

# CATEGORICAL AND COMBINATORIAL APPROACHES TO THE DETERMINATION OF EXCHANGE MATRICES

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## 1. INTRODUCTION

Cluster algebras, defined by Fomin–Zelevinsky [5] at the beginning of the century, are a class of commutative algebras generated by certain elements called *cluster variables*. Cluster variables are grouped into distinguished subsets called *clusters*. Each cluster comes equipped with a quiver, forming a *seed*. Through a combinatorial procedure, starting from an initial seed, all the other seeds can be constructed. Cluster algebra structures are ubiquitous in mathematics (see [12]). For example, in the context of the representation theory of quantum affine algebras, starting from Hernandez–Leclerc’s work [8], many Grothendieck rings of module categories have been endowed with cluster algebras structures. In 2024, Kashiwara–Kim–Oh–Park [11] wrote one of the most comprehensive works on the subject, working with certain monoidal subcategories  $\mathcal{C}_{\mathfrak{g}}^{[a,b],s}$  of the category of finite-dimensional modules of a quantum affine algebra. In order to describe their cluster structures, they introduced the combinatorics of *chain of  $i$ -boxes*, sequences of intervals (the  *$i$ -boxes*) satisfying certain conditions. Each chain of  $i$ -boxes is associated with a *monoidal seed*, which consists of a collection of simple modules satisfying certain properties and an exchange matrix. Monoidal seeds correspond to seeds of the cluster algebra structures on the Grothendieck rings. However, for most of their seeds, Kashiwara–Kim–Oh–Park do not explicitly provide the exchange matrices. At this time, three solutions have been proposed: in [3], we computed Kashiwara–Kim–Oh–Park’s exchange matrices in a context of additive categorification; in [10], Kashiwara–Kim provided deduced explicit formulas by relying on monoidal categorification techniques; finally, together with Qin–Wei [4], we recovered Kashiwara–Kim’s formulas using the combinatorics of signed words. In this report, we discuss the solutions proposed in [3] and [4].

## 2. MONOIDAL CATEGORIFICATION

**2.1. The categories  $\mathcal{C}_{\mathfrak{g}}^{[a,b],s}$ .** Let  $\mathfrak{g}$  be a finite-dimensional simple complex Lie algebra and denote the set of vertices of its Dynkin diagram by  $I_{\mathfrak{g}} = \{1, \dots, n\}$ . For simplicity, in this review we will further assume that the Lie algebra  $\mathfrak{g}$  is of ADE type. Let  $U_q(L\mathfrak{g})$  be the *quantum loop algebra* over the complex numbers associated to  $\mathfrak{g}$  (see [7, Definition 2.1]). For each vertex  $i$  of the Dynkin diagram of  $\mathfrak{g}$ , we choose  $\tilde{\varepsilon}_i \in \{0, 1\}$  such that we have  $|\tilde{\varepsilon}_i - \tilde{\varepsilon}_j| = 1$  whenever the vertices  $i$  and  $j$  are linked by an edge. Define  $\hat{I}_{\mathfrak{g}}$  to be the set

$$\hat{I}_{\mathfrak{g}} = \{(i, p) \in I \times \mathbb{Z} \mid p - \tilde{\varepsilon}_i \in 2\mathbb{Z}\}.$$

Let  $\mathcal{C}_{\mathfrak{g}}^{\mathbb{Z}}$  be the associated Hernandez–Leclerc category (see [8]). It is a monoidal Serre subcategory of the category of finite-dimensional  $U_q(L\mathfrak{g})$ -modules. It follows from the theory of  $q$ -characters (see [6]) that its simple objects are parametrized by the monic monomials in the variables  $Y_{i,p}$ , for  $(i, p)$  in  $\hat{I}_{\mathfrak{g}}$ . Given such monomial  $m$ , we denote the

associated simple object by  $L(m)$ . For example, for  $(i, p) \in \widehat{I}_{\mathfrak{g}}$  and an integer  $k \geq 1$ , the *Kirillov–Reshetikhin module*  $W_{k,p}^i \in \mathcal{C}_{\mathfrak{g}}^{\mathbb{Z}}$  is defined as

$$W_{k,p}^i = L(Y_{i,p} Y_{i,p+2} \cdots Y_{i,p+(k-1)}).$$

Two simple objects  $M$  and  $N$  of  $\mathcal{C}_{\mathfrak{g}}^{\mathbb{Z}}$  *commute* if there is an isomorphism  $M \otimes N \cong N \otimes M$  and they *strongly commute* if  $M \otimes N$  is simple. Moreover, the simple object  $M$  is *real* if it strongly commutes with itself. Let  $w_0$  be the longest element of the Weyl group of  $\mathfrak{g}$  and denote its length by  $l_0$ .

Let  $\mathfrak{s} = (i_k, p_k)_{k \in \mathbb{Z}}$  be an ordering of the elements of  $\widehat{I}_{\mathfrak{g}}$ . For  $s \in \mathbb{Z}$ , we denote by  $s^+$  (resp.  $s^-$ ) the smallest index  $k > s$  such that  $i_k = i_s$  (resp. the greatest index  $k < s$  such that  $i_k = i_s$ ). We assume that the sequence  $\mathfrak{s}$  is *admissible*, that is, it satisfies the following conditions:

- (i)  $p_{s^+} = p_s + 2$  for any  $s \in \mathbb{Z}$ ;
- (ii) for  $s, t \in \mathbb{Z}$ , if  $i_s \sim i_t$  are adjacent vertices and  $t^- < s < t < s^+$ , then  $p_t = p_s + 1$ ;
- (iii) for any  $k \in \mathbb{Z}$ , we have  $s_{i_k} \cdots s_{i_{k+l_0-1}} = w_0$ .

For  $a \leq b \in \mathbb{Z}$ , write  $[a, b] = \{k \in \mathbb{Z} \mid a \leq k \leq b\}$  (in the original sources, the case of infinite intervals is also treated). Following [11, Def. 6.16], the category  $\mathcal{C}_{\mathfrak{g}}^{[a,b],\mathfrak{s}}$  is defined as the monoidal Serre subcategory of  $\mathcal{C}_{\mathfrak{g}}^{\mathbb{Z}}$  generated by the *fundamental modules*  $L(Y_{i_k, p_k})$ , for  $k \in [a, b]$ .

**2.2. Monoidal categorification.** Let  $[a, b]$  be an integer interval. Following [11], a *monoidal seed* in  $\mathcal{C}_{\mathfrak{g}}^{[a,b],\mathfrak{s}}$  is a triple

$$\mathcal{S} = (\{M_i\}_{i \in K}, B, K),$$

where

- $K$  is a finite index set with a decomposition  $K = K^{\text{ex}} \amalg K^{\text{fr}}$ , where the indices in  $K^{\text{ex}}$  are called *exchangeable* and those in  $K^{\text{fr}}$  are called *frozen*;
- $B = (b_{ij})_{K \times K^{\text{ex}}}$  is a skew-symmetric matrix, called *exchange matrix*;
- $\{M_i\}_{i \in K}$  is a strongly commuting family of real simple objects in  $\mathcal{C}_{\mathfrak{g}}^{[a,b],\mathfrak{s}}$ .

Moreover, a monoidal seed  $\mathcal{S}$  in  $\mathcal{C}_{\mathfrak{g}}^{[a,b],\mathfrak{s}}$  is *admissible* if, for any  $k \in K^{\text{ex}}$ , there exists a real simple object  $M'_k \in \mathcal{C}_{\mathfrak{g}}^{[a,b],\mathfrak{s}}$  strongly commuting with  $M_i$  for any  $i \in K \setminus \{k\}$  and occurring in a short exact sequence

$$0 \rightarrow \bigotimes_{b_{ik} > 0} M_i^{b_{ik}} \rightarrow M_k \otimes M'_k \rightarrow \bigotimes_{b_{ik} < 0} M_i^{-b_{ik}} \rightarrow 0.$$

If  $\mathcal{S}$  is admissible, its *mutation* in direction  $k$  is the monoidal seed

$$\mu_k(\mathcal{S}) = (\{M_i\}_{i \neq k} \cup \{M'_k\}, \mu_k(B), K).$$

The monoidal seed  $\mathcal{S}$  is *completely admissible* if it admits successive mutations in all possible directions. The category  $\mathcal{C}_{\mathfrak{g}}^{[a,b],\mathfrak{s}}$  is a *monoidal categorification* of a cluster algebra  $\mathcal{A}$  if there is a ring isomorphism  $\varphi : \mathcal{K}(\mathcal{C}_{\mathfrak{g}}^{[a,b],\mathfrak{s}}) \xrightarrow{\sim} \mathcal{A}$  such that there is a completely admissible monoidal seed  $\mathcal{S} = (\{M_i\}_{i \in K}, B, K)$  whose image  $(B, (\varphi([M_i]))_{i \in K})$  under  $\varphi$  is a seed of  $\mathcal{A}$  and, for  $i \in K^{\text{fr}}$ , the elements  $\varphi([M_i])$  are the frozen cluster variables (we refer to [12] for unexplained terminology).

**2.3. Monoidal categorification via  $i$ -boxes.** Forgetting the second component, we associate to the admissible sequence  $\mathfrak{s} = (i_k, p_k)_{k \in \mathbb{Z}}$  the sequence  $\widehat{w}_0 = (i_k)_{k \in \mathbb{Z}}$ .

**Definition 2.1** ([11, Section 4]).

- An  $i$ -box of color  $j \in I_{\mathfrak{g}}$  is a finite integer interval  $[c, d]$  such that  $j = i_c = i_d$ .
- The  $i$ -cardinality of an  $i$ -box  $[c, d]$  is the number of times that its color appears in the sub-interval of  $\widehat{w}_0$  corresponding to  $[c, d]$ .

For a finite interval  $[c, d]$ , we write  $[c, d\}$  and  $\{c, d]$  for the largest  $i$ -boxes contained in  $[c, d]$  with color  $i_c$  and  $i_d$  respectively. Sometimes, for an  $i$ -box  $[c, d]$  of color  $i$ , we will write  $[c, d]_i$ . Moreover, given an  $i$ -box  $[c, d]$ , we will consider the  $i$ -boxes  $\tau[c, d] = [c^+, d^+]$  and  $\tau^-[c, d] = [c^-, d^-]$ .

**Example 2.2.** Assume that  $\mathfrak{g}$  is of type  $A_3$  and consider the sequence

$$\begin{array}{cccccccccccccccc} \widehat{w}_0 & \cdots & 3 & 1 & 2 & 3 & 1 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & \cdots \\ \mathbb{Z} & \cdots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{array}$$

- The interval  $[0, 0]$  is an  $i$ -box of color 2 and  $i$ -cardinality 1.
- The interval  $[-1, 1]$  is an  $i$ -box of color 1 and  $i$ -cardinality 2.
- The interval  $[-3, 6]$  is an  $i$ -box of color 2 and  $i$ -cardinality 4.

Let  $l$  be the length of the interval  $[a, b]$ .

**Definition 2.3** ([11, Def. 5.1]). A *chain* of  $i$ -boxes of range  $[a, b]$  is a sequence of  $i$ -boxes  $\mathfrak{C} = (\mathfrak{c}_k)_{1 \leq k \leq l}$  contained in  $[a, b]$  such that, for any  $1 \leq s \leq l$ , the union  $\bigcup_{1 \leq k \leq s} \mathfrak{c}_k$  is an interval of length  $s$  and the  $i$ -box  $\mathfrak{c}_s$  is the largest  $i$ -box of its own color contained in it.

**Example 2.4.** Keep the sequence  $\widehat{w}_0$  of Example 2.2. We consider the following chain of  $i$ -boxes and the associated chain of increasing intervals. In case one interval extends to the next one on the right (resp. on the left) we add an arrow labelled by  $R$  (resp.  $L$ ).

$$\begin{array}{ccccccccc} \mathfrak{C} = (\mathfrak{c}_k)_{1 \leq k \leq 6} & [4]_1 & [5]_3 & [3]_2 & [2, 5]_3 & [3, 6]_2 & [1, 4]_1 \\ \cup_{s \leq k} \mathfrak{c}_s & [4] & [4, 5] & [3, 5] & [2, 5] & [2, 6] & [1, 6] \\ & & \xrightarrow{R} & \xrightarrow{L} & \xrightarrow{L} & \xrightarrow{R} & \xrightarrow{L} \end{array}$$

The previous example shows that the datum of a chain of  $i$ -boxes  $\mathfrak{C} = (\mathfrak{c}_k)_{1 \leq k \leq l}$  is equivalent to the data of the singleton  $\mathfrak{c}_1$  and a sequence  $E \in \{L, R\}^{l-1}$ , such that, for  $1 \leq k < l$ ,  $E_k$  is equal to  $L$  respectively  $R$  if adding  $\mathfrak{c}_{k+1}$  increases the range of the interval  $\bigcup_{1 \leq k \leq s} \mathfrak{c}_k$  by one unit on the left respectively on the right. We call the sequence  $E$  the *sequence of expansion operators* associated with the chain of  $i$ -boxes  $\mathfrak{C}$ .

**Example 2.5.** We write  $\mathfrak{C}_{-}^{[a, b]}$  for the chain of  $i$ -boxes

$$\mathfrak{C}_{-}^{[a, b]} = (\mathfrak{c}_k^-)_{1 \leq k \leq b-a+1} = ([s, b])_{b \geq s \geq a}.$$

Notice that the associated pair is  $(b; (E_k)_{1 \leq k \leq b-a})$ , where  $E_k = L$  for any  $k$ .

**Definition 2.6** ([11, §5.2]). Let  $\mathfrak{C} = (\mathfrak{c}_k) = (c, (E_k)_{1 \leq k \leq l})$  be a chain of  $i$ -boxes.

- For  $1 \leq s < l$ , the  $i$ -box  $\mathfrak{c}_s$  is *movable* if  $s = 1$  or  $s \geq 2$  and  $E_{s-1} \neq E_s$ .
- For a movable  $i$ -box  $\mathfrak{c}_s$ , the *box move* at  $s$ , denoted by  $\nu_s$ , is the operation sending  $\mathfrak{C}$  to the chain  $\nu_s \mathfrak{C}$  defined as follows

- Move of type 1: if the color of  $\mathbf{c}_{s+1}$  is different from the color of  $\mathbf{c}_s$ , then  $\nu_s \mathfrak{C}$  is obtained from  $\mathfrak{C}$  by switching the positions of  $\mathbf{c}_s$  and  $\mathbf{c}_{s+1}$ .
- Move of type 2: if the color of  $\mathbf{c}_{s+1}$  is equal to the color of  $\mathbf{c}_s$ , then  $\nu_s \mathfrak{C}$  is obtained from  $\mathfrak{C}$  by replacing  $\mathbf{c}_s$  with the unique  $i$ -box among  $\tau^- \mathbf{c}_s$  and  $\tau \mathbf{c}_s$  contained in  $\mathbf{c}_{s+1}$ .

We call a finite composition of box moves a *chain transformation*.

**Example 2.7.** We keep the sequence  $\widehat{w}_0$  of Example 2.2.

Consider the following chain of  $i$ -boxes on the interval  $[1, 6]$ :

$$\begin{aligned}
 \mathfrak{C} &= [4]_1 \xrightarrow{R} [5]_3 \xrightarrow{L} [3]_2 \xrightarrow{L} [2, 5]_3 \xrightarrow{R} [3, 6]_2 \xrightarrow{L} [1, 5]_3 \\
 \text{move of type 1: } \nu_2(\mathfrak{C}) &= [4]_1 \xrightarrow{L} [3]_2 \xrightarrow{R} [5]_3 \xrightarrow{L} [2, 5]_3 \xrightarrow{R} [3, 6]_2 \xrightarrow{L} [1, 5]_3 \\
 \text{move of type 2: } \nu_3 \nu_2(\mathfrak{C}) &= [4]_1 \xrightarrow{L} [3]_2 \xrightarrow{L} [2]_3 \xrightarrow{R} [2, 5]_3 \xrightarrow{R} [3, 6]_2 \xrightarrow{L} [1, 5]_3
 \end{aligned}$$

**Remark 2.8.** Any chain of  $i$ -boxes of range  $[a, b]$  can be obtained through a chain transformation from the chain of  $i$ -boxes  $\mathfrak{C}_-^{[a, b]}$ . In more detail, let  $\mathfrak{C} = (\mathbf{c}_k)$  be a chain of  $i$ -boxes on the interval  $[a, b]$ . Let  $k_1 < k_2 < \dots$  be the elements of the set

$$\mathcal{R} = \{k \in [1, l-1] \mid E_k = R\}.$$

We set  $\mathfrak{C}_0 = \mathfrak{C}_-^{[a, b]}$  and, for any  $1 \leq s \leq |\mathcal{R}|$ , we define  $\mathfrak{C}_s$  as the chain of  $i$ -boxes of range  $[a, b]$  with associated sequence of expansion operators

$$E_k = \begin{cases} R, & \text{if } k \in \mathcal{R}, k \geq k_{|\mathcal{R}|-s+1} \\ L, & \text{otherwise.} \end{cases}$$

Then we have, for any  $1 \leq s \leq |\mathcal{R}|$ , the chain transformation

$$(1) \quad \nu_{k_{|\mathcal{R}|-s+1}} \circ \nu_{k_{|\mathcal{R}|-s}} \circ \dots \circ \nu_2 \circ \nu_1 : \mathfrak{C}_{s-1} \mapsto \mathfrak{C}_s.$$

By composition, we obtain a chain transformation

$$(2) \quad \mathfrak{C}_-^{[a, b]} \mapsto \mathfrak{C}.$$

We call a chain transformation of the form (1) *basic* and one of the form (2) *canonical*. By composing a canonical chain transformation with the inverse of a canonical chain transformation, we obtain a chain transformation between any pair of chains of  $i$ -boxes with the same range.

Let  $\mathbf{c} = [c, d]_i$  be an  $i$ -box of  $i$ -cardinality  $m$ . Following [11, Rem. 6.15], we associate to the  $i$ -box  $\mathbf{c}$  the Kirillov-Reshetikhin module

$$M(\mathbf{c}) = W_{m, p_c}^i.$$

For a chain of  $i$ -boxes  $\mathfrak{C} = (\mathbf{c}_k)$  of range  $[a, b]$ , set  $K(\mathfrak{C}) = \{k \mid 1 \leq k \leq l\}$  and let  $K(\mathfrak{C})^{\text{fr}}$  be the set formed by the indices corresponding to the maximal  $i$ -boxes in  $\mathfrak{C}$ , for any color.

**Theorem 2.9** ([11, Thm. 8.1]). *For each chain of  $i$ -boxes  $\mathfrak{C}$  of range  $[a, b]$ , there is an exchange matrix  $B(\mathfrak{C})$  such that the monoidal seed*

$$(3) \quad \mathcal{S}(\mathfrak{C}) = (M(\mathfrak{C}), B(\mathfrak{C}), K(\mathfrak{C}))$$

*in  $\mathcal{C}_{\mathfrak{g}}^{[a,b],s}$  is completely admissible and defines a monoidal categorification of the cluster algebra  $\mathcal{A}$  associated with  $B(\mathfrak{C})$ .*

At the end of [11], Kashiwara–Kim–Oh–Park state the following problem for any chain of  $i$ -boxes  $\mathfrak{C}$

(KKOP Problem) determine explicitly the exchange matrix  $B(\mathfrak{C})$ .

We denote the quiver of the matrix  $B(\mathfrak{C})$  by  $Q(\mathfrak{C})$ . The quiver  $Q(\mathfrak{C}_-^{[a,b]})$  is already known from the work of Hernandez–Leclerc [8]. Let  $\widehat{Q}$  be the quiver with set of vertices  $\widehat{I}_{\mathfrak{g}}$  and arrows:

- $(i, p) \rightarrow (i, p - 2)$  for any  $(i, p) \in \widehat{I}_{\mathfrak{g}}$ .
- $(i, p) \rightarrow (j, p + 1)$  for any  $(i, p) \in \widehat{I}_{\mathfrak{g}}$  and  $j$  adjacent to  $i$ .

We denote by  $\widehat{Q}^{[a,b],s}$  the full subquiver of  $\widehat{Q}$  on the set of vertices  $(i_k, p_k)_{k \in \mathbb{Z}}$ .

**Proposition 2.10** ([9]). *The quiver  $Q(\mathfrak{C}_-^{[a,b]})$  equals  $\widehat{Q}^{[a,b],s}$ .*

The next result describes the effect of a box move in the associated monoidal seeds.

**Lemma 2.11** ([11]). *Let  $\mathfrak{C} = (\mathfrak{c}_k)_{1 \leq k \leq l}$  be a chain of  $i$ -boxes. Let  $s \in K^{\text{ex}}$  such that  $\mathfrak{c}_s$  is movable and write  $\mathfrak{c}_{s+1} = [a_{s+1}, b_{s+1}]$ .*

(i) *If the move at  $\mathfrak{c}_s$  is of type 2, then*

$$\mathcal{S}(\nu_s(\mathfrak{C})) = (\{M(\mathfrak{c}_k)\}_{k \in K(\mathfrak{C}) \setminus \{s\}} \sqcup \{M(\mathfrak{c}'_s)\}, \mu_s(B(\mathfrak{C})), K(\mathfrak{C})),$$

*where*

$$M(\mathfrak{c}'_s) = \begin{cases} [a_{s+1}, b_{s+1}^-] & \text{if } \mathfrak{c}_s = [a_{s+1}^+, b_{s+1}] \\ [a_{s+1}^+, b_{s+1}] & \text{if } \mathfrak{c}_s = [a_{s+1}, b_{s+1}^-]. \end{cases}$$

(ii) *If the move at  $\mathfrak{c}_s$  is of type 1, then*

$$\mathcal{S}(\nu_s(\mathfrak{C})) = (\{M(\mathfrak{c}'_k)\}_{k \in K(\mathfrak{C})}, B'(\mathfrak{C}), K(\mathfrak{C})),$$

*where*

$$M(\mathfrak{c}'_k) = M(\mathfrak{c}_{\sigma_s(k)}), \quad B'(\mathfrak{C})_{ij} = B(\mathfrak{C})_{\sigma_s(i), \sigma_s(j)}$$

*and  $\sigma_s$  denotes transposition of  $s$  and  $s + 1$ .*

**2.4. An algorithmic solution.** Using Lemma 2.11, we can define an algorithm to compute the matrices  $B(\mathfrak{C})$ . Let  $\mathfrak{C}$  be a chain of  $i$ -boxes of range  $[a, b]$  associated with a pair  $(c, (E_k)_{1 \leq k \leq l-1})$ . We recursively construct the quiver  $Q(\mathfrak{C})$  starting from the quiver  $\widehat{Q}^{[a,b],s}$  as follows: Let  $i_1 < i_2 < \dots$  be the elements of the set

$$\mathcal{R} = \{k \in [1, l-1] \mid E_k = R\}.$$

We set  $\mathfrak{C}_0 = \mathfrak{C}_-^{[a,b]}$  and, for any  $1 \leq s \leq |\mathcal{R}|$ , as in Remark 2.8, we define the chain of  $i$ -boxes  $\mathfrak{C}_s$  of range  $[a, b]$  associated to the sequence of expansion operators

$$E_k = \begin{cases} R, & \text{if } k \in \mathcal{R}, k \geq i_{|\mathcal{R}|-s+1} \\ L, & \text{otherwise.} \end{cases}$$

Notice that  $\mathfrak{C}_{|\mathcal{R}|} = \mathfrak{C}$  and that, for any  $1 \leq s \leq |\mathcal{R}|$ , the chain of  $i$ -boxes  $\mathfrak{C}_s$  associated with

$$(b-s, (\underbrace{L, \dots, L}_{(i_{|\mathcal{R}|-s+1}-1)}, R, E_{i_{|\mathcal{R}|-s+1}+1}, \dots, E_{l-1}))$$

can be obtained from the chain  $\mathfrak{C}_{(s-1)}$  associated with

$$(b-s+1, (\underbrace{L, \dots, L}_{(i_{|\mathcal{R}|-s+1}-1)}, L, E_{i_{|\mathcal{R}|-s+1}+1}, \dots, E_{l-1}))$$

through a basic chain transformation. Let  $\mathfrak{c}_{j_1}, \dots, \mathfrak{c}_{j_m}$  be the  $i$ -boxes of  $\mathfrak{C}_{s-1} = (\mathfrak{c}_k)_k$  of color  $i_{b-s+1}$  and satisfying  $j_j \leq i_{|\mathcal{R}|-s+1} + 1$ .

Then (see [3, Lemma 4.12]) the quiver  $Q(\mathfrak{C}_s)$  is obtained from the quiver  $Q(\mathfrak{C}_{s-1})$  by mutating, in order, at the vertices associated to the  $i$ -boxes  $\mathfrak{c}_{j_1}, \dots, \mathfrak{c}_{j_{m-1}}$ . Proceeding by recursion we obtain the quiver  $Q(\mathfrak{C})$ .

### 3. A SOLUTION VIA ADDITIVE CATEGORIFICATION

For Let  $k$  be an algebraically closed field and denote by  $D = \text{Hom}_k(-, k)$  the duality over  $k$ . Let  $\mathcal{C}$  be a 2-Calabi–Yau  $k$ -category with suspension functor  $\Sigma$ . For an object  $X$  of  $\mathcal{C}$ , we denote by  $\text{add}(X)$  the full subcategory of  $\mathcal{C}$  whose objects are the direct factors of finite direct sums of copies of  $X$ . In the additive setting, the role of clusters is played by *cluster-tilting* objects, that is basic maximally rigid objects. Let  $T_i$  be an indecomposable direct factor of a cluster-tilting object  $T \cong T_i \oplus T_0$ . Up to isomorphism, there is a unique indecomposable object  $T_i^*$  not isomorphic to  $T_i$  such that the object  $T_i^* \oplus T_0$  is cluster-tilting. Moreover, as avatar of the exchange relation, there are the *exchange triangles*, non split triangles

$$(4) \quad T_i \xrightarrow{u} B \xrightarrow{v} T_i^* \rightarrow \Sigma T_i \text{ and } T_i^* \xrightarrow{u'} B' \xrightarrow{v'} T_i \rightarrow \Sigma T_i^*,$$

satisfying some additional conditions. If the quivers  $Q$  and  $Q'$  of the endomorphism algebras of  $T$  and  $\mu_i(T)$  do not have loops nor 2-cycles, then  $Q'$  is the mutation of  $Q$  at the vertex  $i$ .

Moreover, for any object  $X$  of  $\mathcal{C}$ , there is the *approximation triangle* of  $X$  by  $T$ , a certain triangle of the form

$$(5) \quad \Sigma^{-1}X \rightarrow T'' \rightarrow T' \rightarrow X,$$

such that  $T''$  and  $T'$  are objects of  $\text{add}(T)$ .

Our solution via additive categorification to Kashiwara–Kim–Oh–Park’s problem relies on the following result by Palu:

**Theorem 3.1** (Palu’s generalized mutation rule [13]). *Assume that  $\mathcal{C}$  is algebraic and let  $T = \bigoplus_{i \in J} T_i$  and  $T' = \bigoplus_{i \in J} T'_i$  be two basic cluster tilting objects in  $\mathcal{C}$ . Consider, for each summand  $T'_j$  of  $T'$ , the approximation triangle of  $T'_j$  by  $T$ :*

$$\Sigma^{-1}T'_j \rightarrow \bigoplus_i T_i^{\beta_{ij}} \rightarrow \bigoplus_i T_i^{\alpha_{ij}} \rightarrow T'_j.$$

Define the matrix  $P = (p_{ij})_{i,j \in J}$  by  $p_{ij} = \alpha_{ij} - \beta_{ij}$ . Then the matrix  $P$  is invertible and the matrices associated to the quivers of the endomorphism algebras of the cluster tilting objects  $T$  and  $T'$  are linked by the following formula:

$$B_{T'} = P^{-1}B_TP^{-t},$$

where  $P^{-t}$  denotes the inverse of the transpose of  $P$ .

In order to exploit Palu's formula, we need the following:

- (1) A suitable 2-CY category  $\mathcal{C}$ .
- (2) For any chain of  $i$ -boxes  $\mathfrak{C}$ , a cluster tilting object  $T(\mathfrak{C})$  in  $\mathcal{T}$  whose associated endoquiver is  $B(\mathfrak{C})$ .
- (3) For any cluster-tilting object  $T(\mathfrak{C})$ , the approximation triangles with respect to  $T(\mathfrak{C}_{-}^{[a,b]})$ .

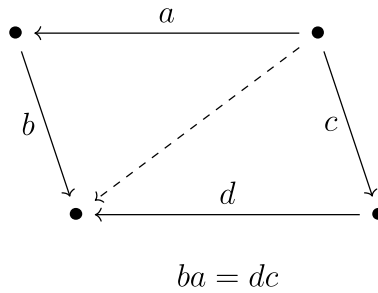
In order to address the first point, we will look at Amiot's cluster categories [1]. We refer to [12] for the definition and basic properties of cluster category. We endow  $\widehat{Q}^{[a,b],s}$  with the potential  $W^{[a,b]}$  given by the alternated sum of the cordless cycles of  $\widehat{Q}^{[a,b],s}$ . Let  $J^{[a,b]}$  be the associated Jacobi algebra.

**Proposition 3.2** ([3, Prop 6.1]).

- (1) The Jacobi algebra  $J^{[a,b]}(\underline{w}_0)$  is finite-dimensional.
- (2) The quiver with potential  $(\widehat{Q}^{[a,b],s}, W^{[a,b]})$  is rigid.

It follows from the general theory that the cluster category associated to the quiver with potential  $(\widehat{Q}^{[a,b],s}, W^{[a,b]})$  is Hom-finite, Krull–Schmidt, 2-Calabi–Yau and it contains a canonical cluster-tilting object, whose associated quiver is  $\widehat{Q}^{[a,b],s}$ . This cluster category is suitable for proving these general properties. Nevertheless, to construct and to work with our cluster tilting objects, an alternative model is more convenient.

For a given arrow  $\alpha : a \rightarrow b$  of  $\widehat{Q}^{[a,b],s}$ , where  $i_a \neq j_b$ , we define its *orbit* as the set of arrows  $\{\beta : c \rightarrow d \mid i_c = i_a, i_d = i_b\}$ . For each pair of lines  $i, j$  such that the associated Dynkin indices are adjacent, remove from the quiver  $\widehat{Q}^{[a,b],s}$  the orbit of the second arrow from the right connecting the line  $i$  and the line  $j$  and replace each removed arrow with a relation like in the figure:

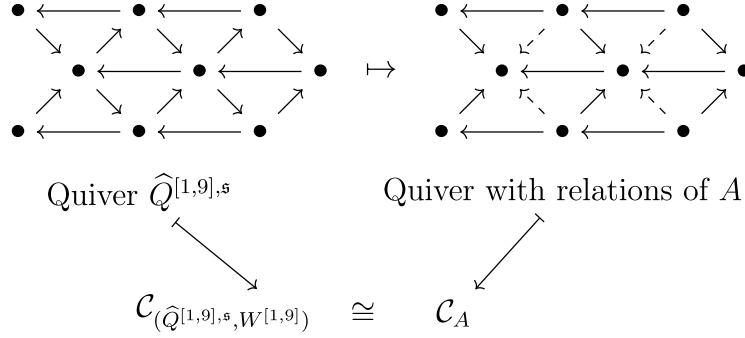


We denote by  $A$  the algebra associated to the resulting quiver with relations.

**Proposition 3.3** ([3, Prop 6.8]). *The algebra  $A$  is of global dimension at most 2.*

Following Amiot's theory, the previous results allows us to define the generalized cluster category  $\mathcal{C}_A$  as the orbit category of the derived category of  $A$ . In particular, there is a functor from the module category of  $A$  to  $\mathcal{C}_A$ . Moreover, by construction, this category is triangle equivalent to the cluster category associated to the quiver with potential  $(\widehat{Q}^{[a,b],s}, W^{[a,b]})$  and, therefore, it inherits all its properties.

**Example 3.4.** Let  $\mathfrak{g}$  be of type  $A_3$  and  $[a, b] = [1, 9]$ . The following diagram shows the construction of the algebra  $A$  and Amiot's equivalence.



Working in the category  $\mathcal{C}_A$ , we define a cluster-tilting object for each chain of  $i$ -boxes  $\mathfrak{C}$  in  $[a, b]$ . We denote by  $m_i$  the  $m$ -th vertex from the right on the line  $i$  of  $\widehat{Q}^{[a,b],s}$  and by  $P_{m_i}$  the associated indecomposable projective  $A$ -module. Let  $i$  be in  $I_{\mathfrak{g}}$  and  $m, k$  be non-negative integers such that  $(m+k)_i$  is a vertex of  $\widehat{Q}^{[a,b],s}$ . We define the additive  $KR$ -module  $KR(i, k, m)$  as the quotient

$$P_{m_i} \rightarrow P_{(m+k)_i} \rightarrow KR(i, k, m) \rightarrow 0.$$

Let  $\mathfrak{c}$  be an  $i$ -box contained in  $[a, b]$  of color  $i$  and cardinality  $k$ . Then there exists an  $i$ -box  $[c, b]_i$  in  $\mathfrak{C}_-^{[a,b]}$  and a non-negative integer  $m$  such that  $(m+k)_i$  is a vertex of  $\widehat{Q}^{[a,b],s}$  and  $\tau^{-m}[c, b]_i = \mathfrak{c}$ . We associate to the  $i$ -box  $\mathfrak{c}$  the additive  $KR$ -module  $T(\mathfrak{c}) = KR(i, k, m)$ .

For a chain of  $i$ -boxes  $\mathfrak{C}$  with range  $[a, b]$  we define the  $A$ -module  $T(\mathfrak{C})$  by

$$T(\mathfrak{C}) = \bigoplus_{\mathfrak{c} \in \mathfrak{C}} T(\mathfrak{c}).$$

Viewing  $T(\mathfrak{C})$  as an object of the cluster category, we have the following.

**Theorem 3.5.** *Let  $\mathfrak{C}$  be a chain of  $i$ -boxes on  $[a, b]$ . Then the object  $T(\mathfrak{C})$  of  $\mathcal{C}_A$  is cluster-tilting.*

When  $\mathfrak{C} = \mathfrak{C}_-^{[a,b]}$ , we have  $T(\mathfrak{C}_-^{[a,b]}) = A$ , the canonical cluster tilting object. In particular, its endoquiver is  $Q(\mathfrak{C}_-^{[a,b]})$ . Moreover, working in the module category and projecting to the cluster category, we have the exchange triangles

$$KR(i, k, m) \rightarrow KR(i, k-1, m+1) \oplus KR(i, k+1, m) \rightarrow KR(i, k, m+1) \xrightarrow{\Delta}$$

and the approximation triangles

$$\Sigma^{-1}KR(i, k, m) \rightarrow P_{m_i} \rightarrow P_{(m+k)_i} \rightarrow KR(i, k, m).$$

Finally, by showing that for any box move of type 2 at a vertex  $s$  we have  $T(\nu_s(\mathfrak{C})) = \mu_s(T(\mathfrak{C}))$ , we conclude that the endoquiver of  $T(\mathfrak{C})$  is the desired quiver  $Q(\mathfrak{C})$ . Therefore, by applying Palu's theorem, we obtain an expression for the associated matrices:

$$B(T(\mathfrak{C})) = P(\mathfrak{C})^{-1}B(T(\mathfrak{C}_-^{[a,b]}))P(\mathfrak{C})^{-t},$$

where the matrix  $P(\mathfrak{C})$  is obtained through the approximation triangles.

## 4. A SOLUTION VIA SIGNED WORDS

**4.1. Combinatorics of signed words.** Let  $l$  be in  $\mathbb{N}$  and let  $C = (c_{ij})_{i,j \in I_{\mathfrak{g}}}$  be the Cartan matrix of  $\mathfrak{g}$ . A signed word of length  $l$  on  $I_{\mathfrak{g}}$  is a sequence  $\underline{\mathbf{h}} = (\varepsilon_k h_k)_{1 \leq k \leq l}$  where, for any  $k$ ,  $\varepsilon_k \in \{\pm 1\}$  and  $h_k \in I$ . Denote  $\mathbf{h}_k = \varepsilon_k h_k$  and  $|\mathbf{h}_k| = h_k$ . For any  $1 \leq j, k \leq l$ , let  $\underline{\mathbf{h}}_{[j,k]}$  denote the sequence  $(\mathbf{h}_s)_{s \in [j,k]}$ .

We denote  $K(\underline{\mathbf{h}}) = [1, l]$  and consider the set of frozen indices

$$K(\underline{\mathbf{h}})^{\text{fr}} = \{s \in K(\underline{\mathbf{h}}) \mid \exists t \in [s+1, l], h_t = h_s\}$$

For any  $k \in K(\underline{\mathbf{h}})$ , we define  $k[1]$  as

$$(6) \quad k[1] = \min(\{k' \in [k+1, l] \mid |\mathbf{h}_{k'}| = |\mathbf{h}_k|\} \sqcup \{+\infty\}).$$

Following [2], for any signed word  $\underline{\mathbf{h}}$ , let  $B(\underline{\mathbf{h}}) = (\tilde{b}_{jk})_{(j,k) \in K(\underline{\mathbf{h}}) \times K(\underline{\mathbf{h}})^{\text{fr}}}$  be the matrix

$$(7) \quad \tilde{b}_{jk} = \begin{cases} \varepsilon_k & k = j[1], \\ -\varepsilon_j & j = k[1], \\ \varepsilon_k c_{|\mathbf{h}_j|, |\mathbf{h}_k|} & \varepsilon_{j[1]} = \varepsilon_k, j < k < j[1] < k[1], \\ \varepsilon_k c_{|\mathbf{h}_j|, |\mathbf{h}_k|} & \varepsilon_k = -\varepsilon_{k[1]}, j < k < k[1] < j[1], \\ -\varepsilon_j c_{|\mathbf{h}_j|, |\mathbf{h}_k|} & \varepsilon_{k[1]} = \varepsilon_j, k < j < k[1] < j[1], \\ -\varepsilon_j c_{|\mathbf{h}_j|, |\mathbf{h}_k|} & \varepsilon_j = -\varepsilon_{j[1]}, k < j < j[1] < k[1], \\ 0 & \text{otherwise.} \end{cases}$$

According to [14], we consider the following operations on signed words and the effects on the associated matrices (notice the similarity with the combinatorics of  $i$ -boxes). Let  $\underline{\mathbf{h}} = (\mathbf{h}_k)_{k \in [1, l]}$  be a signed word.

- The *left reflection* of  $\underline{\mathbf{h}}$  is the operation sending  $\underline{\mathbf{h}}$  to the signed word  $\underline{\mathbf{h}}' = (-\mathbf{h}_1, \mathbf{h}_{[2, l]})$ . Then  $B(\underline{\mathbf{h}}') = B(\underline{\mathbf{h}})$ .
- Let  $j$  be in  $[1, l-1]$  such that  $\mathbf{h}_j$  and  $\mathbf{h}_{j+1}$  have different signs. Then the *flip* of  $\underline{\mathbf{h}}$  at  $j$  is the operation sending  $\underline{\mathbf{h}}$  to the signed word  $\underline{\mathbf{h}}' = (\mathbf{h}_{[1, j-1]}, \mathbf{h}_{j+1}, \mathbf{h}_j, \mathbf{h}_{[j+2, l]})$ . Then

$$B(\underline{\mathbf{h}}') = \begin{cases} \sigma_{j, j+1} B(\underline{\mathbf{h}}) & \text{if } h_j \neq h'_j, \\ \mu_j(B(\underline{\mathbf{h}})) & \text{if } h_j = h'_j. \end{cases}$$

**Example 4.1.** Suppose that  $\mathfrak{g}$  is of type  $A_3$  and consider the signed word

$$\underline{\mathbf{h}} = (-1, -2, 2, 2, 3, -3, 1, 2).$$

- Left reflection of  $\underline{\mathbf{h}}$ :  $\underline{\mathbf{h}}' = (\mathbf{1}, -2, 2, 2, 3, -3, 1, 2)$ .
- Flip of  $\underline{\mathbf{h}}'$  at  $j = 5$ :  $\underline{\mathbf{h}}'' = (1, -2, 2, 2, \mathbf{-3}, \mathbf{3}, 1, 2)$ .

Let  $\mathfrak{C} = (\mathbf{c}_k)_{k \in [1, l]}$  be a chain of  $\mathbf{i}$ -boxes on  $[a, b]$  and let  $(E_k)_{1 \leq k < l}$  be the associated sequence of expansion operators. Following [4], the associated signed word  $\underline{\mathbf{h}}(\mathfrak{C}) = (\mathbf{h}_k)_{1 \leq k \leq l}$  is defined as follows

- we set  $\underline{\mathbf{h}}_1$  equal to the color  $h_1$  of the  $\mathbf{i}$ -box  $\mathbf{c}_1$ ;
- for any  $2 \leq k \leq l$ , we set  $\underline{\mathbf{h}}_k = \varepsilon_k h_k$  where  $h_k$  is the color of the  $\mathbf{i}$ -box  $\mathbf{c}_k$  and the sign  $\varepsilon_k$  is defined by

$$\begin{cases} 1, & \text{if } E_{k-1} = L, \\ -1, & \text{if } E_{k-1} = R. \end{cases}$$

**Example 4.2.** Consider, in the  $A_3$  case, the following chain of  $i$ -boxes on the interval  $[1, 6]$ :

$$\mathfrak{C} \quad [4]_1 \xrightarrow{R} [5]_3 \xrightarrow{L} [3]_2 \xrightarrow{L} [2, 5]_3 \xrightarrow{R} [3, 6]_2 \xrightarrow{L} [1, 4]_1$$

Then the associated signed word is

$$\underline{\mathbf{h}}(\mathfrak{C}) = (1, -3, 2, 3, -2, 3).$$

It is easy to see from the definitions that  $B(\mathfrak{C}_-^{[a,b]}) = B(\underline{\mathbf{h}}(\mathfrak{C}_-^{[a,b]}))$ . Moreover, if  $B(\mathfrak{C}) = B(\underline{\mathbf{h}}(\mathfrak{C}))$ , then  $B(\nu_s(\mathfrak{C})) = B(\underline{\mathbf{h}}(\nu_s(\mathfrak{C})))$ . Combining these two points leads to the equality  $B(\mathfrak{C}) = B(\underline{\mathbf{h}}(\mathfrak{C}))$  [4, Thm. 1.1].

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