Segre cubic 4-fold

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Abstract: The cyclic triple covering of the projective 4-space with branch the Segre cubic is characterized by 10 cusps as the Segre 3-fold is so by 10 nodes. The Fano variety of lines is birationally equivalent to the Hilbert square of a K3 surface studied by Vinberg(1983). I will discuss the binational automorphism group of this holomorphic symplectic 4-fold.

§1 Main motivation and side job

Main motivation: generalize Aut(K3) to Bir-Aut(HK)
(replacing nef cone with movable cone).

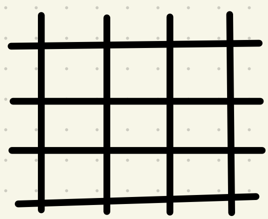
Side job: Kummer theory

(I start with) Kummer quartic surface, $\Phi: A/\pm 1 \xrightarrow{|2\Theta|} P^3$

(1) A= J(C), C: curve of genus 2, Φ : embedding
$(16_6 - 16_6)$ configuration of nodes and tropes

(2) $A = E_1 \times E_2$: Φ is of degree 2 onto a smooth quadric $P^1 \times P^1$

$$\tau^2 = f_4(x)g_4(y)$$



branch locus

Order 3 analogue

(A, σ) : Abelian surface & symplectic automorphism of order 3

Quotient A/σ is a K3 surface with 9 cusps, $9A_2$

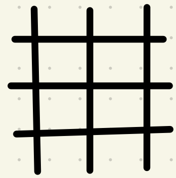
(1) $A=J(C)$, $C: \tau^2 = q(x^3)$, $\sigma: x \mapsto \omega x$, $\omega^3 = 1$

(Barth-)Bertin-Vanhaecke sextic. $A/\sigma \subset \mathbb{P}^4$, Image = (2, 3) c.i.

$(9_4 - 9_4)$ configuration of 9 cusps and 9 conics

(2) $A = E \times E$, E with CM of order 3

$A/\sigma \to \mathbb{P}^1 \times \mathbb{P}^1$, $\tau^3 = (x^2 - x)(y^2 - y)$


branch

$[A/\sigma \hookrightarrow \mathbb{P}^4]$ = (quadric cone) $\cap$ (3) c.i.

Mini history of Aut S and Picard lattice in U + (Leech)

	(1)	(2)
Kummer	Kondo(1998), $(A_3+6A_1)^\perp$	Keum-Kondo(2000), $(2D_4)^\perp$
ord 3 ver.	Not yet, $(A_5+A_2)^\perp$	Vinberg(1983), $E_6^\perp$

A typo corrected on 5/12/25(M).

§2. K3^[2] 4-folds

Simplest higher dimensional analogue of a K3 surface is the Hilbert square of a K3 (and its deformations)

Let

$S^{[2]} \to S^{(2)}$ be the minimal resolution of symmetric product.

$S^{[2]}$ = moduli of ideal sheaves of 2 points (allowing inf. nears)
$= M_S(1, 0, -1)$

$\mathrm{Pic}\, S^{[2]} = \mathrm{Pic}\, S \oplus \mathbb{Z}\delta$, $\delta = (1, 0, 1)$
$= (1, 0, -1)$ in $\mathbb{Z} \oplus \mathrm{Pic}\, S \oplus \mathbb{Z}$

Notation. α on S is identified with $(0, \alpha, 0)$ on $S^{[2]}$.

Relation with cubic 4-folds and K3 sextic $S = (2, 3)$, c.i. in $\mathbb{P}^4$.

$S : q(x, y, z, u, v) = d(x, y, z, u, v) = 0$ in $\mathbb{P}^4$
$X = X_S : q(x, y, z, u, v)w + d(x, y, z, u, v) = 0$ in $\mathbb{P}^5$

This cubic 4-fold is singular at (100000).

Fact: Fano variety of lines $F(X_S)$ is birational with $S^{[2]}$

(1) S : Bertin-Vanhaecke $\Rightarrow X_S$ has $A_1 + 9A_2$

(2) S : Vinberg $\Rightarrow X_S$ has $10A_2$ since the quadric hull of S is a cone

Observation: X_S is a 4-dimensional analogue of Segre cubic 3-fold when S : Vinberg.

$$Y : \sum_1^6 y_i = \sum_1^6 y_i^3 = 0 \subset \mathbb{P}^5$$

$$X_S : \left(\sum_1^6 x_i\right)\left(\sum_1^6 x_i^2\right) = 2\sum_1^6 x_i^3 \subset \mathbb{P}^5$$

Segre cubic 3-fold Y is characterized by $10A_1$
——— 4-fold X ——— $10A_2$

X_S is a triple cyclic covering of $\mathbb{P}^4$ with branch Segre cubic.

Both have an action of symmetric group $\mathfrak{S}_6$ of degree 6

$$\mathrm{Aut}(S \subset \mathbb{P}^4) \cong C_3.[(\mathfrak{S}_3 \times \mathfrak{S}_3).C_2]$$

$$\cap \qquad\qquad \cap \quad \text{Normalizer of } \sigma$$

$$\mathrm{Aut}(X_5 \subset \mathbb{P}^5) \cong \mathfrak{S}_6 \ni (123)(456) =: \sigma$$

§3 Vinberg's theorem and generalization to K3^[2] 4-folds

Theorem (Vinberg)

Aut S = (Free product of 12 involutions) $\rtimes$ Aut(S$\subset$P^4)

$\boxed{\text{Main Theorem Bir-Aut(S^[2]) is semi-direct product of <84+120 involutions> by symmetric group } \mathfrak{S}_7 \text{ of degree 7.}}$

Surprisingly,

$\mathfrak{S}_6$ -action on S extends to a birational $\mathfrak{S}_7$-action on S^[2].

Reason: a symmetry of degree 7 induced by Lagrangian fibration

S^[2] $\dashrightarrow$ $\mathbb{P}$^2 (of $3\tilde{A}$_6-type).

Mordell-Weil group has a 7-torsion (birational) section.

§4 Review of Vinberg's proof

$T_S = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Quick review on Aut S, Picard number 20,

Pic S = U + 2 E_8 + A_2 $\quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, A_2 $= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

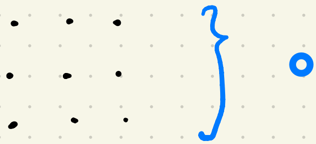
Orhtogonal group O^+(U+2E_8+A_2)

= <24 (-2)-reflections, 12 (-6)-reflections> $\rtimes$ Aut($S \subset$ P^4)

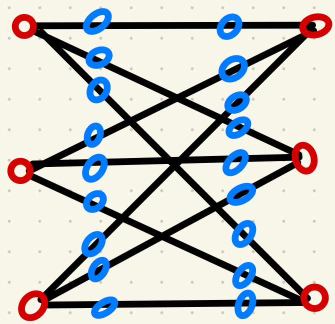
24=6 + 9 •2

6P^1's

9A_2's



Dual graph of 24 P^1's



6 circuits of length 18

S $\rightarrow$ P^1 elliptic fibration

with I_18 fiber, MW group Z $\oplus$ Z/6

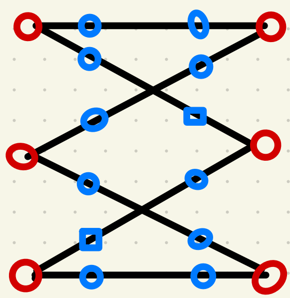
$\Longrightarrow$ 2 anti-symplectic involutions

(-1)-mult. &

its composite with 2-torsion translation

Modulo Aut($S \subset$ P^4), these act as

(-6)-reflection.



§5 Proof of main theorem

Look at the action of Bir-Aut S^[2] on Pic S ⊕ Zδ, and on the movable cone in it. The rest is basically the same as (Vinberg's) K3, but get complicated in two points:

(1) the orthogonal group is no more more reflective
(2) Divisibility should be taken into account. There are two types of (-2)-divisors:
  a) (-2) effective divisor Im[E × S → S^[2]] for (-2)-curve E on S (divisibility 1)
  b) half δ of exceptional divisor class (divisibility 2)

(1) is overcome by Conway-Borcherds domain CB in the positive cone.
CB domain is surrounded by 309 walls:

$$309 = \underbrace{35 + 70}_{} \quad + 84 \quad + 120$$

(-2)-walls (divisibility 1 & 2, respectively) ↑ ① ↑ ②

(-6)-walls ↑ ③

(-42)-walls ↑ ④

★ $$O^+(U+2E\_8+A\_2+A\_1) = \left\langle \begin{array}{l} 105\ (-2)\text{-reflections} \\ 84\ (-6)\text{-reflections} \\ 120\ \text{quasi-reflections} \end{array} \right\rangle \rtimes \mathfrak{S}_7$$

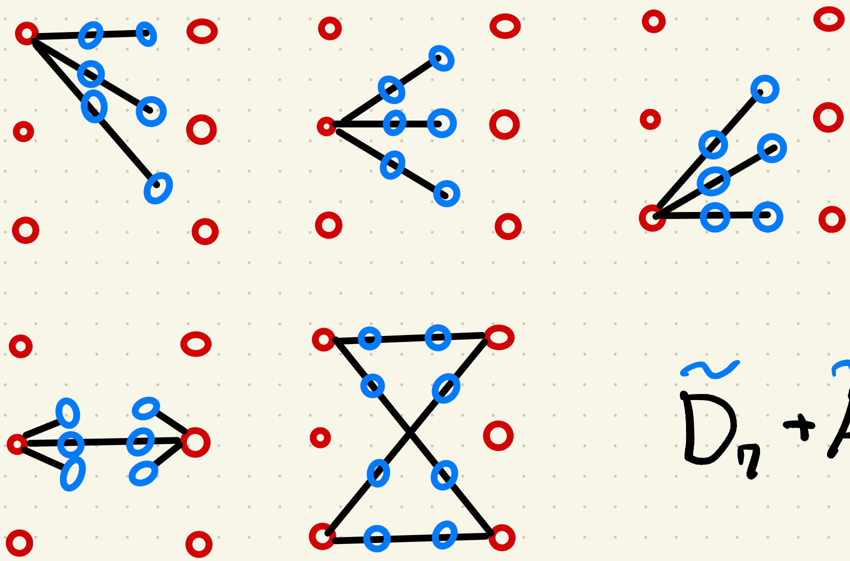
symmetry of CB domain

Geometrization of ★

① Basic (-2)-divisors (of divsibility 1) # = 35 = 24 + 2 + 9

24 are "pull-back" of (-2)'s from S.
Extra 11(=2+9) are (1, f, 1) for elliptic pencil |f| on S of minimal Coxter number (= 12). Since f is isotopic, this divisor has Beauville-square (-2). Geometrically, (1, f, 1) is the Zariski closure of locus of {a, b} with $a \neq b$ and $\Phi(a) = \Phi(b)$.

Two f's are of type 3E_6 and 9 of type D_7+A_11.



$3\tilde{E}_6$

$\tilde{D}_7 + \tilde{A}_{11}$

The dual graph of these 35 (-2) divisors is the 4-valent odd graph O_4.

② (-2) divisors of divisibility 2 $\rightleftarrows$ 70 edges of O_4

③ (-6)-walls $\rightleftarrows$ Induced automorphism from S

④ (-42)-walls: Non-induced automorphism $\rightleftarrows$ (-1) multiplication of Lagrangian fibration of type $3\tilde{A}_5$ (mod $\mathfrak{S}_7$)

Final answer: Aut S^[2] = <84+120 involutions> $\rtimes\ \mathfrak{S}_7$.

References

(added in June 2023)

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§1 Main motivation and side job

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§3 Vinberg's theorem and generalization to K3^[2] 4-folds

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