

# AN EXTENSION OF THEOREMS ON QUASI-SYMMETRIC FUNCTIONS BY GESSEL AND ASSAF-SEARLES

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ABSTRACT. We formulate a  $q$ -analog of theorems on the expansions of a Schur polynomial into quasi-symmetric polynomials by Gessel (1984) and Assaf-Searles (2017). For this purpose, we introduce the statistic with its name *depth*. It allows us to better understand a  $q$ -analog of major index and Kostka numbers together.

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## 1. INTRODUCTION

The aim of this note is to formulate a  $q$ -analog of theorems on the expansions of a Schur polynomial into quasi-symmetric polynomials by Gessel [5] (1984) and Assaf-Searles [1] (2017). For this purpose, we will introduce the statistic with its name *depth*. It allows us to better understand a  $q$ -analog of major index and Kostka numbers together.

Gessel found the following expansion of a Schur polynomial

$$s_{\lambda}(x_1, \dots, x_n) = \sum_{T \in \text{ST}(\lambda)} F_{\text{des } T}(x_1, \dots, x_n)$$

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where  $\text{ST}(\lambda)$  denotes the set of all standard Young tableaux (SYT) of a partition shape  $\lambda$ ,  $\text{des } T$  is the *descent composition* of  $T$  (not *descent set*) and  $F_{\text{des } T}(x_1, \dots, x_n)$  the fundamental quasi-symmetric polynomial as explained more below. As an improvement of this, Assaf-Searles [1, Theorem 2.7] proved

$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in \text{QY}_n(\lambda)} F_{\text{wt } T}(x_1, \dots, x_n)$$

where  $\text{QY}_n(\lambda)$  denotes the set of all quasi-Yamanouchi tableaux of shape  $\lambda$  with entries at most  $n$  and  $\text{wt } T$  is the weight vector of  $T$ . They further pointed out that each  $F_{\text{wt } T}(x_1, \dots, x_n)$  is nonzero.

Our main theorem claims that there exists the *natural*  $q$ -analog of these expansions. Let  $X = (x_1, x_2, \dots)$  be a set of infinite many variables.

**Theorem 1.1.** For a partition  $\lambda$  and any  $n \in \mathbf{N}$ , there exists some statistic which we call *depth*,  $\text{dep} : \text{ST}(\lambda) \cup \text{QY}_n(\lambda) \rightarrow \mathbf{Z}_{\geq 0}$  such that if we define a  $q$ -analog of  $s_\lambda(X)$  as

$$s_\lambda(X, q) = \sum_{T \in \text{SSYT}(\lambda)} q^{\text{dep } T} x^T,$$

then

$$s_\lambda(X, q) = \sum_{T \in \text{ST}(\lambda)} q^{\text{dep } T} F_{\text{des } T}(X)$$

and

$$s_\lambda(x_1, \dots, x_n, q) = \sum_{T \in \text{QY}_n(\lambda)} q^{\text{dep } T} F_{\text{wt } T}(x_1, \dots, x_n).$$

By natural, we mean  $\text{dep}$  is equi-distributed with major index over  $\text{ST}(\lambda)$  as it appears naturally as the graded character of Specht modules.

To give a proof, we will recall basic ideas in the next section. We assume familiarity on Young tableaux as in Fulton [4]. Below,  $\lambda$  is always a partition and  $\ell(\lambda)$  means the number of rows of  $\lambda$ . Identify  $\lambda$  with a Young diagram in English style. The symbol  $\text{SSYT}(\lambda)$  means the set of all semistandard Young tableaux of shape  $\lambda$ .

## 2. PRELIMINARIES

A *weak (strong) composition* is a finite sequence of nonnegative (positive) integers as we often write  $a = (a_1, \dots, a_m)$ . For simplicity, we sometimes abbreviate  $a_1 \dots a_m$  to mean  $(a_1, \dots, a_m)$ . The *length* of  $a$  is  $m = \ell(a)$ . The *size* of  $a$  is

$$|a| = a_1 + \dots + a_m.$$

By  $\text{comp}$  ( $\text{Comp}$ ) we mean the set of all weak (strong) compositions. Further,  $a \models n$  means  $a \in \text{comp}$  and  $|a| = n$ . Say  $a$  dominates  $b$  (and write  $a \geq b$ ) if

$$a_1 + \dots + a_i \geq b_1 + \dots + b_i$$

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for all  $i$ . This is the *dominance order* (a partial order) for compositions.

Let  $a, b \in \text{Comp}$  and  $c \in \text{comp}$ . Define  $b = \text{flat}(c)$  if we obtain  $b$  with deleting all 0 parts of  $c$ . Say  $b$  *refines*  $a$  if we can obtain  $b$  from  $a$  with repeating to split one number into two adjacent positive entries. Denote by  $\text{ref}(a)$  the set of all strong compositions which refine  $a$ . For example, if  $a = 23$ , then

$$\text{ref}(a) = \{23, 221, 212, 2111, 113, 1121, 1112, 11111\}.$$

Let  $X = (x_1, x_2, \dots)$  be infinitely many variables. To any composition  $a$ , associate the weight monomial  $x^a = x_1^{a_1} \cdots x_{\ell(a)}^{a_{\ell(a)}}$ .

**Definition 2.1.** Let  $a, b \in \text{Comp}$ . The *monomial quasi-symmetric function* for  $b$  is

$$M_b(X) = \sum_{\substack{c \in \text{comp} \\ \text{flat}(c)=b}} x^c.$$

The *fundamental quasi-symmetric function* for  $a$  is

$$F_a(X) = \sum_{b \in \text{ref}(a)} M_b(X).$$

With  $x_{n+1} = x_{n+2} = \cdots = 0$ , define corresponding quasi-symmetric polynomials  $M_b(x_1, \dots, x_n)$  and  $F_a(x_1, \dots, x_n)$ .

To describe connections with such functions and semistandard Young tableaux (SSYT), we need another definition; a skew shape is a *horizontal strip* if it contains at most one box in each column.

**Remark 2.2.** In this article, a *box* always means *only a position* in a diagram.

Let  $B_1, B_2$  be boxes in a diagram. Write  $B_1 <_{nE} B_2$  if  $B_2$  lies northEast to  $B_1$ , that is, if  $B_1 = (i_1, j_1)$  and  $B_2 = (i_2, j_2)$ , then  $i_1 \geq i_2$  and  $j_1 < j_2$ .

Suppose now  $B = (B_1, \dots, B_k)$  is a sequence of boxes and moreover there is an associated composition  $a = (a_1, \dots, a_k)$  such that  $a_j$  is at  $B_j$ .

**Definition 2.3.** Say  $H = (B, a)$  is a *horizontal band* if

- $k \geq 1$ ,
- $B_1 <_{nE} \cdots <_{nE} B_k$ ,
- $a_j \leq a_{j+1}$  for all  $j$ .

For convenience, let  $|H| = k$ .

For an SSYT  $T$ , there exists a unique sequence of maximal horizontal bands  $H(T) = (H_1, \dots, H_l)$  such that they contain each entry of  $T$  exactly once and if all entries in  $H_{i+1}$  are strictly greater than those in  $H_i$ ; maximal means adding a box with its entry in  $T$  to  $H_i$  cannot be any other horizontal band in  $T$ . We call such a sequence the *minimal parsing* of  $T$ . This term is due to Maas-Gariépy [8]. It does not seem that there is any traditional name for this sequence.

**Definition 2.4.** The *descent composition* of an SSYT  $T$  is the strong composition

$$\text{des } T = (|H_1|, |H_2|, \dots, |H_l|)$$

where  $H(T) = (H_1, \dots, H_l)$ .

For  $\alpha \models n$ , its *depth* is  $\text{dep}(\alpha) = \sum_{i=1}^{\ell(\alpha)} (i-1)\alpha_i$ . This is a nonnegative integer at most  $\binom{n}{2}$ . In particular,  $\text{dep}(\lambda)$  makes sense for a partition  $\lambda \vdash n$ .

**Example 2.5.**

$$\text{dep}(352) = 0 \times 3 + 1 \times 5 + 2 \times 2 = 9.$$

$$\text{dep}(431) = 0 \times 4 + 1 \times 3 + 2 \times 1 = 5.$$

**Definition 2.6.** For an SSYT  $T$ , define its *depth* as  $\text{dep}(T) = \text{dep}(\text{des } T)$ .

For example,

$$\text{dep} \left( \begin{array}{|c|c|c|c|} \hline 1 & 1 & 4 & 4 \\ \hline 2 & 4 & & \\ \hline 3 & & & \\ \hline \end{array} \right) = \text{dep}(214) = 9.$$

Given an SSYT  $T$  with its descent composition  $\alpha = (\alpha_1, \dots, \alpha_l)$ , its minimal parsing allows us to define the *standardization*  $T^{\text{st}}$ : all entries in the first maximal horizontal band are relabeled by  $1, \dots, \alpha_1$  (from left), in the second, by  $\alpha_1 + 1, \dots, \alpha_1 + \alpha_2$  and so on. The *destandardization* of  $T$  ( $T^{\text{dst}}$ ) is the tableau obtained from  $T$  with replacing all numbers in  $H_i(T)$  by  $i$ .

**Example 2.7.** Let  $T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 5 \\ \hline 3 & 8 & & \\ \hline 8 & & & \\ \hline \end{array}$ . Its minimal parsing is

$$\left( \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & 5 \\ \hline 3 & \square & & \\ \hline \square & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & 8 & & \\ \hline 8 & & & \\ \hline \end{array} \right)$$

and  $\text{des } T = (3, 2, 2)$ ,  $T^{\text{st}} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 7 & & \\ \hline 6 & & & \\ \hline \end{array}$ ,  $T^{\text{dst}} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & & \\ \hline 3 & & & \\ \hline \end{array}$  and  $\text{dep } T = 0+2+4 = 6 = \text{dep } T^{\text{st}} = \text{dep } T^{\text{dst}}$ .

**Definition 2.8.** Say  $Q \in \text{SSYT}(\lambda)$  is *quasi-Yamanouchi* if  $\text{des } Q = \text{wt } Q$ . Denote the set of all such tableaux (with entries at most  $n$ ) by  $\text{QY}(\lambda)$  ( $\text{QY}_n(\lambda)$ ).

**Example 2.9.**  $\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 & \\ \hline 4 & & & \\ \hline \end{array}$  is quasi-Yamanouchi with both of its descent composition and weight 3221.

There is the canonical identification  $\text{QY}(\lambda) \cong \text{ST}(\lambda)$ : standardization  $T \mapsto T^{\text{st}}$ , and destandardization  $T \mapsto T^{\text{dst}}$  are des-preserving bijections and the inverse map of each other.

For an SSYT  $T$ , let  $\text{wt } T = (\mu_1, \dots, \mu_m)$  where  $\mu_i$  is the number of  $i$  in  $T$ ;  $x^T = x^{\text{wt } T}$  is the weight monomial for  $T$ . Notice that  $\text{sh } T$  is a partition,  $\text{des } T$  is a strong composition and  $\text{wt } T$  is a weak composition of all the same size. Throughout this article, the following perspective is crucial:

**Lemma 2.10** (Fundamental lemma of tableaux). If  $T$  is an SSYT, then

$$\text{sh } T \geq \text{des } T \geq_{\text{ref}} \text{wt } T.$$

*Proof.* Let  $\lambda = (\lambda_1, \dots, \lambda_j) = \text{sh } T$ ,  $\alpha = (\alpha_1, \dots, \alpha_k) = \text{des } T$ ,  $\mu = (\mu_1, \dots, \mu_l) = \text{wt } T$  and  $j \leq k \leq l$ . Also, let  $T_i$  denote the  $i$ -th row of  $T$  and  $H_i$  the  $i$ -th maximal horizontal band of  $T$  (and so  $\alpha_i = |H_i|$ ). By construction of  $H(T)$ , all entries in  $H_1, \dots, H_i$  stay in  $T_1 \cup \dots \cup T_i$ . This means  $\lambda_1 + \dots + \lambda_i \geq \alpha_1 + \dots + \alpha_i$  and hence  $\lambda \geq \alpha$ . This proves the first claim. Next, consider all entries of  $H_i$ . They constitute to a part of  $\text{wt } T$ . Note that  $H_i$  and  $H_{i'}$  ( $i < i'$ ) never share any common number and all entries in  $H_i$  are strictly less than ones in  $H_{i'}$ . It follows  $\alpha \geq_{\text{ref}} \mu$ .  $\square$

### 3. PROOF OF THE MAIN THEOREM

**Definition 3.1.** Say  $T, U \in \text{SSYT}(\lambda)$  are *standard equivalent* ( $T \sim U$ ) if  $T^{\text{st}} = U^{\text{st}}$ . A *quasi-crystal*  $\mathbf{Q}$  is an equivalent class of this equivalent relation on  $\text{SSYT}(\lambda)$ .

Notice that  $T^{\text{st}} = U^{\text{st}} \iff T^{\text{dst}} = U^{\text{dst}}$ .

**Remark 3.2.** Maas-Gariépy [8] introduced a quasi-crystal to start a theory of a *crystal skeleton* (a quotient set  $\text{SSYT}(\lambda)/\sim$  with certain graph structure) in 2023. Afterward, Cain–Malheiro–Rodrigues–Rodrigues (2025) [3] and Brauner–Corteel–Daugherty–Schilling (2025) [2] follow her project.

By definition of  $\text{des}$  and  $\text{dep}$ , they are invariant on a quasi-crystal  $\mathbf{Q}$ . Thus, it makes sense to talk about  $\text{des}(\mathbf{Q})$  and  $\text{dep}(\mathbf{Q})$  as  $\text{des}(T)$  and  $\text{dep}(T)$  for a (any)  $T \in \mathbf{Q}$ .

**Lemma 3.3** ([8]). Let  $\mathbf{Q}$  be a quasi-crystal in  $\text{SSYT}(\lambda)$  with  $\text{des}(\mathbf{Q}) = \alpha$ . Then,

$$\sum_{T \in \mathbf{Q}} x^T = F_\alpha(X).$$

As a consequence, we have

$$\sum_{T \in \mathbf{Q}} q^{\text{dep } T} x^T = q^{\text{dep } \alpha} F_\alpha(X).$$

It is now natural to think of *deep Schur function*

$$s_\lambda(X, q) = \sum_{T \in \text{SSYT}(\lambda)} q^{\text{dep}(T)} x^T.$$

**Theorem 3.4.** For a partition  $\lambda$  and any  $n \in \mathbf{N}$ , we have

$$s_\lambda(X, q) = \sum_{T \in \text{ST}(\lambda)} q^{\text{dep } T} F_{\text{des } T}(X)$$

with  $q = 1$  and  $x_{n+1} = x_{n+2} = \cdots = 0$  recovering Gessel's expansion. Also, we have

$$s_\lambda(x_1, \dots, x_n, q) = \sum_{T \in \text{QY}_n(\lambda)} q^{\text{dep } T} F_{\text{wt } T}(x_1, \dots, x_n) \quad (q^{\text{dep } T} F_{\text{wt } T}(x_1, \dots, x_n) \neq 0)$$

with  $q = 1$  recovering Assaf-Searles' expansion.

*Proof.* Think  $s_\lambda(X)$  as a summation over all quasi-crystals in  $\text{SSYT}(\lambda)$ . Notice that each quasi-crystal  $\mathbf{Q}$  contains exactly one SYT of shape  $\lambda$ , say  $T$ , so that the total sum of  $q$ -weight on  $\mathbf{Q}$  is  $q^{\text{dep } T} F_{\text{des } T}(X)$  as mentioned just above.

To prove the second part, let us consider  $T \in \text{QY}_n(\lambda)$ , say  $\alpha = \text{des } T$ . Then  $\text{wt } T = \text{des } T = \alpha$  and  $\ell(\alpha) \leq n$  because  $T$  has entries at most  $n$ . Thus,

$$F_{\text{wt } T}(x_1, \dots, x_n) = F_\alpha(x_1, \dots, x_n) = x_1^{\alpha_1} \cdots x_{\ell(\alpha)}^{\alpha_{\ell(\alpha)}} + \cdots \neq 0.$$

□

**Example 3.5.** Observe that

$$s_{32}(X, q) = q^2 F_{32}(X) + q^3 F_{23}(X) + q^4 F_{221}(X) + q^5 F_{131}(X) + q^6 F_{122}(X),$$

$$s_{32}(x_1, x_2, q) = q^2 F_{32}(x_1, x_2) + q^3 F_{23}(x_1, x_2)$$

(see Example 4.3). Also, we have

$$s_{22}(X, q) = q^2 F_{22}(X) + q^4 F_{121}(X) \quad (\text{missing } q^3 \text{ term}),$$

$$s_{22}(x_1, x_2, q) = q^2 F_{22}(x_1, x_2)$$

as SYTs of shape 22 are only  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}.$

We just gave a proof of our main theorem. However, this is not the end of story. In the sequel, we discuss some more details on depth with connections to charge, Kostka numbers and fake degree polynomials as they all play a significant role in the theory on representations of  $S_n$  and  $\text{GL}(n)$ .

#### 4. CHARGE, DEPTH, FAKE DEGREE, KOSTKA NUMBER, MAJOR INDEX

We recall definitions of the fake degree and Kostka numbers. Let  $\lambda, \mu \vdash n$ .

**Definition 4.1.** Let  $\text{ST}(\lambda)$ ,  $\text{SSYT}(\lambda, \mu)$  denote the set of standard Young tableaux of shape  $\lambda$  and the set of semistandard Young tableaux of shape  $\lambda$  with a partition weight  $\mu$ , respectively. The *fake degree* for  $\lambda$  and *Kostka number* for  $(\lambda, \mu)$  are

$$f_\lambda = |\text{ST}(\lambda)|, \quad K_{\lambda\mu} = |\text{SSYT}(\lambda, \mu)|.$$

**Remark 4.2.** If  $\lambda \not\geq \mu$  in the dominance order, then  $K_{\lambda\mu} = 0$ . Thus, whenever we discuss  $K_{\lambda\mu}$ , we may assume  $\lambda \geq \mu$ .

**Example 4.3.** As the following tableaux indicate, we have  $f_{32} = 5$  and  $K_{431,3221} = 5$ .

|                                                                                                                                                                |          |          |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
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| <b>1</b>                                                                                                                                                       | 3        | <b>4</b> |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 5        |          |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | <b>2</b> | <b>4</b> |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 3                                                                                                                                                              | 5        |          |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| <table><tr><td>1</td><td>1</td><td>1</td><td>2</td></tr><tr><td>2</td><td>3</td><td>3</td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr></table> | 1        | 1        | 1 | 2 | 2 | 3 | 3                                                                                                   |   | 4 |          |   |   | <table><tr><td>1</td><td>1</td><td>1</td><td>2</td></tr><tr><td>2</td><td>3</td><td>4</td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr></table> | 1                                                                                                          | 1        | 1        | 2 | 2 | 3 | 4 |                                                                                                            | 3        |   |          |   | <table><tr><td>1</td><td>1</td><td>1</td><td>3</td></tr><tr><td>2</td><td>2</td><td>3</td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr></table> | 1 | 1                                                                                                          | 1 | 3        | 2        | 2 | 3 |  | 4 |  |  |  | <table><tr><td>1</td><td>1</td><td>1</td><td>3</td></tr><tr><td>2</td><td>2</td><td>4</td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 3 | 2 | 2 | 4 |  | 3 |  |  |  | <table><tr><td>1</td><td>1</td><td>1</td><td>4</td></tr><tr><td>2</td><td>2</td><td>3</td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 4 | 2 | 2 | 3 |  | 3 |  |  |  |
| 1                                                                                                                                                              | 1        | 1        | 2 |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 3        | 3        |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 4                                                                                                                                                              |          |          |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1        | 1        | 2 |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 3        | 4        |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 3                                                                                                                                                              |          |          |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1        | 1        | 3 |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 2        | 3        |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 4                                                                                                                                                              |          |          |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1        | 1        | 3 |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 2        | 4        |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 3                                                                                                                                                              |          |          |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1        | 1        | 4 |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 2        | 3        |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 3                                                                                                                                                              |          |          |   |   |   |   |                                                                                                     |   |   |          |   |   |                                                                                                                                                                |                                                                                                            |          |          |   |   |   |   |                                                                                                            |          |   |          |   |                                                                                                                                                                |   |                                                                                                            |   |          |          |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |

**Remark 4.4.** In particular,  $K_{\lambda(1^n)} = f_\lambda$ . Indeed,  $f_\lambda = \dim S^\lambda$  where  $S^\lambda$  is the *Specht module*, an irreducible representation of  $S_n$  for  $\lambda \vdash n$ . Also,  $\dim M^\mu = \sum_{\lambda \vdash n} K_{\lambda\mu}$  where  $M^\mu$  is the *permutation module* of  $S_n$  for  $\mu \vdash n$  with the multiplicity of  $S^\lambda$  in  $M^\mu$  being  $K_{\lambda\mu}$ . These facts come from the theory on tabloids and generalized tableaux; see Sagan [10, Chapter 2].

There are classical  $q$ -analogs of these numbers. To explain those, we need two particular statistics, major index and charge.

**Definition 4.5.** Let  $I \subseteq \{1, \dots, n-1\}$ . The major index of  $I$  is

$$\text{maj}(I) = \sum_{i \in I} i.$$

For  $T \in \text{ST}(\lambda)$ ,  $\lambda \vdash n$ , let

$$\text{Des } T = \{i \in \{1, \dots, n-1\} \mid i+1 \text{ is strictly lower than } i \text{ in } T\}$$

be the descent set of  $T$ . The *major index* of  $T$  is  $\text{maj}(T) = \text{maj}(\text{Des } T)$ . More generally, for semistandard  $T$ , define  $\text{Des}(T) = \text{Des}(T^{\text{st}})$  and  $\text{maj } T = \text{maj}(T^{\text{st}})$ .

**Example 4.6.** Observe that  $T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 7 & \\ \hline 8 & & & \\ \hline \end{array}$  with  $\text{Des } T = \{3, 5, 7\}$  and  $\text{maj } T = 3 + 5 + 7 = 15$ .

Following Macdonald [9, p.242] (and Lascoux-Schützenberger [7]), we define charge for a permutation (a standard word), a word and an SSYT.

- (1) Suppose first  $w$  is a permutation, say  $w \in S_n$ . We are going to define  $c(w)$  with inductive labelings  $c_i(w) \in \{0, \dots, i-1\}$ . For convenience, introduce the set of *left descents*

$$D_L(w) = \{i \in \{1, \dots, n-1\} \mid w^{-1}(i+1) < w^{-1}(i)\}$$

as in the context of Coxeter groups. First, set  $c_1(w) = 0$ . For  $i \geq 1$ , let  $c_{i+1}(w) = c_i(w)$  if  $i \notin D_L(w)$ . Otherwise,  $c_{i+1}(w) = c_i(w) + 1$ . Define  $c(w) = \sum_{i=1}^n c_i(w) \in \{0, \dots, \binom{n}{2}\}$ . This is the charge of a permutation  $w$ .

- (2) Now let  $w$  be a word, i.e., a finite sequence of positive integers. Assume that the weight of  $w$ , say  $\mu = (\mu_1, \dots, \mu_m)$ , is a partition. We extract a sequence of standard words  $(w_i)$  from  $w$  as follows. Reading from the *left*, choose the first 1 that occurs in  $w$ , then the first 2 to the right of the 1 chosen, and so on. If at any stage, there is no  $j + 1$  to the right of the  $j$  chosen, go back to the beginning. This procedure selects a standard subword  $w_1$  of  $w$ , length  $\mu'_1$  (the first part of the conjugate partition  $\mu'$  of  $\mu$ ). Now erase all letters in  $w_1$  from  $w$  and repeat the procedure to obtain another standard subword  $w_2$  of length  $\mu'_2$ , and so on. This algorithm ends after finite steps, of course. The charge of  $w$  is  $c(w) = \sum c(w_i)$ .
- (3) Finally, for an SSYT  $T$  in English style, let  $w(T)$  be the reading word from *right to left, top to bottom* (which is different from Fulton [4]). Define the charge of  $T$  to be  $c(T) = c(w(T))$ .

**Definition 4.7.** Let  $q$  be a variable. The *fake degree polynomial* for  $\lambda$  and *Kostka (Kostka-Foulkes) polynomial* for  $(\lambda, \mu)$  are

$$f_\lambda(q) = \sum_{T \in \text{ST}(\lambda)} q^{\text{maj}(T)}, \quad K_{\lambda\mu}(q) = \sum_{T \in \text{SSYT}(\lambda, \mu)} q^{c(T)}.$$

**Example 4.8.** As the first five tableaux in Example 4.3 show (bold letters indicate descents), we have

$$f_{32}(q) = q^2 + q^3 + q^4 + q^5 + q^6.$$

Note that possibly  $\text{maj}(T) \neq c(T)$  for some  $T \in \text{ST}(\lambda)$ . However, since  $f_\lambda$  is a special case of  $K_{\lambda\mu}$ , we prefer to discuss  $q$ -analogs  $f_\lambda$  and  $K_{\lambda\mu}$  in terms of some *common* statistic. Indeed, the depth plays this role. It is actually a nice grading in the following sense.

**Proposition 4.9** ([6]). Let  $\text{Comp}(n)$  denote the set of all strong compositions of size  $n$ . The poset  $(\text{Comp}(n), \leq, \text{dep})$  is graded with lowest depth 0 and highest  $\binom{n}{2}$ . In particular, if  $\alpha \geq \beta$  in this poset, then  $\text{dep}(\alpha) \leq \text{dep}(\beta)$ .

**Remark 4.10.** The subposet  $(\text{Par}(n), \leq, \text{dep})$  for  $n > 6$  is *not* graded.

As we see here

$$\text{maj} \left( \begin{array}{|c|c|c|c|} \hline 1 & 2 & 6 & 7 \\ \hline 3 & 5 & & \\ \hline 4 & & & \\ \hline \end{array} \right) = 2 + 3 = 5, \text{dep} \left( \begin{array}{|c|c|c|c|} \hline 1 & 2 & 6 & 7 \\ \hline 3 & 5 & & \\ \hline 4 & & & \\ \hline \end{array} \right) = \text{dep}(214) = 9,$$

it is possible that  $\text{dep}(T) \neq \text{maj}(T)$  for some  $T$ . Nonetheless, the following equidistribution holds (via what we call *evacuation*) as the author found [6].

**Lemma 4.11.**  $f_\lambda(q) = \sum_{T \in \text{ST}(\lambda)} q^{\text{maj}(T)} = \sum_{T \in \text{ST}(\lambda)} q^{\text{dep}(T)}$ .

Now it is natural to think of the  $q$ -analog of  $K_{\lambda\mu}$  with depth.

**Definition 4.12.** Let  $\lambda, \mu \vdash n$ . Define

$$\overline{K}_{\lambda\mu}(q) = \sum_{T \in \text{SSYT}(\lambda, \mu)} q^{\text{dep}(T)}.$$

Let us call this the *deep Kostka polynomial*.

**Remark 4.13.** This polynomial is different from  $K_{\lambda\mu}(q)$ . For  $\lambda \geq \mu$ ,  $\deg K_{\lambda\mu}(q) = \text{dep}(\mu) - \text{dep}(\lambda)$  [9, p.243] while  $\deg \overline{K}_{\lambda\mu}(q) = \text{dep}(\mu)$  as we show just below.

**Example 4.14.** Descent compositions of the last five tableaux in Example 4.3 are 3221, 323, 341, 3221, 323 hence with depth 9, 8, 6, 9, 8, respectively. Thus,

$$\overline{K}_{431,3221}(q) = q^6 + 2q^8 + 2q^9.$$

**Proposition 4.15.** Suppose  $\lambda, \mu \vdash n$  and  $\lambda \geq \mu$ . If  $T \in \text{SSYT}(\lambda, \mu)$ , then  $\text{dep } \lambda \leq \text{dep } T \leq \text{dep } \mu$ . Moreover, there exists some  $T_0 \in \text{SSYT}(\lambda, \mu)$  such that  $\text{dep } T_0 = \text{dep } \mu$  and hence  $\deg \overline{K}_{\lambda\mu}(q) = \text{dep } \mu$ .

*Proof.* The first assertion follows from Lemma 2.10 and Proposition 4.9. Now let us construct  $T_0$  with its minimal parsing  $(H_1, \dots, H_m)$  ( $\mu = (\mu_1, \dots, \mu_m)$ ). First, set  $H_1 = \underbrace{1 \cdots 1}_{\mu_1}$ . For  $i \geq 2$ , assume that  $H_1, \dots, H_{i-1}$  are constructed. Define  $H_i$

to be a maximal horizontal band such that all of its entries are  $i$ ;  $|H_i| = \mu_i$ ; about positions of its entries, seek all possible locations (from top-most rows, from left) on just outside of  $H_1 \cup \dots \cup H_{i-1}$  and place those  $i$ 's. Indeed,  $H_i$  stays in the first  $i$  rows of  $\lambda$  (this is possible because  $\lambda \geq \mu$ ). The example with  $\lambda = 431, \mu = 3221$ ,

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & & & \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 & \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 & \\ 4 & & & \end{array}$$

makes our argument apparent. By construction,  $\text{wt } T_0 = \text{des } T_0 = \mu$  and so  $\text{dep } T_0 = \text{dep } \mu$ .  $\square$

We have now two polynomials  $f_\lambda(q)$  and  $\overline{K}_{\lambda\mu}(q)$  with the common statistic depth and  $q = 1$  recovers  $f_\lambda, K_{\lambda\mu}$ . This is a progress. However, one sum is about  $\text{ST}(\lambda)$  and the other is about  $\text{SSYT}(\lambda, \mu)$ . It is better to discuss everything within the same class of tableaux. This is also possible as we justify next.

Let  $\text{ST}(n)$  be the set of all SYTs of size  $n$ . For  $\lambda, \mu \vdash n$ , define

$$\text{ST}(\lambda, \mu) = \{T \in \text{ST}(n) \mid \lambda = \text{sh } T \geq \text{des } T \geq_{\text{ref}} \mu\}.$$

**Theorem 4.16.** The depth satisfies the following.

$$f_\lambda(q) = \sum_{T \in \text{ST}(\lambda)} q^{\text{dep}(T)}, \quad \overline{K}_{\lambda\mu}(q) = \sum_{T \in \text{ST}(\lambda, \mu)} q^{\text{dep}(T)}.$$

For a proof, we need more subtle destandardization.

**Definition 4.17.** Let  $T \in \text{ST}(\lambda, \mu)$  and say

$$\alpha = \text{des } T = (\alpha_1, \dots, \alpha_l), \quad \mu = (\mu_1, \dots, \mu_m)$$

and there exists some  $1 = i_1 < i_2 < \cdots < i_l \leq m$  such that

$$\begin{aligned}\alpha_j &= \mu_{i_j} + \cdots + \mu_{i_{j+1}-1}, \quad (1 \leq j \leq l-1) \\ \alpha_l &= \mu_{i_l} + \cdots + \mu_m,\end{aligned}$$

so that  $\mu$  refines  $\alpha$ . The  $\mu$ -destandardization of  $T$  is  $T^{\mu\text{-dst}} \in \text{SSYT}(\lambda, \mu)$  obtained with replacing entries of  $H_j(T)$  by numbers  $\mu_{i_j} i_j$ 's,  $\dots$ ,  $(\mu_{i_{j+1}} - 1) (i_{j+1} - 1)$ 's.

It is easy to check that  $T \sim T^{\mu\text{-dst}}$ ,  $T^{(1^n)\text{-dst}} = T^{\text{st}}$  and  $T^{\alpha\text{-dst}} = T^{\text{dst}}$  with notation just as above.

**Lemma 4.18.** The map  $M : \text{ST}(\lambda, \mu) \rightarrow \text{SSYT}(\lambda, \mu)$ ,  $T \mapsto T^{\mu\text{-dst}} = M(T)$  is a dep-preserving bijection.

*Proof.* Suppose  $M(T) = M(U)$  for  $T, U \in \text{ST}(\lambda, \mu)$ . Then  $T \sim M(T) = M(U) \sim U$ . Here  $T, U$  are standard and each quasi-crystal contains exactly one standard Young tableau. Thus,  $T = U$ . Next, suppose  $V \in \text{SSYT}(\lambda, \mu)$ . It follows from Lemma 2.10 that

$$\mathrm{sh} V = \lambda \geq \mathrm{des} V \geq_{\mathrm{ref}} \mathrm{wt} V = \mu.$$

Note that  $V^{\text{st}}$  satisfies

$$\lambda = \operatorname{sh} V = \operatorname{sh} V^{\operatorname{st}} \geq \operatorname{des} V^{\operatorname{st}} = \operatorname{des} V \geq_{\operatorname{ref}} \mu,$$

that is,  $V^{\text{st}} \in \text{ST}(\lambda, \mu)$ . Now by definition of  $M$ ,  $M(V^{\text{st}}) = V$  is obvious. We showed that  $M$  is a bijection. Finally,  $T \sim M(T)$  implies  $\text{dep } T = \text{dep } M(T)$ .  $\square$

By this lemma, we proved Theorem 4.16. Let us now see something similar with quasi-Yamanouchi tableaux.

**Definition 4.19.** For  $\lambda, \mu \vdash n$ , define

$$\mathrm{QY}(\lambda, \mu) = \{Q \in \mathrm{QY}(\lambda) \mid \mathrm{des} Q \geq_{\mathrm{ref}} \mu\}.$$

**Lemma 4.20.**  $\text{ST}(\lambda, \mu) \rightarrow \text{QY}(\lambda, \mu)$ ,  $T \mapsto T^{\text{dst}}$  is a dep-preserving bijection.

*Proof.* The dep-preserving bijection  $T \mapsto T^{\text{dst}}$ ,  $\text{ST}(\lambda) \rightarrow \text{QY}(\lambda)$  restricts to the desired map  $\text{ST}(\lambda, \mu) \rightarrow \text{QY}(\lambda, \mu)$ .  $\square$

**Example 4.21.** Observe all tableaux in  $\text{SSYT}(\lambda, \mu)$ ,  $\text{ST}(\lambda, \mu)$  and  $\text{QY}(\lambda, \mu)$  for  $\lambda = 431, \mu = 3221$ .

|   |   |   |   |
|---|---|---|---|
| 1 | 2 | 3 | 5 |
| 4 | 6 | 7 |   |
| 8 |   |   |   |

|   |   |   |   |
|---|---|---|---|
| 1 | 2 | 3 | 5 |
| 4 | 7 | 8 |   |
| 6 |   |   |   |

|   |   |   |   |
|---|---|---|---|
| 1 | 2 | 3 | 7 |
| 4 | 5 | 6 |   |
| 8 |   |   |   |

|   |   |   |   |
|---|---|---|---|
| 1 | 2 | 3 | 7 |
| 4 | 5 | 8 |   |
| 6 |   |   |   |

|   |   |   |   |
|---|---|---|---|
| 1 | 2 | 3 | 8 |
| 4 | 5 | 7 |   |
| 6 |   |   |   |

|                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------|---|---|---|---|---|---|---|--|---|--|--|--|----------------------------------------------------------------------------------------------------------------------------------------------------------------|---|---|---|---|---|---|---|--|---|--|--|--|----------------------------------------------------------------------------------------------------------------------------------------------------------------|---|---|---|---|---|---|---|--|---|--|--|--|----------------------------------------------------------------------------------------------------------------------------------------------------------------|---|---|---|---|---|---|---|--|---|--|--|--|----------------------------------------------------------------------------------------------------------------------------------------------------------------|---|---|---|---|---|---|---|--|---|--|--|--|
| <table><tr><td>1</td><td>1</td><td>1</td><td>2</td></tr><tr><td>2</td><td>3</td><td>3</td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 2 | 2 | 3 | 3 |  | 4 |  |  |  | <table><tr><td>1</td><td>1</td><td>1</td><td>2</td></tr><tr><td>2</td><td>3</td><td>4</td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 2 | 2 | 3 | 4 |  | 3 |  |  |  | <table><tr><td>1</td><td>1</td><td>1</td><td>3</td></tr><tr><td>2</td><td>2</td><td>3</td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 3 | 2 | 2 | 3 |  | 4 |  |  |  | <table><tr><td>1</td><td>1</td><td>1</td><td>3</td></tr><tr><td>2</td><td>2</td><td>4</td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 3 | 2 | 2 | 4 |  | 3 |  |  |  | <table><tr><td>1</td><td>1</td><td>1</td><td>4</td></tr><tr><td>2</td><td>2</td><td>3</td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 4 | 2 | 2 | 3 |  | 3 |  |  |  |
| 1                                                                                                                                                              | 1 | 1 | 2 |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 3 | 3 |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 4                                                                                                                                                              |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1 | 1 | 2 |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 3 | 4 |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 3                                                                                                                                                              |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1 | 1 | 3 |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 2 | 3 |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 4                                                                                                                                                              |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1 | 1 | 3 |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 2 | 4 |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 3                                                                                                                                                              |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1 | 1 | 4 |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 2 | 3 |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 3                                                                                                                                                              |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| <table><tr><td>1</td><td>1</td><td>1</td><td>2</td></tr><tr><td>2</td><td>3</td><td>3</td><td></td></tr><tr><td>4</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 2 | 2 | 3 | 3 |  | 4 |  |  |  | <table><tr><td>1</td><td>1</td><td>1</td><td>2</td></tr><tr><td>2</td><td>3</td><td>3</td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 2 | 2 | 3 | 3 |  | 3 |  |  |  | <table><tr><td>1</td><td>1</td><td>1</td><td>2</td></tr><tr><td>2</td><td>2</td><td>2</td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 2 | 2 | 2 | 2 |  | 3 |  |  |  | <table><tr><td>1</td><td>1</td><td>1</td><td>3</td></tr><tr><td>2</td><td>2</td><td>4</td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 3 | 2 | 2 | 4 |  | 3 |  |  |  | <table><tr><td>1</td><td>1</td><td>1</td><td>3</td></tr><tr><td>2</td><td>2</td><td>3</td><td></td></tr><tr><td>3</td><td></td><td></td><td></td></tr></table> | 1 | 1 | 1 | 3 | 2 | 2 | 3 |  | 3 |  |  |  |
| 1                                                                                                                                                              | 1 | 1 | 2 |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 3 | 3 |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 4                                                                                                                                                              |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1 | 1 | 2 |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 3 | 3 |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 3                                                                                                                                                              |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1 | 1 | 2 |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 2 | 2 |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 3                                                                                                                                                              |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1 | 1 | 3 |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 2 | 4 |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 3                                                                                                                                                              |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 1                                                                                                                                                              | 1 | 1 | 3 |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 2                                                                                                                                                              | 2 | 3 |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |
| 3                                                                                                                                                              |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |                                                                                                                                                                |   |   |   |   |   |   |   |  |   |  |  |  |

**Theorem 4.22.**

$$f_\lambda(q) = \sum_{T \in \text{QY}(\lambda)} q^{\text{dep}(T)}, \quad \overline{K}_{\lambda\mu}(q) = \sum_{T \in \text{QY}(\lambda, \mu)} q^{\text{dep}(T)}.$$

**Corollary 4.23.**  $K_{\lambda\mu} = |\text{ST}(\lambda, \mu)| = |\text{QY}(\lambda, \mu)| = |\text{SSYT}(\lambda, \mu)|$ . In particular,  $f_\lambda = f_\lambda(1) = K_{\lambda(1^n)} = K_{\lambda(1^n)}(1) = \overline{K}_{\lambda(1^n)}(1)$ .

## 5. MULTI-VARIABLE ANALOG

We can consider a multi-variable analog of fake degree and Kostka polynomials because the bijections  $\text{ST}(\lambda, \mu) \rightarrow \text{SSYT}(\lambda, \mu)$ ,  $\text{ST}(\lambda, \mu) \rightarrow \text{QY}(\lambda, \mu)$  are in fact des-preserving. The author [6] introduced in 2025 *crystal skeleton polynomial* for  $\lambda$  with

$$\text{Sk}_\lambda(x) = \sum_{Q \in \text{QY}(\lambda)} x^{\text{des } Q}$$

as a multi-variable fake degree polynomial. Rather than  $\text{ST}(\lambda)$ , we carefully choose  $\text{QY}(\lambda)$  for this sum because  $\text{des } Q = \text{wt } Q$  whenever  $Q \in \text{QY}(\lambda)$  so that  $\text{Sk}_\lambda(x)$  makes more sense as a partial sum of  $s_\lambda(X)$ . As manifested, one has

$$\text{Sk}_\lambda(1, q, \dots, q^{n-1}) = f_\lambda(q)$$

and in particular  $\text{Sk}_\lambda(1, 1, \dots, 1) = f_\lambda$ .

From this point of view, we might define the multi-variable Kostka polynomial as

$$\overline{K}_{\lambda\mu}(x_1, \dots, x_n) = \sum_{Q \in \text{QY}(\lambda, \mu)} x^{\text{des } Q}$$

such that

$$\overline{K}_{\lambda\mu}(1, q, \dots, q^{n-1}) = \overline{K}_{\lambda\mu}(q).$$

However, we could not locate any references which deal with such polynomials. We take another opportunity for this topic.

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