2/3, 23/1/4, 24/1/3, 34/1/2 and 1/2/3/4.

There is a natural projection $\text{sh} : \Pi(n) \rightarrow \text{Par}(n)$, $\text{sh}(\pi) = \text{sort}(|\pi_k|) \vdash n$ where sort is the operation to arrange a weakly decreasing sequence from a given one. This is the *shape* of π .

Now let $\lambda \vdash n$, say $\lambda = (\lambda_1, \dots, \lambda_l) = (n^{k_n}, \dots, 1^{k_1})$. Define the *e-factorial* of λ as

$$\lambda!_e = \lambda! = \lambda_1! \cdots \lambda_l!$$

and the *m-factorial* of λ is

$$\lambda!_m = \lambda^! = k_1! \cdots k_n!.$$

Also, for $\pi \in \Pi(n)$, define $\pi!_e = \pi! = \text{sh}(\pi)!_e$, $\pi!_m = \pi! = \text{sh}(\pi)!_m$. These factorials will naturally show up in a coefficient of commuting Gebhard–Sagan m - and e -functions below.

Remark 2.3. Such little unfamiliar symbols $\pi!$, $\pi!$ appeared in Dahlberg–van Willigenburg [1, p.3], for example. Gebhard–Sagan [4, p.228] previously used $|\pi|$ to mean $\pi!$. Here we stick to $!_m$ and $!_e$ for our formalism with m - and e -objects.

Example 2.4. Let $\pi = 1467/38/59/2 \in \Pi(9)$. Then

$$\text{sh}(\pi) = 4221, \pi!_e = 4!2!2!1! = 96, \pi!_m = 2! = 2.$$

TABLE 2. m -expansion of e_λ , $\lambda \vdash 1, 2, 3, 4, 5$.

e_1	$= m_1$
e_2	$= m_{11}$
e_{11}	$= m_2 + 2m_{11}$
e_3	$= m_{111}$
e_{21}	$= m_{21} + 3m_{111}$
e_{111}	$= m_3 + 3m_{21} + 6m_{111}$
e_4	$= m_{1111}$
e_{31}	$= m_{211} + 4m_{1111}$
e_{22}	$= m_{22} + 2m_{211} + 6m_{1111}$
e_{211}	$= m_{31} + 2m_{22} + 5m_{211} + 12m_{1111}$
e_{1111}	$= m_4 + 4m_{31} + 6m_{22} + 12m_{211} + 24m_{1111}$
e_5	$= m_{1^5}$
e_{41}	$= m_{2111} + 5m_{1^5}$
e_{32}	$= m_{221} + 3m_{2111} + 10m_{1^5}$
e_{311}	$= m_{311} + 2m_{221} + 7m_{2111} + 20m_{1^5}$
e_{221}	$= m_{32} + 2m_{311} + 5m_{221} + 12m_{2111} + 30m_{1^5}$
e_{2111}	$= m_{41} + 3m_{32} + 7m_{311} + 12m_{221} + 27m_{2111} + 60m_{1^5}$
e_{1^5}	$= m_5 + 5m_{41} + 10m_{32} + 20m_{311} + 30m_{221} + 60m_{2111} + 120m_{1^5}$

TABLE 3. m -expansion of e_λ , $\lambda \vdash 6$.

e_6	$= m_1^6$
e_{51}	$= m_{21^4} + 6m_1^6$
e_{42}	$= m_{2211} + 4m_{21^4} + 15m_1^6$
e_{411}	$= m_{3111} + 2m_{2211} + 9m_{21^4} + 30m_1^6$
e_{33}	$= m_{222} + 2m_{2211} + 6m_{21^4} + 20m_1^6$
e_{321}	$= m_{321} + 3m_{3111} + 3m_{222} + 8m_{2211} + 22m_{21^4} + 60m_1^6$
e_{3111}	$= m_{411} + 3m_{321} + 10m_{3111} + 6m_{222} + 18m_{2211} + 48m_{21^4} + 120m_1^6$
e_{222}	$= m_{33} + 3m_{321} + 6m_{3111} + 6m_{222} + 15m_{2211} + 36m_{21^4} + 90m_1^6$
e_{2211}	$= m_{42} + 2m_{411} + 2m_{33} + 8m_{321} + 18m_{3111}$ $+ 15m_{222} + 34m_{2211} + 78m_{21^4} + 180m_1^6$
e_{21^4}	$= m_{51} + 4m_{42} + 9m_{411} + 6m_{33} + 22m_{321} + 48m_{3111}$ $+ 36m_{222} + 78m_{2211} + 168m_{21^4} + 360m_1^6$
e_{1^6}	$= m_6 + 6m_{51} + 15m_{42} + 30m_{411} + 20m_{33} + 60m_{321}$ $+ 120m_{3111} + 90m_{222} + 180m_{2211} + 360m_{21^4} + 720m_1^6$

2.2. Palette m -function. In addition to m_λ and e_λ , we here introduce new functions. It is convenient to call a sequence $a = (a_1, \dots, a_n) \in \mathbf{N}^n$ a *coloring* (as on the complement of the complete graph K_n).

Definition 2.5. The *palette m -function* for $\pi \in \Pi(n)$ is

$$m_\pi = \sum x_{a_1} \cdots x_{a_n}$$

where a coloring $a = (a_1, \dots, a_n)$ satisfies $a_i = a_j$ if and only if i, j are in the same block of π .

This is a commuting version of m_π in [4, p.228].

Example 2.6.

$$m_{13/2} = \sum_{\substack{a_1=a_3=c_1 \\ a_2=c_2 \\ c_1 \neq c_2}} x_{a_1} x_{a_2} x_{a_3}, \quad m_{134/25} = \sum_{\substack{a_1=a_3=a_4=c_1 \\ a_2=a_5=c_2 \\ c_1 \neq c_2}} x_{a_1} x_{a_2} x_{a_3} x_{a_4} x_{a_5}.$$

Proposition 2.7. m_π is symmetric.

Proof. We have $m_\pi = \pi!_m m_{\text{sh}(\pi)}$ [4, p.229] and $m_{\text{sh}(\pi)}$ is symmetric. □

2.3. Palette e -function. For $\pi, \sigma \in \Pi(n)$, say π is *compatible* with σ if $\pi \wedge \sigma = \widehat{0}$. Let

$$C(\sigma) = \{\pi \in \Pi(n) \mid \pi \wedge \sigma = \widehat{0}\}.$$

Notice that this set is always nonempty since $\widehat{0} \in C(\sigma)$. In particular, $C(\widehat{0}) = \Pi(n)$ and $C(\widehat{1}) = \{\widehat{0}\}$.

Observation 2.8. The following are equivalent:

- (1) $\pi \wedge \sigma = \widehat{0}$.
- (2) For any k, l , we have $0 \leq |\pi_k \cap \sigma_l| \leq 1$.

Example 2.9. $\pi = 3798/256/14$ is compatible with $\sigma = 123/457/69/8$.

Proposition 2.10. $C(\sigma)$ is an order ideal of $\Pi(n)$.

Proof. Let $\pi^{(2)} \in C(\sigma)$ and $\pi^{(1)} \leq \pi^{(2)}$. It is enough to show that $\pi^{(1)} \in C(\sigma)$. Let $i \neq j$ and assume a block σ_l contains both. Suppose now, toward a contradiction, that there exists some block $\pi_{k_1}^{(1)}$ such that $i, j \in \pi_{k_1}^{(1)}$. Since $\pi^{(1)}$ is a refinement of $\pi^{(2)}$, there must exist some block $\pi_{k_2}^{(2)}$ which contains $\pi_{k_1}^{(1)}$. However, we have $i, j \in \pi_{k_1}^{(1)} \subseteq \pi_{k_2}^{(2)}$, a contradiction to $\pi^{(2)} \in C(\sigma)$ and $i, j \in \sigma_l$. $\square$

Definition 2.11. The *palette e -function* for $\sigma \in \Pi(n)$ is

$$e_\sigma = \sum_{\pi \in C(\sigma)} m_\pi.$$

In other words,

$$e_\sigma = \sum x_{a_1} \cdots x_{a_n}$$

where $(a_1, \dots, a_n) \in \mathbf{N}^n$ satisfies $a_i \neq a_j$ if i, j are in the same block of σ .

Again, this is a commuting version of e_σ in [4, p.229].

Proposition 2.12. e_σ is symmetric.

Proof. We have $e_\sigma = \sigma!_e e_{\text{sh}(\sigma)}$ [4, p.229] and $e_{\text{sh}(\sigma)}$ is symmetric. $\square$

Example 2.13. Observe that $C(13/2) = \{12/3, 23/1, 1/2/3\}$ and hence

$$e_{13/2} = m_{12/3} + m_{23/1} + m_{1/2/3}.$$

Remark 2.14. $e_{12 \cdots n} = m_{1/2 \cdots /n}$, $e_{1/2 \cdots /n} = \sum_{\pi \in \Pi(n)} m_\pi$. More generally,

$$\sigma \leq \tau \implies e_\sigma \geq e_\tau$$

where $f \leq g$ in Λ_n means $g - f$ is a formal power series with all nonnegative coefficients.

Example 2.15. Let $\sigma = 134/25$ and $\pi \in C(\sigma)$.

- Case 1: $\ell(\pi) = 3$. There exist exactly $3!2!/2!=6$ palettes $\pi \in C(\sigma)$ because $\pi^{(1)} = 12/35/4 \in C(\sigma)$ and this property is invariant under any permutation of 1, 3, 4 and 2, 5. It is not 12 because we do not care the order of blocks of size 2.
- Case 2: $\ell(\pi) = 4$. Similarly, $\pi^{(2)} = 12/3/4/5 \in C(\sigma)$ and this is invariant, again, under any permutation of 1, 3, 4 and 2, 5.
- Case 3: $\ell(\pi) = 5$. $\pi^{(3)} = 1/2/3/4/5$ is the only choice.

Altogether, we have

$$\begin{aligned}
e_{134/25} &= \sum_{\pi \in C(134/25)} m_\pi = \frac{1}{2!} \sum_{\substack{\{a_1, a_2, a_3\}=\{1,3,4\} \\ \{b_1, b_2\}=\{2,5\}}} m_{a_1 b_1 / a_2 b_2 / a_3} \\
&\quad + \frac{1}{2!} \sum_{\substack{\{a_1, a_2, a_3\}=\{1,3,4\} \\ \{b_1, b_2\}=\{2,5\}}} m_{a_1 b_1 / a_2 / b_2 / a_3} + m_{1/2/3/4/5} \\
&= \frac{1}{2!} (3!2!(2!m_{221})) + \frac{1}{2!} (3!2!(3!m_{2111})) + 5!m_{1^5}.
\end{aligned}$$

Overall, it coincides with

$$3!2!e_{32} = 12(m_{221} + 3m_{2111} + 10m_{1^5})$$

as in Table 2.

Remark 2.16. In this context, palettes π and σ play dual roles. Let us call π an *m-palette* and σ an *e-palette*.

3. PALETTE, MOSAIC OF A COLORING

3.1. Palette of a coloring.

Definition 3.1. For $\kappa \in K(G)$, define its *palette* $\text{pal}(\kappa) = \pi(\kappa) \in \Pi(n)$ by

$$\kappa^{-1}(c_1)/\cdots/\kappa^{-1}(c_N)$$

where $c_1, \dots, c_N$ are all distinct positive integers such that

$$\kappa([n]) = \{c_1, \dots, c_N\}.$$

The *degree* of a color c at κ is $\deg_\kappa c = |\kappa^{-1}(c)|$. Colors c_k, c_l are *similar* at κ if $\deg_\kappa c_k = \deg_\kappa c_l$. Also, say vertices i, j are *sim-colored* at κ if $\deg_\kappa \kappa(i) = \deg_\kappa \kappa(j)$.

Definition 3.2. Define the *palette set* of G as

$$\Pi(G) = \{\pi(\kappa) \in \Pi(n) \mid \kappa \in K(G)\}.$$

Further, denote by $\overline{\Pi}(G)$ the set of all maximal elements of $\Pi(G)$.

Say $\kappa, \rho \in K(G)$ are *palette equivalent* ($\kappa \sim_{\text{pal}} \rho$) if $\text{pal}(\kappa) = \text{pal}(\rho)$. For $\pi \in \Pi(G)$, let $K_\pi(G) = \{\kappa \in K(G) \mid \text{pal}(\kappa) = \pi\}$.

Lemma 3.3.

$$\sum_{\kappa \in K_\pi(G)} x_\kappa = \begin{cases} m_\pi & \text{if } \pi \in \Pi(G), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First, suppose $\pi \in \Pi(G)$, say $\pi = \text{pal}(\rho)$, $\rho \in K(G)$. Any coloring $\kappa = (\kappa(1), \dots, \kappa(n))$ on G satisfying $i =_\pi j \iff \kappa(i) = \kappa(j)$ has the palette $\pi = \text{pal}(\rho)$ and hence is proper. Thus,

$$\sum_{\kappa \in K_\pi(G)} x_\kappa = \sum_{i=_\pi j \iff \kappa(i)=\kappa(j)} x_\kappa = m_\pi.$$

If $\pi \notin \Pi(G)$, then $K_\pi(G) = \emptyset$ and the sum is 0. $\square$

We have discussed m -positivity above. Here let us see some variants. Say $F(x) \in \Lambda_n$ is *palette- m -positive* if

$$F(x) = \sum_{\pi \in \Pi(n)} a_\pi m_\pi, \quad a_\pi \in \mathbf{Z}_{\geq 0}.$$

By Lemma 3.3, X_G must be palette- m -positive; indeed, we can say more. It is *dominant-palette- m -positive* if

$$F(x) = \sum_{\pi \in I} a_\pi m_\pi, \quad a_\pi \in \mathbf{Z}_{>0}$$

for some order ideal $I \subseteq \Pi(n)$.

Lemma 3.4 (dominance property of proper colorings). $\Pi(G)$ is an order ideal.

Proof. Suppose $\pi \leq \omega$, $\omega \in \Pi(G)$, say $\omega = \text{pal}(\rho)$ for $\rho \in K(G)$. It suffices to show that $\pi = \text{pal}(\kappa)$ for some $\kappa \in K(G)$. Write

$$\omega = \rho^{-1}(c_1) / \dots / \rho^{-1}(c_N)$$

with all distinct colors (c_k) . Since $\pi \leq \omega$, there are distinct blocks of π such that

$$\rho^{-1}(c_1) = \pi_1 \sqcup \dots \sqcup \pi_{k_1},$$

$$\dots,$$

$$\rho^{-1}(c_N) = \pi_{k_1+\dots+k_{N-1}+1} \sqcup \dots \sqcup \pi_{k_1+\dots+k_N}.$$

Now choose all distinct colors $b_1, \dots, b_{k_1+\dots+k_N}$. Define $\kappa(i) = b_l$ if $i \in \pi_l$. Notice that this κ is proper. Indeed, if $i \sim j$ in G , then i, j must belong to distinct blocks of ρ and hence of κ , i.e., $\kappa(i) \neq \kappa(j)$. $\square$

Theorem 3.5. X_G is dominant-palette- m -positive.

Proof. It just follows from Lemma 3.4 that

$$X_G = \sum_{\pi \in \Pi(G)} \sum_{\kappa \in K_\pi(G)} x_\kappa = \sum_{\pi \in \Pi(G)} m_\pi$$

is dominant-palette- m -positive (and hence m -positive). $\square$

Call $\Pi(G)$ the *palette ideal* of G .

Example 3.6. Let $G = 1 - 2 - 3 - 4$. We have

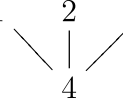
$$\Pi(G) = \{13/24, 14/2/3, 13/2/4, 24/1/3, 1/2/3/4\}.$$

Hence

$$X_G = m_{13/24} + m_{14/2/3} + m_{13/2/4} + m_{24/1/3} + m_{1/2/3/4}.$$

In terms of monomial symmetric functions, this is $2m_{22} + 6m_{211} + 24m_{1111}$.

Example 3.7. Take G to be the claw $K_{31} =$



$$\Pi(G) = \{123/4, 12/3/4, 13/2/4, 23/1/4, 1/2/3/4\}$$

so that

$$\begin{aligned} X_G &= \underbrace{m_{123/4}}_{m_{31}} + \underbrace{m_{12/3/4} + m_{13/2/4} + m_{23/1/4}}_{3 \times 2!m_{211}} + \underbrace{m_{1/2/3/4}}_{4!m_{1111}} \\ &= m_{31} + 6m_{211} + 24m_{1111} \end{aligned}$$

missing the m_{22} term. With the relations

$$e_4 = m_{1111}, \quad e_{31} = m_{211} + 4m_{1111},$$

$$e_{22} = m_{22} + 2m_{211} + 6m_{1111}, \quad e_{211} = m_{31} + 2m_{22} + 5m_{211} + 12m_{1111},$$

we can check that

$$X_G(x) = 4e_4 + 5e_{31} - 2e_{22} + e_{211}$$

which is *not* e -positive.

Remark 3.8. We could say $F(x)$ is *dominant- m -positive* if $F(x) = \sum_{\lambda \in I} a_\lambda m_\lambda$, $a_\lambda \in \mathbf{Z}_{>0}$ for some order ideal I of $(\text{Par}(n), \leq_{\text{dom}})$. We just saw that $X_{1-2-3-4}$ is dominant- m -positive while $X_{K_{31}}$ is not.

3.2. Mosaic.

Definition 3.9. A sequence $\mu = (\mu_1, \dots, \mu_N)$ is a *mosaic* of $[n]$ if it satisfies all of the following.

- (1) Each μ_k is a sequence of nonempty subsets of $[n]$.
- (2) If $A, B \in \mu_k$ (abuse of notation to mean A, B appear in the sequence μ_k), then $|A| = |B|$.
- (3) If $A \in \mu_k, B \in \mu_l, k < l$, then $|A| \geq |B|$.
- (4) If $A \in \mu_k, B \in \mu_l, A \neq B$, then $A \cap B = \emptyset$.
- (5) $\sqcup_k \sqcup_{A \in \mu_k} A = [n]$.

Let $M(n)$ denote the set of all such mosaics. Each μ_k is a *component* of μ while each $A \in \mu_k$ is a (color) *block*. Any component of length 1 can be identified as a block itself. As before, $i =_\mu j$ means i, j belong to the same block of μ .

Example 3.10.

$$\mu = ((\{2, 3, 5\}), (\{4, 6\}, \{1, 7\}), (\{9\}, \{8\})) \in M(9).$$

The natural shorthand notation is $\mu = (235)(46|17)(9|8)$. Whenever there is no confusion, we even suppress some ().

Definition 3.11. The *degree* of μ_k is $\deg \mu_k = |A|$ for all (any) $A \in \mu_k$. Say a component μ_k is a *stripe* if $\deg \mu_k \geq 2$.

For example, $\mu = (26|18|45)(7|3)$ consists of two stripes of degree 2 and 1, respectively.

For blocks A, B of μ , declare $A \equiv_\mu B$ if A, B are in the same stripe of μ . Similarly, declare $i \equiv_\mu j$ if $i \in A, j \in B$ for some blocks $A \equiv_\mu B$.

Definition 3.12. Given μ , define the *inside order* (stripe order) $A <_\mu B$ on $2^{[n]}$ if $A \equiv_\mu B$ and A appears before B in a stripe. Similarly, $i <_\mu j$ on $[n]$ if $i \in A, j \in B$ for some blocks $A <_\mu B$. The *outside order* for blocks of μ is $A <^\mu B$ if $A <_\mu B$ or $A \in \mu_k, B \in \mu_l$ for some $k < l$. Similarly, the *outside order* for entries of μ is $i <^\mu j$ if $i <_\mu j$ or $i \in A, j \in B$ for some blocks $A <^\mu B$.

Remark 3.13. In particular, $i <_\mu j \implies i <^\mu j$.

Definition 3.14. The *mosaic function* for $\mu \in M(n)$ is

$$m_\mu = \sum x_{a_1} \cdots x_{a_n}$$

where a coloring $(a_1, \dots, a_n) \in \mathbb{N}^n$ satisfies:

- $a_i = a_j$ if and only if i, j are in the same block of μ . [palette condition]
- $i <_\mu j \implies a_i < a_j$. [a -color order respects μ -stripe order]

Observe that m_μ is a partial sum of $m_{\text{pal}(\mu)}$.

Example 3.15. We have

$$m_{(24|15|37)6} = \sum x_{a_1} \cdots x_{a_7}$$

where $a_2 = a_4 = c_1, a_1 = a_5 = c_2, a_3 = a_7 = c_3, a_6 = c_4$, c_k 's are all distinct and moreover $c_1 < c_2 < c_3$ (without any other restriction to the magnitude of c_4).

Example 3.16. Let us confirm $m_{(24|15|37)6} = m_{(15|24|37)6}$. Let a, b be colorings such that $a_2 = a_4 = b_1 = b_5 = c_1, a_1 = a_5 = b_2 = b_4 = c_2, a_3 = a_7 = b_3 = b_7 = c_3, a_6 = b_6 = c_4$, c_k 's are all distinct and $c_1 < c_2 < c_3$. By definition,

$$m_{(24|15|37)6} = \sum_a x_{a_1} \cdots x_{a_7}, \quad m_{(15|24|37)6} = \sum_b x_{b_1} \cdots x_{b_7}.$$

Each of corresponding monomials in the sums coincides as $x_{a_1} \cdots x_{a_7} = x_{b_1} \cdots x_{b_7}$. Therefore, $m_{(24|15|37)6} = m_{(15|24|37)6}$.

There is a natural projection $\text{pal} : M(n) \rightarrow \Pi(n)$, $\mu \mapsto \text{pal}(\mu)$.

Definition 3.17. For $\mu, \nu \in M(n)$, define $\mu \sim_{\text{pal}} \nu$ (*palette equivalent*) if $\text{pal}(\mu) = \text{pal}(\nu)$.

Theorem 3.18. $\mu \sim_{\text{pal}} \nu \implies m_\mu = m_\nu$.

Proof. We may assume that ν is obtained from μ with flipping only two adjacent blocks in a stripe, say

$$\mu = (\mu_1, \dots, \mu_N),$$

$\nu_l = \mu_l$ ($l \neq k$) and

$$\mu_k = (A_1, \dots, A_p, A_{p+1}, \dots, A_q),$$

$$\nu_k = (A_1, \dots, A_{p+1}, A_p, \dots, A_q)$$

with $A_p \cap A_{p+1} = \emptyset$ and $|A_p| = |A_{p+1}|$. Now fix a bijection $f : A_p \rightarrow A_{p+1}$. For a coloring $a = (a_i)$ with $i =_\mu j \iff a_i = a_j$ and $i <_\mu j \implies a_i < a_j$, define $b = (b_i)$ by

$$b_i = \begin{cases} a_{f(i)} & \text{if } i \in A_p, \\ a_{f^{-1}(i)} & \text{if } i \in A_{p+1}, \\ a_i & \text{otherwise.} \end{cases}$$

Therefore,

$$m_\mu = \sum_a x_{a_1} \cdots x_{a_n} = \sum_b x_{b_1} \cdots x_{b_n} = m_\nu.$$

□

Observation 3.19. For $\pi \in \Pi(n)$, $|\text{pal}^{-1}(\pi)| = \pi!_m$.

Theorem 3.20. $m_\mu = m_{\text{sh}(\mu)}$. In particular, m_μ is symmetric.

Proof. Suppose $\mu \in \text{pal}^{-1}(\pi)$. Then we have

$$(\text{sh } \pi)!_m m_{\text{sh}(\pi)} = m_\pi = \sum_{\nu: \nu \sim_{\text{pal}} \mu} m_\nu = \pi!_m m_\mu$$

with $(\text{sh } \pi)!_m = \pi!_m$. □

Remark 3.21. Suppose $\lambda = (\lambda_1, \dots, \lambda_l) = \text{sh}(\mu)$. The *leading term* of $m_\lambda (= m_\mu)$ is $x_1^{\lambda_1} \cdots x_l^{\lambda_l}$. On m_μ side, this comes from the particular coloring (a_i) with $a_i = k$ if $i \in \mu_k$.

3.3. Mosaic of a coloring.

Definition 3.22. Let $\kappa \in K(G)$. Say κ uses exactly N colors, $c_1, \dots, c_N$. We are going to define its *mosaic* $\text{mos}(\kappa) = \mu(\kappa) \in M(n)$. Up to size, sort and assemble $\kappa^{-1}(c_1), \dots, \kappa^{-1}(c_N)$ to make up components of $\text{mos}(\kappa)$. Moreover, arrange stripe orders by

$$A <_{\mu(\kappa)} B \iff \text{for some } i, j \text{ such that } i \in A, j \in B, \kappa(i) < \kappa(j).$$

Example 3.23. Consider this coloring

$$\kappa : \quad 1_{c_1} \text{---} \overbrace{2_{c_4} \text{---} 3_{c_2}} \text{---} 4_{c_4} \text{---} 5_{c_2} \text{---} \overbrace{6_{c_1} \text{---} 7_{c_4}} \text{---} 8_{c_3} \text{---} 9_{c_2}$$

where $c_1 < c_2 < c_3 < c_4$. Thus,

$$\mu(\kappa) = (\underbrace{359}_{c_2} | \underbrace{247}_{c_4}) \underbrace{16}_{c_1} / \underbrace{8}_{c_3}.$$

The set of mosaics on G is

$$M(G) = \{\mu(\kappa) \in M(n) \mid \kappa \in K(G)\}.$$

Since we have $c_1|c_2|\cdots|c_n \in M(G)$ for any permutation $c_1, \dots, c_n$ of $1, \dots, n$, this is always nonempty.

Example 3.24. Let $G = 1 - 2 - 3 - 4$. We have

$$M(G) = \{(13|24), (24|13), 13(2|4), 13(4|2), 24(1|3), 24(3|1), \\ 14(2|3), 14(3|2), (1|2|3|4), \dots, (4|3|2|1)\}.$$

Theorem 3.25. If $\mu \in M(G)$, $\nu \in M(n)$ and $\nu \sim_{\text{pal}} \mu$, then $\nu \in M(G)$.

Proof. Let $\mu \in M(G)$ and we may assume that ν is obtained from μ with flipping only two adjacent blocks in a stripe, say

$$\mu = (\mu_1, \dots, \mu_N),$$

$\nu_l = \mu_l$ ($l \neq k$) and

$$\begin{aligned} \mu_k &= (A_1, \dots, A_p, A_{p+1}, \dots, A_q), \\ \nu_k &= (A_1, \dots, A_{p+1}, A_p, \dots, A_q) \end{aligned}$$

with $A_p \cap A_{p+1} = \emptyset$ and $|A_p| = |A_{p+1}|$. Fix a bijection $f : A_p \rightarrow A_{p+1}$. Now, say $\mu = \text{mos}(\kappa)$ for some $\kappa \in K(G)$. Define $\rho : [n] \rightarrow \mathbf{N}$ by

$$\rho(i) = \begin{cases} \kappa(f(i)) & i \in A_p, \\ \kappa(f^{-1}(i)) & i \in A_{p+1}, \\ \kappa(i) & \text{otherwise.} \end{cases}$$

By construction, ρ is a proper coloring on G and $\nu = \text{mos}(\rho)$. □

Say κ, ρ are *mosaic equivalent* ($\kappa \sim_{\text{mos}} \rho$) if $\text{mos}(\kappa) = \text{mos}(\rho)$. Let

$$K^\mu(G) = \{\kappa \in K(G) \mid \text{mos}(\kappa) = \mu\}.$$

As in the palette case, we see that

$$\sum_{\kappa \in K^\mu(G)} x_\kappa = \begin{cases} m_\mu & \text{if } \mu \in M(G), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $\mu \sim_{\text{pal}} \nu$, then $K^\mu(G) \neq \emptyset \iff K^\nu(G) \neq \emptyset$.

3.4. Future work. This is not the end of story. We are planning to develop our theory at another opportunity as follows.

- Find a natural graded poset (lattice) structure on $M(n)$ as $(\Pi(n), \leq)$.
- Introduce somehow a new object “ensemble” ε as an analogy of e -palette; the theory of P -partition may work. Discuss then compatibility of a mosaic and ensemble.
- Define the ensemble function e_ε as a sum of appropriate $m'_\mu s$.
- From a compatible pair (μ, ε) , construct a dual Knuth array. This is the key idea to prove that $e_\varepsilon = e_{\text{sh}(\varepsilon)}$.

Acknowledgment.

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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