

DUALITY-PRESERVING EXTENSIONS OF THE WORLEY–SAGAN INSERTION AND HAIMAN’S MIXED INSERTION FOR THE HYPEROCTAHEDRAL GROUP

東京大学大学院数理科学研究科 中桐 正人

MASATO NAKAGIRI

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES,
THE UNIVERSITY OF TOKYO

ABSTRACT. The Worley–Sagan insertion and Haiman’s mixed insertion are insertion algorithms on shifted Young tableaux; each yields a Robinson–Schensted-type correspondence between the symmetric group S_n and pairs of shifted Young tableaux of the same shape with n cells. It is well known that these two insertions are dual: if a permutation π corresponds to (P, Q) via the Worley–Sagan insertion, then π^{-1} corresponds to (Q, P) via Haiman’s mixed insertion. In this talk we present extensions of these insertions that preserve this duality. Our extensions map signed permutations to pairs of shifted tableaux. Note that our extension of the Worley–Sagan insertion differs from restricting Sagan’s “Knuth version” to signed permutations. To prove duality for our extensions, we embed them into the doubly mixed insertion introduced by Shimozono and White on unshifted tableaux via a doubling construction for shifted tableaux, and then appeal to the self-duality of the doubly mixed insertion established by Shimozono and White. The contents of this article are based on [N].

1. BASIC NOTIONS

We first establish the fundamental notations and the alphabet used throughout this paper. Let $\mathbb{P} = \{1, 2, 3, \dots\}$ be the set of positive integers, and let $\bar{\mathbb{P}} = \{\bar{1}, \bar{2}, \bar{3}, \dots\}$ be the set of barred integers. The set $\mathbb{P} \cup \bar{\mathbb{P}}$ is endowed with the following total ordering:

$$\bar{1} < 1 < \bar{2} < 2 < \bar{3} < 3 < \dots$$

We refer to an element of $\mathbb{P}$ as an **unbarred letter** and an element of $\bar{\mathbb{P}}$ as a **barred letter**. Furthermore, we define the absolute value operator $|\cdot|: \mathbb{P} \cup \bar{\mathbb{P}} \rightarrow \mathbb{P}$, which removes the bars, as follows: for any element $\alpha \in \mathbb{P} \cup \bar{\mathbb{P}}$,

$$|\alpha| := \begin{cases} a & \text{if } \alpha = a \in \mathbb{P}, \\ a & \text{if } \alpha = \bar{a} \in \bar{\mathbb{P}}. \end{cases}$$

Definition 1.1. Let $n \in \mathbb{P}$. A sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is called a *distinct partition* of n if it satisfies $\lambda_1 > \lambda_2 > \dots > \lambda_l > 0$ and $\sum_{i=1}^l \lambda_i = n$. We denote by $\mathcal{DP}(n)$ the set of all distinct partitions of n .

Definition 1.2. For a distinct partition $\lambda \in \mathcal{DP}(n)$, the *shifted Young diagram* $D'(\lambda)$ is defined as the set of cells:

$$D'(\lambda) := \{(i, j) \in \mathbb{P} \times \mathbb{P} \mid i \leq j \leq i + \lambda_i - 1, 1 \leq i \leq l\}.$$

Each element $(i, j) \in D'(\lambda)$ is called a *cell*. A cell of the form (i, i) is specifically referred to as a *diagonal cell*.

Definition 1.3. Let $\lambda \in \mathcal{DP}(n)$. The set $\text{ST}_n(\lambda)$ of *shifted standard tableaux* of shape λ consists of all bijections $T : D'(\lambda) \rightarrow \{1, 2, \dots, n\}$ such that the entries are strictly increasing along each row and each column.

Furthermore, we introduce barred versions of these tableaux as follows:

Definition 1.4. Let $T \in \text{ST}_n(\lambda)$.

- (1) We define $\widetilde{\text{ST}}_n(\lambda)$ as the set of tableaux obtained by adding bars to some off-diagonal entries of T . Note that the diagonal entries must remain unbarred.
- (2) We define $\widetilde{\text{ST}}'_n(\lambda)$ as the set of tableaux obtained by adding bars to any entries of T , including diagonal entries.

In both cases, the entries must respect the total ordering on $\mathbb{P}'$.

Example 1.5. Consider the distinct partition $\lambda = (3, 2, 1) \in \mathcal{DP}(6)$. Examples of elements from each of the sets defined above are shown below:

- An element $T \in \text{ST}_6(\lambda)$:

1	2	4
	3	5
		6

- An element $\tilde{T} \in \widetilde{\text{ST}}_6(\lambda)$:

1	2	$\bar{4}$
	3	5
		6

- An element $\tilde{T}' \in \widetilde{\text{ST}}'_6(\lambda)$:

$\bar{1}$	2	4
	$\bar{3}$	5
		$\bar{6}$

2. WORLEY–SAGAN INSERTION AND HAIMAN’S MIXED INSERTION

2.1. Worley–Sagan insertion. The Worley–Sagan insertion (WS) [Sa, Wo] provides a fundamental correspondence between permutations and pairs of shifted tableaux.

Proposition 2.1 ([Sa, Wo]). *The Worley–Sagan insertion establishes a bijection:*

$$S_n \rightarrow \coprod_{\lambda \in \mathcal{DP}(n)} \text{ST}_n(\lambda) \times \widetilde{\text{ST}}_n(\lambda)$$

Specifically, it maps each signed permutation $\pi \in S_n$ to a pair of same-shape tableaux $(P_{\text{WS}}(\pi), Q_{\text{WS}}(\pi))$.

2.2. Worley–Sagan insertion. Consider a permutation $\pi \in S_n$ written in two-line notation as:

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}.$$

The Worley–Sagan insertion tableau $P_{\text{WS}}(\pi)$ is defined by the following iterative process:

$$P_{\text{WS}}(\pi) := (\dots ((\emptyset \xleftarrow{\text{WS}} a_1) \xleftarrow{\text{WS}} a_2) \dots) \xleftarrow{\text{WS}} a_n.$$

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The symbol $\xleftarrow{\text{WS}}$ denotes the operation of inserting a letter into a shifted tableau according to the Worley–Sagan algorithm. Specifically, $T \xleftarrow{\text{WS}} a$ represents the tableau obtained by inserting the letter a into T . A crucial feature of this insertion is the rule for diagonal cells: if an entry in a diagonal cell (i, i) is bumped, the insertion mode switches from row insertion to column insertion for the remainder of that step.

The recording tableau $Q_{\text{WS}}(\pi)$ is constructed by starting from an empty shifted shape and, at each step k , labeling the new cell added by the operation $\xleftarrow{\text{WS}} a_k$ with either k or $\bar{k}$. If the insertion concludes with a row bump, we use the label k ; if it concludes with a column bump, we use $\bar{k}$. Consequently, $Q_{\text{WS}}(\pi)$ belongs to $\widetilde{\text{ST}}_n(\lambda)$.

2.3. Haiman’s mixed insertion. Haiman’s mixed insertion (HM) [H] provides a counterpart to the WS algorithm, where the roles of the standard and primed tableaux in the codomain are effectively swapped.

Proposition 2.2. [H] *Haiman’s mixed insertion establishes a bijection:*

$$S_n \xrightarrow{\sim} \coprod_{\lambda \in \mathcal{DP}(n)} \widetilde{\text{ST}}_n(\lambda) \times \text{ST}_n(\lambda).$$

Specifically, it maps each signed permutation $\pi \in S_n$ to a pair of same-shape tableaux $(P_{\text{HM}}(\pi), Q_{\text{HM}}(\pi))$.

If $\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \in S_n$, the insertion tableau $P_{\text{HM}}(\pi)$ is defined iteratively as:

$$P_{\text{HM}}(\pi) := (\dots ((\emptyset \xleftarrow{\text{HM}} a_1) \xleftarrow{\text{HM}} a_2) \dots) \xleftarrow{\text{HM}} a_n,$$

where $\xleftarrow{\text{HM}}$ represents the mixed insertion operation. The fundamental rule of the HM insertion is that the mode of insertion is determined by the barring of the bumped letter: bumped unbarred letters are inserted into the next **row**, while bumped barred letters are inserted into the next **column**. Furthermore, if a letter in a diagonal cell is bumped, the entry is primed and then inserted into the next column.

The recording tableau $Q_{\text{HM}}(\pi)$ is obtained by placing the label k in the cell added during the k -th insertion step a_k . Note that the labels in $Q_{\text{HM}}(\pi)$ are always unbarred, hence $Q_{\text{HM}}(\pi) \in \text{ST}_n(\lambda)$.

The following proposition describes the fundamental relationship between these two algorithms.

Proposition 2.3 (Duality between the Worley–Sagan and Haiman mixed insertions [H]). *For any permutation $\pi \in S_n$, we have:*

$$(P_{\text{WS}}(\pi), Q_{\text{WS}}(\pi)) = (Q_{\text{HM}}(\pi^{-1}), P_{\text{HM}}(\pi^{-1})).$$

3. EXTENSIONS OF WORLEY–SAGAN INSERTION AND HAIMAN’S MIXED INSERTION FOR SIGNED PERMUTATIONS

The set of signed permutations B_n can be identified with the set of two-line arrays of the form:

$$B_n = \left\{ \begin{pmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} \mid \alpha_k \in \mathbb{P} \cup \bar{\mathbb{P}}, \{|\alpha_1|, |\alpha_2|, \dots, |\alpha_n|\} = \{1, 2, \dots, n\} \right\}.$$

The author extended the Worley–Sagan insertion and Haiman’s mixed insertion to the elements of B_n in a manner that preserves the duality between the two algorithms. Since

the extension of the Worley–Sagan insertion incorporates the characteristic features of mixed insertion, we refer to it as the **Worley–Sagan mixed insertion (WSM)**. Furthermore, we call the extension of Haiman’s mixed insertion the **generalized mixed insertion (GM)**.

These two extended algorithms are defined based on two types of bumping procedures:

- **ordinary mixed bumping**: In this procedure, a bumped **unbarred** letter is inserted into the next **row**, while a bumped **barred** letter is inserted into the next **column**.
- **dual mixed bumping**: In this procedure, a bumped **unbarred** letter is inserted into the next **column**, while a bumped **barred** letter is inserted into the next **row**.

3.1. Worley–Sagan mixed insertion. The Worley–Sagan mixed insertion (WSM) [N] is an extension of the Worley–Sagan insertion for signed permutations, incorporating the mechanics of mixed insertion.

Proposition 3.1 ([N]). *The Worley–Sagan mixed insertion establishes a bijection:*

$$B_n \xrightarrow{\sim} \coprod_{\lambda \in \mathcal{DP}(n)} \widetilde{ST}'_n(\lambda) \times \widetilde{ST}_n(\lambda).$$

Specifically, it maps each signed permutation $\pi \in B_n$ to a pair of same-shape tableaux $(P_{\text{WSM}}(\pi), Q_{\text{WSM}}(\pi))$.

The insertion tableau $P_{\text{WSM}}(\pi)$ is obtained by the following iterative process:

$$P_{\text{WSM}}(\pi) := (\dots ((\emptyset \xleftarrow{\text{WSM}} \alpha_1) \xleftarrow{\text{WSM}} \alpha_2) \dots) \xleftarrow{\text{WSM}} \alpha_n,$$

where $\xleftarrow{\text{WSM}}$ denotes the Worley–Sagan mixed insertion operation. The rules for the operation $\xleftarrow{\text{WSM}} \alpha$ are described as follows.

The insertion process begins based on whether the letter α is barred: if α is an **unbarred** letter, the insertion starts with **ordinary** mixed bumping; conversely, if α is a **barred** letter, the insertion starts with **dual** mixed bumping.

A distinctive feature of the $\xleftarrow{\text{WSM}}$ operation is that the insertion rule (i.e., whether to follow **ordinary** or **dual** mixed bumping) may **switch** during the process. We switch the current rule if and only if one of the following situations occurs:

- (1) The current rule is **ordinary** mixed bumping and a diagonal **unbarred** letter β is bumped. In this case, the current rule is switched to **dual** mixed bumping, and β is inserted into the next column.
- (2) The current rule is **dual** mixed bumping and a diagonal **barred** letter β is bumped. In this case, the current rule is switched to **ordinary** mixed bumping, and β is inserted into the next column.

Once the current rule is switched to **ordinary** [resp. **dual**] mixed bumping, it remains in effect until the situation (1) [resp. (2)] occurs again, or the process terminates by placing a letter in an empty cell.

The recording tableau $Q_{\text{WSM}}(\pi)$ is defined by recording the labels at each step $k \in \{1, \dots, n\}$. Let $\xleftarrow{\text{WSM}} \alpha_k$ be the k -th insertion step. If the operation $\xleftarrow{\text{WSM}} \alpha_k$ terminates under **ordinary** mixed bumping, we label the newly added cell with the **unbarred label** k . On the other hand, if $\xleftarrow{\text{WSM}} \alpha_k$ terminates under **dual** mixed bumping, we label the newly

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added cell with the **barred label** $\bar{k}$. By this construction, it follows that $Q_{\text{WSM}}(\pi) \in \widetilde{\text{ST}}_n(\lambda)$.

3.2. Generalized mixed insertion. The generalized mixed insertion (GM) [N] is an extension of Haiman's mixed insertion to signed permutations. Unlike the Worley–Sagan mixed insertion, this algorithm does not switch between ordinary and dual mixed bumping during the insertion process; instead, it occasionally alters the barring of a letter.

Proposition 3.2 ([N]). *The generalized mixed insertion establishes a bijection:*

$$B_n \xrightarrow{\sim} \coprod_{\lambda \in \mathcal{DP}(n)} \widetilde{\text{ST}}_n(\lambda) \times \widetilde{\text{ST}}_n'(\lambda).$$

Specifically, it maps each signed permutation $\pi \in B_n$ to a pair of same-shape tableaux $(P_{\text{GM}}(\pi), Q_{\text{GM}}(\pi))$.

For a signed permutation $\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix} \in B_n$, the insertion tableau $P_{\text{GM}}(\pi)$ is defined iteratively by:

$$P_{\text{GM}}(\pi) := (\dots ((\emptyset \xleftarrow{\text{GM}} \alpha_1) \xleftarrow{\text{GM}} \alpha_2) \dots) \xleftarrow{\text{GM}} \alpha_n.$$

The mechanism of the operation $\xleftarrow{\text{GM}} \alpha$ is determined solely by whether the incoming letter α is unbarred or barred. As mentioned, the procedure does not permit any switching of the bumping rules midway.

- If α is an **unbarred** letter, the operation $\xleftarrow{\text{GM}} \alpha$ strictly follows **ordinary** mixed bumping from start to finish.
- If α is a **barred** letter, the operation $\xleftarrow{\text{GM}} \alpha$ strictly follows **dual** mixed bumping from start to finish.

Furthermore, the addition or removal of bars occurs exclusively when interacting with diagonal cells. Recall that the insertion tableau of the generalized mixed insertion belongs to $\widetilde{\text{ST}}_n(\lambda)$, meaning its diagonal entries must always be strictly unbarred. To preserve this property, we apply the following rules:

- When α is **unbarred** (proceeding under ordinary mixed bumping): If a diagonal letter $b \in \mathbb{P}$ is bumped, we **bar it** to yield $\bar{b}$, and then insert this $\bar{b}$ into the next column.
- When α is **barred** (proceeding under dual mixed bumping): If an incoming letter bumps an entry in a diagonal cell, this incoming letter must be barred, say $\bar{a}$ with $a \in \mathbb{P}$ (since it arrives via row insertion). In this case, we **remove the bar** from $\bar{a}$ to yield a , and place this unbarred a in the diagonal cell.

4. DUALITY BETWEEN THE WORLEY–SAGAN MIXED INSERTION AND THE GENERALIZED MIXED INSERTION

To prove the duality between the Worley–Sagan mixed insertion (WSM) and the generalized mixed insertion (GM), we employ the method of embedding them into the doubly mixed insertion (DM) introduced by Shimozono and White [Sh–Wh]. The doubly mixed insertion is a self-dual insertion algorithm for unshifted tableaux that establishes the following Robinson–Schensted-type correspondence.

Proposition 4.1 ([Sh–Wh]). *The doubly mixed insertion establishes a bijection:*

$$DC_n \xrightarrow{\sim} \coprod_{\lambda \in \mathcal{P}(n)} \text{UCT}(\lambda) \times \text{UCT}(\lambda).$$

Specifically, it maps each doubly colored biword $\pi \in DC_n$ to a pair of same-shape tableaux $(P_{\text{DM}}(\pi), Q_{\text{DM}}(\pi))$. Here, DC_n denotes the set of all doubly colored biwords of length n , $\mathcal{P}(n)$ is the set of all ordinary (not necessarily distinct) partitions of n , and $\text{UCT}(\lambda)$ represents the set of unshifted colored tableaux of shape λ .

For the precise definitions of doubly colored biwords and colored tableaux, we refer the reader to [Sh–Wh].

A crucial property of the doubly mixed insertion is that it is self-dual; that is, it coincides with its own dual. This symmetry is stated in the following proposition:

Proposition 4.2 (Self-duality of the doubly mixed insertion [Sh–Wh]). *For any doubly colored biword $\pi \in DC_n$, we have*

$$(P_{\text{DM}}(\pi), Q_{\text{DM}}(\pi)) = (Q_{\text{DM}}(\pi^{-1}), P_{\text{DM}}(\pi^{-1})).$$

In order to embed the Robinson-Schensted-type correspondences established by WSM (Proposition 3.1) and GM (Proposition 3.2) into the correspondence given by DM (Proposition 4.1), the author [N] introduced two maps $d_1, d_2: B_n \rightarrow DC_{2n}$ that double the length of the biwords. Furthermore, to construct unshifted tableaux by doubling the number of cells in the shifted tableaux, two additional maps $\mathcal{D}_1: \widetilde{\text{ST}}'_n(\lambda) \rightarrow \text{UCT}(\lambda^D)$ and $\mathcal{D}_2: \widetilde{\text{ST}}_n(\lambda) \rightarrow \text{UCT}(\lambda^D)$ were defined. Here, λ^D denotes the unshifted partition whose k -th part λ_k^D is defined as the sum of the length of the k -th row and the length of the k -th column in the shifted Young diagram $D'(\lambda)$. The idea for this proof is inspired by Ishikawa [I].

Theorem 4.3 ([N]). *The following diagram commutes:*

$$\begin{array}{ccc}
 \coprod_{\lambda \in \mathcal{DP}(n)} \widetilde{\text{ST}}'_n(\lambda) \times \widetilde{\text{ST}}_n(\lambda) & \xrightarrow{\text{sw}} & \coprod_{\lambda \in \mathcal{DP}(n)} \widetilde{\text{ST}}_n(\lambda) \times \widetilde{\text{ST}}'_n(\lambda) \\
 \downarrow \mathcal{D}_1 \times \mathcal{D}_2 & & \downarrow \mathcal{D}_2 \times \mathcal{D}_1 \\
 \begin{array}{ccc}
 \begin{array}{c} \uparrow \text{WSM} \\ B_n \end{array} & \xrightarrow{\text{inv}} & \begin{array}{c} \uparrow \text{GM} \\ B_n \end{array} \\
 \downarrow d_1 & & \downarrow d_2 \\
 \begin{array}{c} (4) \\ DC_{2n} \end{array} & \xrightarrow{\text{inv}} & \begin{array}{c} (5) \\ DC_{2n} \end{array} \\
 \downarrow \text{DM} & & \downarrow \text{DM} \\
 \coprod_{\mu \in \mathcal{P}(n)} \text{UCT}(\mu) \times \text{UCT}(\mu) & \xrightarrow{\text{sw}} & \coprod_{\mu \in \mathcal{P}(n)} \text{UCT}(\mu) \times \text{UCT}(\mu)
 \end{array}
 \end{array}$$

(1) $B_n \rightarrow B_n$, (2) $B_n \rightarrow DC_{2n}$, (3) $DC_{2n} \rightarrow DC_{2n}$, (4) $B_n \rightarrow DC_{2n}$, (5) $DC_{2n} \rightarrow B_n$

Here, WSM, GM, and DM denote the bijections introduced in Propositions 3.1, 3.2, and 4.1, respectively. inv is the map that sends a colored permutation π to its inverse π^{-1} , and sw is the map that swaps the order of the tableau pairs, i.e., $\text{sw}(P, Q) = (Q, P)$. $\mathcal{D}_1 \times \mathcal{D}_2$ is the map that sends (P, Q) in $\widetilde{\text{ST}}'_n(\lambda) \times \widetilde{\text{ST}}_n(\lambda)$ for some $\lambda \in \mathcal{DP}(n)$ to $(\mathcal{D}_1(P), \mathcal{D}_2(Q))$ in $\text{UCT}(\lambda^D) \times \text{UCT}(\lambda^D)$. The definition of $\mathcal{D}_2 \times \mathcal{D}_1$ is similar, only the order of operators is swapped.

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Sketch of proof. The duality between the Worley–Sagan mixed insertion and the generalized mixed insertion is precisely equivalent to the commutativity of square (1) in the diagram above.

First, observe that the outermost rectangle is trivially commutative, which is simply the identity $\text{sw} \circ (\mathcal{D}_1 \times \mathcal{D}_2) = (\mathcal{D}_2 \times \mathcal{D}_1) \circ \text{sw}$. By a standard diagram chase, to prove the commutativity of square (1), it therefore suffices to show that the remaining components (2), (3), (4), and (5) of the diagram commute.

- The commutativity of square (2) follows from a straightforward calculation based on the definitions of the embeddings d_1 and d_2 .
- The commutativity of square (3) is exactly the self-duality of the doubly mixed insertion, which was established in Proposition 4.2.
- The fact that WSM and GM can be embedded into DM corresponds to the commutativity of squares (4) and (5), respectively. The proof of these embeddings constitutes the main result of [N].

Since all these subdiagrams commute, square (1) commutes as well, which completes the proof. $\square$

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