

A skew group ring and the universal Bannai–Ito algebra

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Abstract

This article provides an expository account of recent results concerning the relationship between a skew group ring over $\mathbb{Z}/2\mathbb{Z}$ of $U(\mathfrak{sl}_2)$ and the universal Bannai–Ito algebra $\mathfrak{BI}$. Emphasis is placed on the conceptual framework and the interplay between representation theory and algebraic combinatorics. In particular, we elucidate how certain eigenspaces of modules over the tensor product of this skew group ring naturally carry $\mathfrak{BI}$ -module structures, and how odd graphs provide concrete combinatorial realizations of these phenomena.

1 Introduction

The relationship between representation theory and algebraic combinatorics has long been a fruitful source of deep mathematical structures. A classical example is the appearance of $\mathfrak{sl}_2$ in the study of hypercubes and Terwilliger algebras [4, 5, 6], notably as explored in [1]. The purpose of this article is to survey a circle of ideas centered around a skew group ring over $\mathbb{Z}/2\mathbb{Z}$ of $U(\mathfrak{sl}_2)$, its representation theory, and its role in connecting the universal Bannai–Ito algebra $\mathfrak{BI}$ with Leonard triples arising from combinatorial objects such as odd graphs.

The material presented here is based on a talk given at the workshop “Confluence of Combinatorics and Representation Theory” held at the Research Institute for Mathematical Sciences (RIMS), Kyoto University, from October 21 to October 24, 2025. Detailed proofs of the results discussed herein will appear in [3]; our aim here is instead to highlight the main conceptual ideas and structural insights.

2 Skew group rings and Hopf algebras

Let R be a ring with identity and let G be a finite group. Suppose $\varphi : G \rightarrow \text{Aut}(R)$ is a group homomorphism. The *skew group ring* $R \rtimes_{\varphi} G$ consists of all formal sums

$$\sum_{g \in G} a_g g, \quad a_g \in R,$$

with multiplication defined by

$$(a_1 g_1) \cdot (a_2 g_2) = a_1 \varphi(g_1)(a_2) g_1 g_2.$$

When R is an algebra over a field $\mathbb{F}$, the skew group ring $R \rtimes G$ inherits a natural $\mathbb{F}$ -algebra structure.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$. Its universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra with comultiplication Δ , counit ε , and antipode S satisfying

$$\Delta(u) = u \otimes 1 + 1 \otimes u, \quad \varepsilon(u) = 0, \quad S(u) = -u \quad (u \in \mathfrak{g}).$$

Suppose a finite group G acts on $\mathfrak{g}$ by Lie algebra automorphisms. This action extends uniquely to an action on $U(\mathfrak{g})$ by algebra automorphisms, which enables us to define the skew group ring

$$U(\mathfrak{g})_G := U(\mathfrak{g}) \rtimes G.$$

In this setting, $U(\mathfrak{g})_G$ naturally possesses a Hopf algebra structure, where the structure maps are induced by those of $U(\mathfrak{g})$ and the group algebra $\mathbb{C}[G]$.

3 The algebra $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$

Recall that $\mathfrak{sl}_2$ is the Lie algebra over $\mathbb{C}$ with the standard basis E, F, H satisfying

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

There is an involutive automorphism ρ of $\mathfrak{sl}_2$ defined by

$$\rho(E) = F, \quad \rho(F) = E, \quad \rho(H) = -H.$$

Let $\mathbb{Z}/2\mathbb{Z} = \langle \rho \rangle$. According to [3, Theorem 3.3], the corresponding skew group ring

$$U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$$

is the algebra over $\mathbb{C}$ generated by E, F, H, ρ subject to the above relations along with

$$\rho^2 = 1, \quad \rho E = F\rho, \quad \rho H = -H\rho.$$

3.1 Finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ -modules

For each integer $n \geq 0$, let L_n denote the $(n+1)$ -dimensional irreducible $U(\mathfrak{sl}_2)$ -module. According to [3, Equation (36)], the action of ρ can be used to extend L_n to a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ -module in two non-isomorphic ways, denoted by L_n^+ and L_n^- .

A fundamental result established in [3, Theorems 5.6 and 5.7] is that these modules exhaust all finite-dimensional irreducible $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ -modules, and that every finite-dimensional module is completely reducible. The proofs rely on the classical representation theory of $\mathfrak{sl}_2$ along with elementary arguments concerning the action of ρ .

4 The algebra $\mathfrak{B}\mathfrak{I}$ and Leonard triples

The *universal Bannai–Ito algebra* $\mathfrak{B}\mathfrak{I}$ is the algebra over $\mathbb{C}$ generated by X, Y, Z subject to the condition that the elements

$$\{X, Y\} - Z, \quad \{Y, Z\} - X, \quad \{Z, X\} - Y$$

are central in $\mathfrak{BJ}$. As established in [2], the finite-dimensional irreducible $\mathfrak{BJ}$ -modules admit explicit matrix realizations, with their structures depending delicately on a set of scalar parameters.

The study of these modules is intimately connected to the concept of *Leonard triples*. Recall that a Leonard triple consists of three linear operators on a finite-dimensional vector space such that, for each operator, there exists a basis in which it acts diagonally while the other two act as irreducible tridiagonal matrices. Within the framework of $\mathfrak{BJ}$, the generators X, Y, Z realize a Leonard triple on a given finite-dimensional irreducible module if and only if they are all diagonalizable.

As a substantial generalization of Leonard pairs, these triples play a fundamental role in the study of Terwilliger algebras and the theory of orthogonal polynomials.

5 From $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}^{\otimes 2}$ -modules to $\mathfrak{BJ}$ -modules

Let V be a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}^{\otimes 2}$ -module. For any $\theta \in \mathbb{C}$ let $V(\theta)$ denote the eigenspace

$$V(\theta) = \{v \in V \mid \Delta(H)v = \theta v\}.$$

A central observation is that the eigenspace $V(1)$ naturally carries a $\mathfrak{BJ}$ -module structure, where the generators X, Y, Z are expressed in terms of the actions of $\mathfrak{sl}_2$ and ρ [3, Theorem 2.1]. For a finite-dimensional irreducible $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}^{\otimes 2}$ -module V , the nonvanishing of $V(1)$ is equivalent to the irreducibility of this $\mathfrak{BJ}$ -module and implies the emergence of a Leonard triple [3, Theorem 2.2].

6 A combinatorial realization via odd graphs

Odd graphs provide a concrete realization of the theoretical framework developed above. Let O_{d+1} denote the odd graph with vertex set $\binom{\Omega}{d}$, where $|\Omega| = 2d + 1$. The Terwilliger algebra associated with O_{d+1} naturally admits a $\mathfrak{BJ}$ -module structure [3, Theorem 2.3(i)].

Through an explicit correspondence, the $\mathfrak{BJ}$ -module arising from $V(1)$ is identified with the standard module over the Terwilliger algebra of O_{d+1} . Furthermore, the generators X, Y, Z are shown to act as a Leonard triple on each irreducible component [3, Theorem 2.3(ii)].

7 Concluding remarks

The skew group ring $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ serves as a unifying algebraic framework that links representation theory, special functions, and combinatorial structures. The appearance of $\mathfrak{BJ}$ and Leonard triples highlights the rich interplay between these areas, while odd graphs provide an especially transparent realization of the theory.

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