

Branching rules for winding subalgebras of affine Kac–Moody algebras

Duc-Khanh Nguyen

Abstract

In this paper, by using Lakshmibai–Seshadri paths, we establish a branching rule for representations of affine Kac–Moody algebras when restricted to winding subalgebras. As a consequence, the branching multiplicities can be described in terms of paths. We also obtain an analogue of Steinberg’s formula for these branching multiplicities.

2020 Mathematics Subject Classification. 17B67, 17B10, 22E65.

Key words and phrases. Affine Kac–Moody algebras, winding subalgebras, Lakshmibai–Seshadri paths, generalized Kostant’s partition functions.

0 Introduction

One of the central problems in representation theory is to understand how an irreducible module decomposes as a direct sum of irreducible modules over a subalgebra. More precisely, one asks what the support of such a decomposition is and how the corresponding branching multiplicities can be computed. In the representation theory of Kac–Moody algebras, these questions have been answered in several special cases. Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra. In [Lit94], Peter Littelmann established the generalized Littlewood–Richardson rule for the case $\mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$, and also obtained the branching rule for $\mathfrak{h} \subset \mathfrak{g}$ when $\mathfrak{h}$ is a Levi subalgebra of $\mathfrak{g}$, using the language of Lakshmibai–Seshadri paths. In [Ste61], Robert Steinberg gave a combinatorial formula for tensor multiplicities of simple Lie algebras. This formula is analogous to Kostant’s multiplicity formula for symmetrisable Kac–Moody algebras; see [Car05].

In this paper, we investigate the branching rule for winding subalgebras $\mathfrak{g}[u]$ of affine Kac–Moody algebras $\mathfrak{g}$. This inclusion was studied in [KW90] because of its connection with the tensor product case and with a solution of Frenkel’s conjecture [Fre06]. In [KW90], Kac and Wakimoto derived the branching function and its modular transformation for both inclusions $\mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ and $\mathfrak{g}[u] \subset \mathfrak{g}$. In certain special cases, the branching function for tensor products can be realized as the branching function for winding subalgebras. This observation suggests an approach to studying the branching rule for affine Kac–Moody algebras with respect to winding subalgebras by adapting the methods used in the tensor product setting. Some particular cases were previously studied by the author in [Ngu20]. The results obtained in the present paper extend those of [Ngu20] and provide a more combinatorial treatment than that of [KW90]. Our main result is stated as follows. Let P_+ be the set of dominant integral weights. For each $\lambda \in P_+$, let $L(\lambda)$ be the irreducible $\mathfrak{g}$ -module of highest weight λ . Let $\dot{P}_+$ and $\dot{L}(\lambda')$, for $\lambda' \in \dot{P}_+$, denote the corresponding objects for $\mathfrak{g}[u]$. For any $\lambda \in P_+$ and $\lambda' \in \dot{P}_+$, let $c_{\lambda}^{\lambda'}$ be the multiplicity of $\dot{L}(\lambda')$ in the decomposition

$$L(\lambda) = \bigoplus_{\lambda' \in \dot{P}_+} c_{\lambda}^{\lambda'} \dot{L}(\lambda').$$

For each dominant integral weight λ of $\mathfrak{g}$, let $\wp_{\lambda}$ be the set of all Lakshmibai–Seshadri paths of shape λ as defined in [Lit94]. Each path is piecewise linear, starting at $\pi(0) = 0$ and ending at $\pi(1)$. A path is called $\mathfrak{g}[u]$ -dominant if its image is contained in the dominant Weyl chamber of the root system of $\mathfrak{g}[u]$. The branching rule for $\mathfrak{g}[u] \subset \mathfrak{g}$ can be stated as follows.

Theorem 0.1. *Let $\mathfrak{g}$ be an affine Kac–Moody algebra and $\mathfrak{g}[u]$ be a winding subalgebra of $\mathfrak{g}$. Then for any $\lambda \in P_+$, we have*

$$L(\lambda) = \bigoplus_{\pi} \dot{L}(\pi(1)),$$

where the direct sum ranges over all $\mathfrak{g}[u]$ -dominant paths in $\wp_\lambda$. In particular, for each $\lambda' \in \dot{P}_+$ the multiplicity $c_\lambda^{\lambda'}$ is equal to the number of $\mathfrak{g}[u]$ -dominant paths in $\wp_\lambda$ with end point λ' .

Let W denote the Weyl group, and let ρ be the sum of the fundamental weights of $\mathfrak{g}$. Let $\dot{W}$ and $\dot{\rho}$ denote the corresponding objects associated with $\mathfrak{g}[u]$. Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$. For each root $\alpha \in \mathfrak{h}^*$, let m_α denote its multiplicity. The generalized Kostant partition function $\mathcal{P} : \mathfrak{h}^* \rightarrow \mathbb{N}$ is defined by letting $\mathcal{P}(\lambda)$ be the number of ways to express λ as a sum of positive roots, where each positive root α occurs with multiplicity m_α . With this notation, we obtain a combinatorial formula for $c_\lambda^{\lambda'}$.

Theorem 0.2. *Let $\mathfrak{g}$ be an affine Kac–Moody algebra and $\mathfrak{g}[u]$ be a winding subalgebra of $\mathfrak{g}$. Then for any $\lambda \in P_+$ and $\lambda' \in \dot{P}_+$, we have*

$$c_\lambda^{\lambda'} = \sum_{\sigma \in W} \sum_{\tau \in \dot{W}} \epsilon(\sigma\tau) \mathcal{P}(\sigma(\lambda + \rho) + \tau(\dot{\rho}) - (\lambda' + \rho + \dot{\rho})).$$

The paper is organized as follows. In Section 1, we review the basic results on affine Kac–Moody algebras, winding subalgebras, and Lakshmibai–Seshadri paths. In Section 2, we establish our main theorems.

Acknowledgments: This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

1 Preliminaries

1.1 Symmetrisable Kac–Moody algebras

In this subsection, we review some basic definitions and results concerning symmetrisable Kac–Moody algebras. Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra, and let W denote its Weyl group. Let P_+ be the set of dominant integral weights, and let ρ be the sum of the fundamental weights Λ_i . For each $\lambda \in P_+$, let $L(\lambda)$ denote the irreducible highest weight $\mathfrak{g}$ -module with highest weight λ . The character of $L(\lambda)$ is given by the Kac character formula.

Theorem 1.1 (Corollary 19.18, [Car05]). *Let $L(\lambda)$, $\lambda \in P_+$, be an irreducible module for a symmetrisable Kac–Moody algebra. Then*

$$\text{ch}_\lambda = \frac{\sum_{w \in W} \epsilon(w) e_{w(\lambda + \rho)}}{\sum_{w \in W} \epsilon(w) e_{w(\rho)}}.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let Φ be the root system of $\mathfrak{g}$ and Φ_+ be the set of all positive roots. For each $\alpha \in \Phi$, we denote its multiplicity by m_α . We define the generalized Kostant’s partition function $\mathcal{P} : \mathfrak{h}^* \rightarrow \mathbb{N}$ by

$$\mathcal{P}(\zeta) = \# \left\{ (r_{\alpha,i})_{\alpha \in \Phi_+, 1 \leq i \leq m_\alpha} \mid r_{\alpha,i} \in \mathbb{N} \text{ and } \zeta = \sum_{\alpha \in \Phi_+} \sum_{i=1}^{m_\alpha} r_{\alpha,i} \alpha \right\}$$

for each $\zeta \in \mathfrak{h}^*$. The multiplicity $m_\lambda(\mu)$ of each weight μ of $L(\lambda)$ is then given by a combinatorial formula.

Theorem 1.2 (Proposition 19.20, [Car05]). *Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra and $\lambda \in P_+$. Then for each weight μ of $L(\lambda)$ we have*

$$m_\lambda(\mu) = \sum_{w \in W} \epsilon(w) \mathcal{P}(w(\lambda + \rho) - (\mu + \rho)).$$

1.2 Affine Kac–Moody algebras and their winding subalgebras

In this subsection, we consider the case $\mathfrak{g}$ is an affine Kac–Moody algebra. Let $I := \{0, \dots, l\}$ and $A := (a_{ij})_{i,j \in I}$ be the generalized Cartan matrix defining $\mathfrak{g}$. Let $a := {}^t(a_0, \dots, a_l)$ and $c := (c_0, \dots, c_l)$ be the null vectors of relatively prime integers such that $a_i, c_i \geq 0$ and $Aa = cA = 0$. The Coxeter number and dual Coxeter number of $\mathfrak{g}$ are $h := \sum_{i \in I} a_i$ and $h^\vee := \sum_{i \in I} c_i$, respectively. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ with $\Pi := \{\alpha_0, \dots, \alpha_l\}$, $\Pi^\vee := \{h_0, \dots, h_l\}$ be a realisation of $\mathfrak{g}$. Let $K := \sum_{i \in I} c_i h_i$ be the canonical central element and $\delta := \sum_{i \in I} a_i \alpha_i$ be the basic imaginary root of $\mathfrak{g}$.

The winding subalgebras of $\mathfrak{g}$ are defined in [KW90, Wak01, Ngu20] as follows: Fix $u \in \mathbb{Z}_{>0}$ relatively prime to a_0 . Let $\Pi_u := \{\dot{\alpha}_0, \dots, \dot{\alpha}_l\}$ and $\Pi_u^\vee := \{\dot{h}_0, \dots, \dot{h}_l\}$, where

$$\begin{aligned} \dot{\alpha}_0 &:= \frac{u-1}{a_0} \delta + \alpha_0, \dot{\alpha}_i := \alpha_i \text{ for } i > 0, \\ \dot{h}_0 &:= \frac{u-1}{c_0} K + h_0, \dot{h}_i := h_i \text{ for } i > 0. \end{aligned}$$

The winding subalgebra $\mathfrak{g}[u]$ of $\mathfrak{g}$ is defined by the realization $(\mathfrak{h}, \Pi_u, \Pi_u^\vee)$. Since the matrix $(\dot{\alpha}_i(\dot{h}_j))_{i,j \in I}$ is exactly A , the subalgebra $\mathfrak{g}[u]$ is isomorphic to the algebra $\mathfrak{g}$. We denote by an overdot the objects associated with $\mathfrak{g}[u]$:

$$\begin{aligned} \dot{K} &= \sum_{i \in I} c_i \dot{h}_i = uK, & \dot{\delta} &= \sum_{i \in I} a_i \dot{\alpha}_i = u\delta, \\ \dot{h} &= h, & \dot{h}^\vee &= h^\vee, \\ \dot{\Lambda}_i &= \Lambda_i + \left(\frac{1}{u} - 1\right) \frac{c_i}{c_0} \Lambda_0, \\ \dot{\rho} &= \sum_{i \in I} \dot{\Lambda}_i = \rho + \left(\frac{1}{u} - 1\right) h^\vee \Lambda_0, \\ \dot{W} &= \langle \dot{s}_0, \dots, \dot{s}_l \rangle, \text{ where } \dot{s}_i \in \text{Aut}(\mathfrak{h}^*), \dot{s}_i(\lambda) = \lambda - \lambda(\dot{h}_i) \dot{\alpha}_i, \\ \text{ch}_\lambda &= \frac{\sum_{w \in \dot{W}} \epsilon(w) e_{w(\lambda + \dot{\rho})}}{\sum_{w \in \dot{W}} \epsilon(w) e_{w(\dot{\rho})}} \text{ for any } \lambda \in \dot{P}_+. \end{aligned}$$

1.3 Lakshmibai–Seshadri paths and character formula

In this section, we review the basic theory of Lakshmibai–Seshadri paths and the character formula expressed in terms of paths for an arbitrary symmetrisable Kac–Moody algebra, following [Lit94]. Let $(\cdot | \cdot)$ denote the standard symmetric invariant bilinear form on $\mathfrak{g}$. For each real root α , we write $\alpha^\vee := \frac{2\alpha}{(\alpha | \alpha)}$ for the corresponding real coroot. Let $\ell(\cdot)$ denote the length function on W . For a positive real root β , let s_β denote the associated reflection.

1.3.1 Path operators

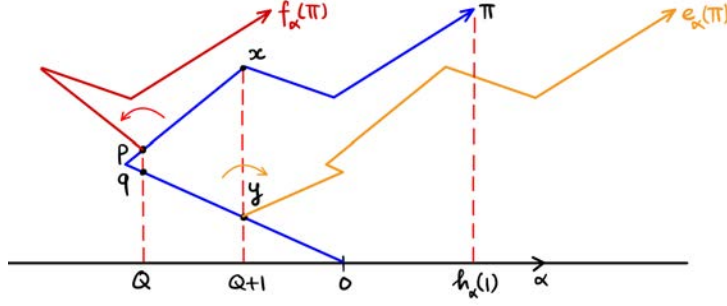
Let $P_{\mathbb{R}} := P \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by $\wp$ the set of all piecewise linear paths $\pi : [0, 1] \rightarrow P_{\mathbb{R}}$ such that $\pi(0) = 0$ with the equivalent relation $\pi \sim \pi'$ if $\pi = \pi'$ up to a reparametrization. For $\pi_1, \pi_2 \in \wp$, we define $\pi_1 * \pi_2$ to be a path still in $\wp$ by

$$(\pi_1 * \pi_2)(t) := \begin{cases} \pi_1(2t) & \text{if } t \in [0, \frac{1}{2}); \\ \pi_1(1) + \pi_2(2t - 1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

For each simple root α and a path π in $\wp$, we define the path $s_\alpha(\pi)$ still in $\wp$ by

$$s_\alpha(\pi)(t) := s_\alpha(\pi(t)).$$

We define operators e_α and f_α on $\wp \cup \{0\}$ as follows.

Figure 1: Path operators e_α and f_α .

1. Let $h_\alpha : [0, 1] \rightarrow \mathbb{R}$ be a function defined by $h_\alpha(t) := (\pi(t)|\alpha^\vee)$. Set

$$Q := \min\{h_\alpha(t) \mid t \in [0, 1]\} \cap \mathbb{Z}$$

and

$$q := \min\{t \in [0, 1] \text{ such that } h_\alpha(t) = Q\},$$

$$p := \max\{t \in [0, 1] \text{ such that } h_\alpha(t) = Q\}.$$

Note that $Q \leq 0$ since $\pi(0) = 0$ and $0 \leq [h_\alpha(1) - Q]$ where $[x]$ means the integral part of a real number

$$[x] = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0; \\ \lceil x \rceil & \text{if } x \leq 0. \end{cases}$$

2. If $Q < 0$, let y be the unique value of t in $[0, q)$ such that $h_\alpha(y) = Q + 1$ and $Q < h_\alpha(t) < Q + 1$ for all $t \in (y, q)$. In this case, let π_1, π_2, π_3 be three paths in $\wp$ defined by

$$\pi_1(t) := \pi(ty); \pi_2(t) := \pi(y + t(q - y)) - \pi(y); \pi_3(t) := \pi(q + t(1 - q)) - \pi(q).$$

In other words, π_1, π_2, π_3 are obtained from restriction of the path π on intervals $[0, y], [y, q], [q, 1]$, then take translations of points $\pi(0), \pi(y), \pi(q)$ to 0. We define the operator e_α on $\wp \cup \{0\}$ by

$$e_\alpha(\pi) := \begin{cases} 0 & \text{if } Q = 0; \\ \pi_1 * s_\alpha(\pi_2) * \pi_3 & \text{if } Q < 0. \end{cases}$$

3. If $[h_\alpha(1) - Q] > 0$, let x be the unique value of t in $(p, 1]$ such that $h_\alpha(x) = Q + 1$ and $Q < h_\alpha(t) < Q + 1$ for all $t \in (p, x)$. In this case, let $\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3$ be three paths in $\wp$ defined by

$$\hat{\pi}_1(t) := \pi(tp); \hat{\pi}_2(t) := \pi(p + t(x - p)) - \pi(p); \hat{\pi}_3(t) := \pi(x + t(1 - x)) - \pi(x).$$

In other words, $\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3$ are obtained from restriction of the path π on intervals $[0, p], [p, x], [x, 1]$, then take translations of points $\pi(0), \pi(p), \pi(x)$ to 0. We define the operator f_α on $\wp \cup \{0\}$ by

$$f_\alpha(\pi) := \begin{cases} 0 & \text{if } [h_\alpha(1) - Q] = 0; \\ \hat{\pi}_1 * s_\alpha(\hat{\pi}_2) * \hat{\pi}_3 & \text{if } [h_\alpha(1) - Q] > 0. \end{cases}$$

Lemma 1.3 (Lemma 1.4, [Lit94]). *If $e_\alpha(\pi) \neq 0$ then $e_\alpha(\pi)(1) = \pi(1) + \alpha$. If $f_\alpha(\pi) \neq 0$ then $f_\alpha(\pi)(1) = \pi(1) - \alpha$.*

Lemma 1.4 (Proposition 1.5, [Lit94]). *$e_\alpha^n(\pi) = 0$ if and only if $n > -Q$ and $f_\alpha^n(\pi) = 0$ if and only if $n > [h_\alpha(1) - Q]$.*

1.3.2 Lakshmibai–Seshadri paths and character formula

For each $\lambda \in P_+$, let W_λ be the stabilizer of λ and let $\leq$ denote the Bruhat order on W/W_λ . We denote $(\underline{\tau}, \underline{a})$ the pair of a sequence in W/W_λ

$$\underline{\tau} : \tau_1 > \tau_2 > \cdots > \tau_r$$

and a sequence in $\mathbb{Q}$

$$\underline{a} : 0 = a_0 < a_1 < \cdots < a_r = 1.$$

We associate this pair with a piecewise linear path $\pi_{(\underline{\tau}, \underline{a})}^\lambda : [0, 1] \rightarrow P_{\mathbb{R}}$ defined by

$$\pi_{(\underline{\tau}, \underline{a})}^\lambda(t) := \sum_{i=1}^{j-1} (a_i - a_{i-1})\tau_i(\lambda) + (t - a_{j-1})\tau_j(\lambda) \text{ for } a_{j-1} \leq t \leq a_j.$$

We call such paths rational W -paths of shape λ . For each pair (τ, σ) in W/W_λ with $\tau > \sigma$ and a rational number a in $(0, 1)$. We call an a -chain of (τ, σ) a sequence in W/W_λ

$$\kappa_0 := \tau > \kappa_1 := s_{\beta_1}\tau > \kappa_2 := s_{\beta_2}s_{\beta_1}\tau > \cdots > \kappa_s := s_{\beta_s}\cdots s_{\beta_1}\tau = \sigma,$$

where $\beta_1, \dots, \beta_s$ are positive real roots such that $a(\kappa_i(\lambda)|\beta_i^\vee) \in \mathbb{Z}$ and $\ell(\kappa_i) - 1 = \ell(\kappa_{i-1})$. For each $\lambda \in P_+$, A Lakshmibai–Seshadri path of shape λ is defined to be a W -path of shape λ $\pi_{(\underline{\tau}, \underline{a})}^\lambda$ such that there exists an a_i -chain for the pair (τ_i, τ_{i+1}) for all $i = 1, \dots, r-1$. The set of all Lakshmibai–Seshadri paths of shape λ is denoted by $\wp_\lambda$.

Lemma 1.5 (Lemma 3.5 and Proposition 4.2, [Lit94]). *We have $\wp_\lambda \subset \wp$ and the set of $\{0\} \cup \wp_\lambda$ is stable under the action of operators e_α and f_α . Moreover, for each $\pi \in \wp_\lambda$, we have $\min_{t \in [0, 1]} \{h_\alpha(t)\}$ is Q .*

Theorem 1.6 (Character formula, [Lit94]). *Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra and $\lambda \in P_+$. Then the set $\wp_\lambda$ is equal to the set of all paths π of the form*

$$\pi = f_{\alpha_1} \circ f_{\alpha_2} \circ \cdots \circ f_{\alpha_s}(\pi_\lambda),$$

where $\alpha_1, \dots, \alpha_s$ are simple roots and π_λ is the straight line in $P_{\mathbb{R}}$ connecting 0 with λ . Furthermore,

$$\text{ch}_\lambda = \sum_{\pi \in \wp_\lambda} e_{\pi(1)}.$$

2 The main results

Let $\mathfrak{g}$ be an affine Kac–Moody algebra and A the generalized Cartan matrix which defines $\mathfrak{g}$. Fix $u \in \mathbb{Z}_{>0}$ relatively prime to a_0 in the null vector a of A . Let $\mathfrak{g}[u]$ be the corresponding winding subalgebra. For each $k \in \mathbb{N}$, let P_+^k be the set of all dominant integral weights of level k . For each integrable irreducible highest weight $\mathfrak{g}$ -module $L(\lambda)$ of level $k > 0$, we can decompose it into a direct sum of integrable irreducible highest weight modules $\dot{L}(\lambda')$ of level uk (since $\dot{K} = uK$). Each $\dot{L}(\lambda')$ appears in $L(\lambda)$ with finite multiplicity, which we denote by $c_\lambda^{\lambda'}$

$$L(\lambda) = \bigoplus_{\lambda' \in \dot{P}_+^{uk}} c_\lambda^{\lambda'} \dot{L}(\lambda').$$

A path $\pi \in \wp_\lambda$ is called $\mathfrak{g}[u]$ -dominant if its image is contained in the dominant Weyl chamber of the root system of $\mathfrak{g}[u]$. We now obtain a branching rule for $\mathfrak{g}[u] \subset \mathfrak{g}$ in the following theorem.

Theorem 2.1. Let $\mathfrak{g}$ be an affine Kac–Moody Lie algebra and $\mathfrak{g}[u]$ be a winding subalgebra of $\mathfrak{g}$. Then for any $\lambda \in P_+$, we have

$$L(\lambda) = \bigoplus_{\pi} \dot{L}(\pi(1)),$$

where the direct sum ranges over all $\mathfrak{g}[u]$ -dominant paths in $\wp_\lambda$. In particular, for each $\lambda' \in \dot{P}_+$ the multiplicity $c_\lambda^{\lambda'}$ is equal to the number of $\mathfrak{g}[u]$ -dominant paths in $\wp_\lambda$ with end point λ' .

Proof. By using Kac’s character formula for $\dot{\text{ch}}_{\lambda'}$, the equality

$$\text{ch}_\lambda = \sum_{\lambda' \in \dot{P}_+^{uk}} c_\lambda^{\lambda'} \dot{\text{ch}}_{\lambda'}$$

becomes

$$\sum_{\mu} m_\lambda(\mu) \sum_{\sigma \in \dot{W}} \epsilon(\sigma) e_{\mu + \sigma(\dot{\rho})} = \sum_{\lambda' \in \dot{P}_+^{uk}} c_\lambda^{\lambda'} \sum_{\sigma \in \dot{W}} \epsilon(\sigma) e_{\sigma(\lambda' + \dot{\rho})}. \quad (1)$$

Replace μ by $\sigma(\mu)$, then the left-hand side of the equation (1) does not change. So we have

$$\sum_{\mu} m_\lambda(\mu) \sum_{\sigma \in \dot{W}} \epsilon(\sigma) e_{\sigma(\mu + \dot{\rho})} = \sum_{\lambda' \in \dot{P}_+^{uk}} c_\lambda^{\lambda'} \sum_{\sigma \in \dot{W}} \epsilon(\sigma) e_{\sigma(\lambda' + \dot{\rho})}. \quad (2)$$

We change the variable both sides of the equation (2) by putting $\eta = \sigma(\mu + \dot{\rho}) - \dot{\rho}$ on the left, $\eta = \sigma(\lambda' + \dot{\rho}) - \dot{\rho}$ on the right, and then replacing σ^{-1} by σ to obtain

$$\sum_{\eta} \sum_{\sigma \in \dot{W}} \epsilon(\sigma) m_\lambda(\sigma(\eta + \dot{\rho}) - \dot{\rho}) e_{\eta + \dot{\rho}} = \sum_{\eta} \sum_{\sigma \in \dot{W}} \epsilon(\sigma) c_\lambda^{\sigma(\eta + \dot{\rho}) - \dot{\rho}} e_{\eta + \dot{\rho}}. \quad (3)$$

As a weight of $\mathfrak{g}[u]$, if η is dominant then $\eta + \dot{\rho}$ is strictly dominant. Then $\sigma(\eta + \dot{\rho}) - \dot{\rho}$ is not dominant unless $\sigma = 1$. Hence $c_\lambda^{\sigma(\eta + \dot{\rho}) - \dot{\rho}} = 0$ unless $\sigma = 1$. So, if we restrict both sides of the equation (3) over $\eta \in \dot{P}_+$ and then delete $e_{\dot{\rho}}$ both sides we get

$$\sum_{\eta \in \dot{P}_+} \sum_{\sigma \in \dot{W}} \epsilon(\sigma) m_\lambda(\sigma(\eta + \dot{\rho}) - \dot{\rho}) e_\eta = \sum_{\eta \in \dot{P}_+} c_\lambda^\eta e_\eta. \quad (4)$$

We see that if there exist $\sigma \in \dot{W}$ such that $\sigma(\mu + \dot{\rho}) - \dot{\rho}$ is a dominant weight then σ and η are uniquely determined by μ . Then we set $p(\mu) := \epsilon(\sigma)$ and $\{\mu\} := \eta$. In the case there does not exist $\sigma \in \dot{W}$ satisfying above condition, we set $p(\mu) := 0$ and $\{\mu\} := 0$. By Theorem 1.6, each weight μ of $L(\lambda)$ is an endpoint $\pi(1)$ of some path $\pi \in \wp_\lambda$. Furthermore, the multiplicity $m_\lambda(\mu)$ is equal to the number of Lakshmibai–Seshadri paths of shape λ with endpoint μ . So, in the language of paths, we can rewrite the equation (4) as

$$\sum_{\pi \in \wp_\lambda} p(\pi(1)) e_{\{\pi(1)\}} = \sum_{\eta \in \dot{P}_+} c_\lambda^\eta e_\eta.$$

By character formula in Theorem 1.6, we obtain

$$\sum_{\pi \in \wp_\lambda} p(\pi(1)) \dot{\text{ch}}_{\{\pi(1)\}} = \sum_{\eta \in \dot{P}_+} c_\lambda^\eta \dot{\text{ch}}_\eta. \quad (5)$$

Let Ω be the set of all π in $\wp_\lambda$ such that π is $\mathfrak{g}[u]$ -dominant and let Ω' be the completion of Ω in $\wp_\lambda$. We see that for each $\pi \in \Omega$, $\pi(1)$ is dominant then $p(\pi(1)) = 1$ and $\{\pi(1)\} = \pi(1)$. By Lemma 2.2 below, we show that for each $\pi \in \Omega'$, there exist $\bar{\pi} \in \Omega'$ such that $p(\bar{\pi}(1)) = -p(\pi(1))$ and $\{\bar{\pi}(1)\} = \{\pi(1)\}$. Hence by equation (5) we have

$$\sum_{\pi \in \Omega} \dot{\text{ch}}_{\pi(1)} = \sum_{\eta \in \dot{P}_+} c_\lambda^\eta \dot{\text{ch}}_\eta.$$

It implies that the set of all $\eta \in \dot{P}_+^{uk}$ such that $\dot{L}(\eta) \subset L(\lambda)$ as $\mathfrak{g}[u]$ -modules is the set of endpoints of paths in Ω . The multiplicity c_λ^η is then equal to the number of paths in Ω with endpoint η . $\square$

Lemma 2.2. *With the notation in the proof of Theorem 2.1. For each $\pi \in \Omega'$, there exist $\bar{\pi} \in \Omega'$ such that $p(\bar{\pi}(1)) = -p(\pi(1))$ and $\{\bar{\pi}(1)\} = \{\pi(1)\}$.*

Proof. For each $\pi \in \Omega'$, there exist $t \in [0, 1]$ such that $\pi(t)$ is not contained in the dominant Weyl chamber of $\mathfrak{g}[u]$. It means that there exist a simple root α of $\mathfrak{g}[u]$ and $t \in [0, 1]$ such that $(\pi(t)|\alpha^\vee) < 0$. By Lemma 1.5, $\min_{t \in [0, 1]} \{(\pi(t)|\alpha^\vee)\} = Q$ is an integer, we can choose s to be the minimum value of $t \in [0, 1]$ and a simple root α of $\mathfrak{g}[u]$ such that $(\pi(s)|\alpha^\vee) = -1$. Let M_π be the subset of $\wp_\lambda$ of all paths π' such that $\pi'(zt) = \pi(st)$ for some $z \in [0, 1]$. Clearly that $M_\pi \subset \Omega'$ for each $\pi \in \Omega'$. By the minimum value of s , two sets $M_\pi, M_{\pi'}$ are the same set or have no common path for each $\pi, \pi' \in \Omega'$. Hence Ω' is the disjoint union of some subsets M_π for $\pi \in \Omega'$.

Fix a path $\pi' \in \Omega'$. We need to show that for each $\pi \in M_{\pi'}$, we can find a path $\bar{\pi}$ in $M_{\pi'}$ that satisfies the statement of the lemma. Or equivalently, for each $(\pi, \sigma) \in M_{\pi'} \times \dot{W}$ such that $\sigma(\pi(1) + \dot{\rho}) - \dot{\rho} \in \dot{P}_+$, there exist $(\bar{\pi}, \bar{\sigma}) \in M_{\pi'} \times \dot{W}$ such that $\epsilon(\sigma) = -\epsilon(\bar{\sigma})$ and $\sigma(\pi(1) + \dot{\rho}) = \bar{\sigma}(\bar{\pi}(1) + \dot{\rho})$. This can be done by choosing $\bar{\sigma} = \sigma s_\alpha$ and $\bar{\pi} = f_\alpha^{1+(\pi(1)|\alpha^\vee)}(\pi)$. Indeed, we have $f_\alpha(M_\pi) \subset M_\pi \cup \{0\}$. Since $(\pi(1)|\alpha^\vee) + 1 \leq (\pi(1)|\alpha^\vee) - Q$, we have $\bar{\pi} \neq 0$ by Lemma 1.4. Finally, $\sigma(\pi(1) + \dot{\rho}) = \bar{\sigma}(\bar{\pi}(1) + \dot{\rho})$ by Lemma 1.3. $\square$

Theorem 2.3. *Let $\mathfrak{g}$ be an affine Kac–Moody algebra and let $\mathfrak{g}[u]$ be a winding subalgebra of $\mathfrak{g}$. Then for any $\lambda \in P_+^k$ and $\lambda' \in \dot{P}_+^{uk}$, we have*

$$c_\lambda^{\lambda'} = \sum_{\sigma \in \dot{W}} \sum_{\tau \in \dot{W}} \epsilon(\sigma\tau) \mathcal{P}(\sigma(\lambda + \rho) + \tau(\dot{\rho}) - (\lambda' + \rho + \dot{\rho})).$$

Proof. By Kac's character formula, the equality

$$\text{ch}_\lambda = \sum_{\lambda' \in \dot{P}_+^{uk}} c_\lambda^{\lambda'} \text{ch}_{\lambda'}$$

becomes

$$\text{ch}_\lambda \sum_{\sigma \in \dot{W}} \epsilon(\sigma) e_{\sigma(\dot{\rho})} = \sum_{\lambda' \in \dot{P}_+^{uk}} c_\lambda^{\lambda'} \sum_{\sigma \in \dot{W}} \epsilon(\sigma) e_{\sigma(\lambda' + \dot{\rho})}. \quad (6)$$

Now we use Theorem 1.2, to rewrite ch_λ in the left-hand side of equation (6), we get

$$\sum_{\mu} \sum_{\tau \in \dot{W}} \sum_{\sigma \in \dot{W}} \epsilon(\tau\sigma) \mathcal{P}(\tau(\lambda + \rho) - (\mu + \rho)) e_{\sigma(\dot{\rho}) + \mu} = \sum_{\lambda' \in \dot{P}_+^{uk}} c_\lambda^{\lambda'} \sum_{\sigma \in \dot{W}} \epsilon(\sigma) e_{\sigma(\lambda' + \dot{\rho})}. \quad (7)$$

We change the variable both sides of the equation (7) by putting $\eta = \sigma(\dot{\rho}) + \mu - \dot{\rho}$ on the left, $\eta = \sigma(\lambda' + \dot{\rho}) - \dot{\rho}$ on the right and then replacing σ^{-1} on the right-hand side by σ to obtain

$$\sum_{\eta} \sum_{\tau \in \dot{W}} \sum_{\sigma \in \dot{W}} \epsilon(\tau\sigma) \mathcal{P}(\tau(\lambda + \rho) + \sigma(\dot{\rho}) - (\eta + \rho + \dot{\rho})) e_{\eta + \dot{\rho}} = \sum_{\eta} \sum_{\sigma \in \dot{W}} \epsilon(\sigma) c_\lambda^{\sigma(\eta + \dot{\rho}) - \dot{\rho}} e_{\eta + \dot{\rho}}. \quad (8)$$

As a weight of $\mathfrak{g}[u]$, if η is dominant then $\eta + \dot{\rho}$ is strictly dominant. Then $\sigma(\eta + \dot{\rho}) - \dot{\rho}$ is not dominant unless $\sigma = 1$. Hence, $c_\lambda^{\sigma(\eta + \dot{\rho}) - \dot{\rho}} = 0$ unless $\sigma = 1$. So, if we restrict both sides of the equation (8) over $\eta \in \dot{P}_+$ then we get

$$\sum_{\eta \in \dot{P}_+} \sum_{\tau \in \dot{W}} \sum_{\sigma \in \dot{W}} \epsilon(\tau\sigma) \mathcal{P}(\tau(\lambda + \rho) + \sigma(\dot{\rho}) - (\eta + \rho + \dot{\rho})) e_{\eta + \dot{\rho}} = \sum_{\eta \in \dot{P}_+} c_\lambda^\eta e_{\eta + \dot{\rho}}.$$

Note that η in both sides belong to $\dot{P}_+^{uk}$. It implies the formula of c_λ^η in terms of Weyl groups and generalized Konstant's partition function for any λ in P_+^k and $\eta \in \dot{P}_+^{uk}$. $\square$

References

- [Car05] R. W. Carter. *Lie algebras of finite and affine type*, volume 96 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2005.
 - [Fre06] Igor B Frenkel. Representations of affine Lie algebras, Hecke modular forms and Korteweg—De vries type equations. In *Lie Algebras and Related Topics: Proceedings of a Conference Held at New Brunswick, New Jersey, May 29–31, 1981*, pages 71–110. Springer, 2006.
 - [KW90] V. G. Kac and M. Wakimoto. Branching functions for winding subalgebras and tensor products. *Acta Appl. Math.*, 21(1-2):3–39, 1990.
 - [Lit94] Peter Littelmann. A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras. *Invent. Math.*, 116(1-3):329–346, 1994.
 - [Ngu20] Duc-Khanh Nguyen. Branching problem on winding subalgebras of affine Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$. *arXiv preprint arXiv:1911.03316*, 2020.
 - [Ste61] Robert Steinberg. A general Clebsch-Gordan theorem. *Bull. Amer. Math. Soc.*, 67:406–407, 1961.
 - [Wak01] Minoru Wakimoto. *Lectures on infinite-dimensional Lie algebra*. World Scientific Publishing Co., Inc., River Edge, NJ, 2001.
- Okinawa Institute of Science and Technology, Onna-son, Okinawa, Japan 904-0495
 E-mail: khanh.mathematic@gmail.com