CUBIC FOURFOLDS WITH ELEVEN CUSPS AND A RELATED MODULI SPACE

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ABSTRACT. First we construct a cubic 4-fold whose singularities are 11 cusps and which has an action of the Mathieu group M_{11}, all over the ternary field $\mathbb{F}_3$. We next consider a certain moduli space of bundles on a supersingular K3 surface of Artin invariant one in characteristic 3. We show that it has 275 (-2) Mukai vectors which form the McLaughlin graph, and ask questions on it and on its relation with our M_{11}-cubic 4-fold.

The classification of finite simple groups singles out 26 sporadic groups. We are interested in realizing some of these very large and complicated groups geometrically, as acting on *K3-like varieties*, namely, a higher dimensional analogue of K3 surfaces, in positive characteristic. In this note we investigate the case of McLaughlin group McL, relating its defining graph with the Fermat quartic surface and a certain cubic 4-fold both in characteristic 3 (cf. Remark 20).

Our model case is the Fermat cubic 4-fold $\sum_1^6 x_i^4 = 0 \subset \mathbb{P}^5_{(x)}$ in characteristic 2. Its automorphism group, that is, the finite unitary group $U_6(2)$, is important in two respects: firstly, it extends to the Fisher group Fi_{22} and secondly, it contains the Mathieu group M_{22}. We show an analogue of the latter for the smallest Mathieu group M_{11} in our main theorem. We note that M_{11} is one of the maximal subgroups of McL and that neither M_{11} or M_{22} has an action on a cubic 4-fold in characteristic 0 (cf. Remark 12).

Now we start to work over an algebraically closed field in characteristic 3, but varieties are mostly defined over $\mathbb{F}_3$ or $\mathbb{F}_9$. A general inseparable triple covering

$$V \to \mathbb{P}^4_{(x)}, \quad \tau^3 = G(x_0, x_1, x_2, x_3, x_4), \quad \deg G = 3. \tag{1}$$

of the projective 4-space is a cubic 4-fold in $\mathbb{P}^5_{(x\tau)}$ with 11 cusps, i.e., simple singularities of type A_2 (Corollary 6). Among such cubic 4-folds, a highly symmetric one is obtained from the Segre cubic 3-fold

$$Seg^3 : \sum_{i=1}^{6} y_i = \sum_{1\le i<j<k\le 6} y_i y_j y_k = 0 \subset \mathbb{P}^5_{(y)},$$

which has the maximal number (=10) of nodes (e.g., [8]), in the following way:

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Example 1. The inseparable triple covering

$$Seg^4 \to \mathbb{P}^4, \quad \tau^3 = \sum_{1\le i<j<k\le 6} y_i y_j y_k, \quad \text{with} \quad \mathbb{P}^4 : \sum_{i=1}^{6} y_i = 0 \subset \mathbb{P}^5_{(y)}, \tag{2}$$

with formal branch Seg^3 has 10 cusps over its nodes, and one more at $(y : \tau) = (111111 : -1)$. The automorphism group $\mathfrak{S}_6$ of Seg^4 (and also of Seg^3) acts on the 11 cusps with two orbits of length 10 and 1.

A little bit surprisingly, there is a *more symmetric* cubic 4-fold with 11 cusps in the sense that the automorphism group, which is M_{11}, acts transitively on the cusps. The following is our main result of this note, and is regarded as a characteristic 3 analogue of the fact that the Fermat cubic 4-fold has an action of the Mathieu group M_{22} over $\mathbb{F}_4$ and the M_{22}-action on a set of 22 planes in it is (triply) transitive ([10], [4, p. 39]):

Theorem 2. *The cubic 4-fold*

$$V : z^3 = \sum_{i\in\mathbb{Z}/5\mathbb{Z}} (x_{i-1}x_i x_{i+1} - x_{i-2}x_i x_{i+2}) \quad \text{in} \quad \mathbb{P}^5_{(xz)} \tag{3}$$

has an action of the Mathieu group M_{11} over $\mathbb{F}_3$. Moreover, V has cusps at 11 $\mathbb{F}_3$-points, on which the M_{11} acts (quadruply) transitively. V is smooth elsewhere.

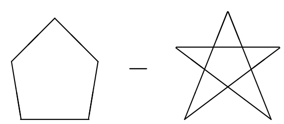

FIGURE 1. Pentagon minus pentagram

In §1 we prepare the singularity of purely inseparable covering of the projective space. In §2, we prove our main theorem by simplifying arguments in Adler[1]. The two cubic 4-folds Seg^4 and V in Theorem 2 are closely related with a supersingular K3 surface of Artin invariant one, whose standard projective model is the Fermat quartic surface. Though it does not have an action of M_{11}, there is a chance for a suitable moduli space of bundles over it to have a birational action of M_{11}. In §3, we give an 8-dimensional candidate and ask two questions.

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1. PRELIMINARY

Let V be an n-dimensional smooth hypersurface of degree d over the complex number. Then the primitive Betti number of its middle cohomology $H^n(V)$ is equal to

$$b_n(V)_{pr} = (d-1)J_n^{(d)}$$

by [9, Corollary 1.12], where we put

$$J_n^{(d)} := \frac{1}{d}((d-1)^{n+1} + (-1)^n).$$

The case $d = 3$, that is,

$$J_n = J_n^{(3)} = 1, 1, 3, 5, 11, 21, 43, 85, \dots \quad \text{for} \quad n = 0, 1, 2, 3, 4, 5, 6, 7, \dots$$

is known as the Jacobsthal sequence (OEIS A001045).

$J_n^{(d)}$ is also equal to the top Chern number $c_n(\Omega_{\mathbb{P}}(d))$ of the twisted sheaf of differentials of the n-dimensional projective space $\mathbb{P}^n$ by the exact sequence

$$0 \to \Omega_{\mathbb{P}}(d) \to \mathcal{O}_{\mathbb{P}}(d-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}}(d) \to 0. \tag{4}$$

This number $c_n(\Omega_{\mathbb{P}}(d))$ has the following meaning in algebraic geometry of positive characteristic. Assume that d is a power of a prime p and consider a hypersurface of the *special* form

$$V : \tau^d - G(x) = 0 \subset \mathbb{P}^{n+1}_{(x_0:\dots:x_n:\tau)}$$

for a homogeneous polynomial G of degree d over an algebraically closed field of characteristic p. This hypersurface is special in the sense that its polar at $(0 : \dots : 0 : 1)$ is identically zero but not a cone in general. The projection from $(0 : \dots : 0 : 1)$ induces a purely inseparable covering $\pi : V \to \mathbb{P}^n$ of degree d. So we can say that a hypersurface $V \subset \mathbb{P}^{n+1}$ has $(0 : \dots : 0 : 1)$ as *its inseparable point*.

The following is obvious.

Lemma 3. *The singular locus of V is bijected by π onto the critical locus of $G(x)$, that is, the common zero locus of all partials $\partial_i G$ of $G(x)$.*

Since $\sum_0^{n+1} x_i \partial_i G = 0$, we have the well-defined differential map

$$d : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(d)) \to H^0(\mathbb{P}^n, \Omega_{\mathbb{P}}(d)), \quad G \mapsto dG \tag{5}$$

by (4). The critical locus of G is the zero locus of dG.

Lemma 4. *The unique singular point of V over a critical point $P \in \mathbb{P}^n$ is a simple singularity of type A_{d-1} if and only if $dG \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}}(d))$ has a reduced isolated zero at P.*

Proof. Since the assertion is local and since we can add the d-th power of linear forms freely to G, we may assume that $G(x)$ is of the form $q(x) + (\text{higher order terms})$, $q(x)$ being quadratic, in a neighborhood of P. ($\tau^d = G(x)$ and $\tau^d = G(x) + L(x)^d$ define coverings which are isomorphic to each other.) Then dG has a reduced isolated zero at P if and only if the quadratic form $q(x)$ is non-degenerate. Hence we have our lemma. □

Proposition 5. *(Kanemitsu) The inseparable covering $\tau^d = G(x)$ of $\mathbb{P}^n$, d being a power of p, has $J_n^{(d)}$ simple singular points of type A_{d-1} for every general (in Zariski topology) form $G(x_0, \dots, x_n)$ of degree d, if the generalized Jacobsthal number $J_n^{(d)}$ is not divisible by p.*

Proof. Since the reducedness and 0-dimensionality are open conditions, it suffices to construct one example for which the zero locus $(dG)_0$ is reduced and 0-dimensional. The form $G(x_0, \dots, x_n) = \sum_{i\in\mathbb{Z}/(n+1)\mathbb{Z}} x_i^{d-1}x_{i+1}$ of Klein type is such an example since a primitive $J_n^{(d)}$-th root ζ of unity exists by our assumption $p \not| J_n^{(d)}$ and G has the automorphism $\mathrm{diag}[\zeta, \zeta^{1-d}, \zeta^{(1-d)^2}, \cdots]$ of order $J_n^{(d)}$. See the next section ($d = p = 3, n = 4$ and $J_4 = 11$), especially Lemma 7, for details. □

Corollary 6. *The inseparable triple covering* (1) *of* $\mathbb{P}^4$ *has 11 cusps for general cubic* $G(x_0, \dots, x_4)$.

The author does not know whether the proposition holds or not when $p|J_n^{(d)}$

2. Cubic 4-fold with action of M_{11} over $\mathbb{F}_3$

Our action of the Mathieu group M_{11} is generated by three transformations. One is a certain signed permutation (9) of order 4 which preserves the right hand side of (3). The other two transformations $\tilde{A}, \tilde{B}$ come from Klein's cubic 3-fold.

2.1. **From Klein's to the pentagon-pentagram cubic.** We start our study Klein's cubic 3-fold

$$U : \sum_{i\in\mathbb{Z}/5\mathbb{Z}} y_i^2 y_{i+1} = 0 \subset \mathbb{P}^4, \tag{6}$$

which is invariant under the transformation $A' = \mathrm{diag}[\zeta, \zeta^9, \zeta^4, \zeta^3, \zeta^5]$ of order 11 and cyclic permutation $B' = y_i \mapsto y_{i+1}$ of order 5, where ζ is a primitive 11-th root of unity, assuming that the base field is of characteristic 3. The minimal polynomial of ζ over $\mathbb{F}_3$ is of degree 5 and there are two possibilities, among which choose $X^5 + X^4 - X^3 + X^2 - 1$. In studying the automorphism of U the following is crucial:

Lemma 7. *The critical locus of Klein's cubic 3-fold* (6) *consists of the 11 points*

$$P_i\,(\zeta^i : \zeta^{9i} : \zeta^{4i} : \zeta^{3i} : \zeta^{5i}), \quad i \in \mathbb{Z}/11\mathbb{Z}$$

and the inseparable triple covering $V' : \tau^3 = \sum_{i\in\mathbb{Z}/5\mathbb{Z}} y_i^2 y_{i+1} \subset \mathbb{P}^4$ *has 11 simple singularities of type* A_2 *over them.*

Proof. By Lemma 3, the singular locus of V' is bijected onto the common zero locus of partials $-y_{i-1}^2 + y_i y_{i+1} = 0$ for $i \in \mathbb{Z}/5\mathbb{Z}$ and hence consists of the 11 points above. Since $J_4 = 11$, our claim follows from and from Lemma 4. □

We make the following change of coordinates of $\mathbb{P}^4$

$$\begin{aligned} y_0 &= \zeta x_0 + \zeta^9 x_1 + \zeta^4 x_2 + \zeta^3 x_3 + \zeta^5 x_4 \\ y_1 &= \zeta^9 x_0 + \zeta^4 x_1 + \zeta^3 x_2 + \zeta^5 x_3 + \zeta x_4 \\ y_2 &= \zeta^4 x_0 + \zeta^3 x_1 + \zeta^5 x_2 + \zeta x_3 + \zeta^9 x_4 \\ y_3 &= \zeta^3 x_0 + \zeta^5 x_1 + \zeta x_2 + \zeta^9 x_3 + \zeta^4 x_4 \\ y_4 &= \zeta^5 x_0 + \zeta x_1 + \zeta^9 x_2 + \zeta^4 x_3 + \zeta^3 x_4 \end{aligned} \tag{7}$$

so that the six critical points P_1, P_9, P_4, P_3, P_5 and P_0 become the five coordinate points and $(-1-1-1-1-1)$, respectively. In this new coordinate system (x), Klein's cubic (6) is defined by

$$U : \sum_{i \in \mathbb{Z}/5\mathbb{Z}} (-x_i^3 + x_{i-1}x_i x_{i+1} - x_{i-2}x_i x_{i+2}) = 0. \tag{8}$$

The cyclic group $\langle B' \rangle$ of order 5 in (y)-coordinate is generated by $B : x_i \mapsto x_{i+1}$ in our new (y)-coordinate. The transformation A' of order 11 becomes

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 & -1 \end{pmatrix},$$

in (y)-coordinate by computation. In particular, it is defined over $\mathbb{F}_3$. (This is not surprising because A' induces a permutation of critical points all of which are defined over $\mathbb{F}_3$.)

Now we are ready to explain that U and V' have extra automorphisms. Firstly, the permutation

$$x_1 \leftrightarrow x_4, \quad x_2 \leftrightarrow x_3$$

of type $(2)^2$ is an automorphism of U since it preserves the pentagon supporting $\sum_{i \in \mathbb{Z}/5\mathbb{Z}} x_{i-1}x_i x_{i+1}$ and also the pentagram supporting $\sum_{i \in \mathbb{Z}/5\mathbb{Z}} x_{i-2}x_i x_{i+2}$ in (8). Together with the cyclic group $\langle B \rangle$, this involution generates a dihedral group D_{10} of order 10.

Secondly, what is more crucial in characteristic 3 is to consider the cyclic permutation

$$x_1 \mapsto x_2 \mapsto x_4 \mapsto x_3 (\mapsto x_1)$$

of type (4) whose square is the involution above. Since this permutation interchanges the pentagon and pentagram above, we consider the signed permutation

$$C : x_1 \mapsto -x_2 \mapsto x_4 \mapsto -x_3 (\mapsto x_1), \quad x_5 \mapsto -x_5 \tag{9}$$

instead. Then C preserves the right hand side of (3). We also observe the following:

Lemma 8. *The pentagon-pentagram cubic form* (8) *is preserved by the linear transformations A and B. It is not preserved by C but transformed under C to*

$$\sum_{i \in \mathbb{Z}/5\mathbb{Z}} (x_i^3 + x_{i-1}x_i x_{i+1} - x_{i-2}x_i x_{i+2}). \tag{10}$$

In particular, it is invariant under the action of $\langle A, B, C \rangle$ modulo cubes of linear forms.

Remark 9. Similar claims, especially the last one, in Lemma 8 were first found in Adler[1, Lemma 3.1] for Klein's cubic.

2.2. **Proof of Theorem 2.** In order to eliminate the modulo cubes ambiguity, we introduce a new independent variable τ and consider the cubic 4-fold

$$V : \tau^3 + \sum_{i\in\mathbb{Z}/5\mathbb{Z}} (-x_i^3 + x_{i-1}x_ix_{i+1} - x_{i-2}x_ix_{i+2}) = 0 \tag{11}$$

in $\mathbb{P}^5$, or equivalently, the cubic 4-fold in Theorem 2 by change of variables $\tau = -z + \sum_{i\in\mathbb{Z}/5\mathbb{Z}} x_i$. We extend the action of A, B, C to $\mathbb{P}^5$ by

$$\tilde{A} : \tau \mapsto \tau, \quad \tilde{B} : \tau \mapsto \tau \quad \text{and} \quad \tilde{C} : \tau \mapsto \tau + \textstyle\sum_{i\in\mathbb{Z}/5\mathbb{Z}} x_i,$$

the last of which is equivalent to $\tilde{C} : z \mapsto z$. By Lemma 8, this action of $\tilde{A}, \tilde{B}$ and $\tilde{C}$ preserves V.

In our (xz)-coordinate system, the singularity of V locates at 11 points

$$\begin{gathered}(x_0 : \ldots : x_4 : z) = \\ (10000; 0), (01000; 0), (00100; 0), (00010; 0), (00001; 0), (-1-1-1-1-1; 0), \\ (01-1-11; 1), (101-1-1; 1), (-1101-1; 1), (-1-1101; 1), (1-1-110; 1),\end{gathered} \tag{12}$$

which are the 11 points $1, 2, 3, 4, 5, 6$ and a, b, c, d, e in the notation of [15, p. 406]. By Coxeter-Todd ([6], [15]), the automorphism group of the 12-pointed projective space

$$(\mathbb{P}^5; 1, 2, \ldots, 6, a, b, \ldots, f)$$

is known to be the Mathieu group M_{12}, where we put $f(00000 : 1)$. Furthermore the permutation action of M_{12} on the 12 points is quintuply transitive.

Lemma 10. *An automorphism of V preserves the point $f(00000 : 1)$.*

Proof. As we saw in §1, the point f is an inseparable point of $V \subset \mathbb{P}^5$. It suffices to show there are no other inseparable points. This is obvious since the five partials $-y_{i-1}^2 + y_iy_{i+1}$ of Klein's cubic are linearly independent. □

An automorphism of V induces a permutation of its singular locus. Hence, by the lemma, the automorphism group of $V \subset \mathbb{P}^5$ is contained in M_{11}, the stabilizer of M_{12} at f. The following completes our proof of Theorem 2.

Lemma 11. *The three linear transformations $\tilde{A}, \tilde{B}$ and $\tilde{C}$ generate M_{11} in $PGL(6, \mathbb{F}_3)$.*

Proof. Le $G \subset M_{11}$ be the subgroup generated by $\tilde{A}, \tilde{B}$ and $\tilde{C}$. Since $\tilde{A}, \tilde{B}, \tilde{C}$ are of order 11, 5, 4, the order of G is divisible by 220. By the classification of maximal subgroups of $\mathfrak{S}_{11}$ (e.g., [4]), G is isomorphic to either M_{11} or $L_2(11)$. The latter is impossible since the subgroup $\langle\tilde{B}, \tilde{C}\rangle \subset G$ is the semi-direct product $5 : 4$ or $\mathrm{Hol}(C_5)$ by our construction but $L_2(11)$ does not contain such a semi-direct product. □

Remark 12. (1) The cubic 4-fold $\tau^3 - \sum_{i\in\mathbb{Z}/5\mathbb{Z}} y_i^2y_{i+1} = 0 \subset \mathbb{P}^5_{(\tau y)}$ is also interesting over the complex number field $\mathbb{C}$ in the sense that its automorphism group $L_2(11)$ is maximal among all finite groups with a symplectic action on a smooth cubic 4-fold ([12]). Similar holds for Klein's cubic 3-fold $\sum_{i\in\mathbb{Z}/5\mathbb{Z}} y_i^2y_{i+1} = 0 \subset \mathbb{P}^4_{(y)}$ ([16]).

(2) The stabilizer group M_{10} of the standard permutation action of M_{11} is also maximal among all finite groups with a symplectic action on a smooth cubic 4-fold over $\mathbb{C}$ ([12]).

3. CONJECTURAL SYMPLECTIC 8-FOLD AS MODULI OF BUNDLES ON FERMAT QUARTIC

3.1. **Two questions.** The Fermat quartic surface $Fer_4 : \sum_1^4 x_i^4 = 0 \subset \mathbb{P}^3_{(x)}$ has an action of the finite unitary group $PGU_4(3)$. The action of a subgroup of index 4, namely, of $U_4(3) := PSU_4(3)$ is symplectic. Though $U_4(3)$ does not contain M_{11} as a subgroup, the moduli space $\overline{M}_{Fer}(v)$ of (semi-)stable sheaves on the Fermat quartic Fer_4 might have a birational action of M_{11}, or even a much larger finite simple group, for suitable Mukai vector $v = (r, *, s) \in \mathbb{Z} \oplus \mathrm{Pic} \oplus \mathbb{Z}$. A hopeful candidate, in view of symmetry of the Leech lattice, is 8-dimensional, i.e., $\langle v^2 \rangle = 6$, and the group containing M_{11} should be the McLaughlin group McL.

Question 13. Does the moduli space $\overline{M}_{Fer}(3, \alpha, -3)$ have a birational action of McL, where α is a (-12)-divisor class attached to Segre's hemisystem (see §3.2)?

Remark 14. As is explained e.g. in [4, p. 100], the McL is the pointwise stabilizer of a triangle ABC of type 322 in the Leech lattice Λ. This means that the orthogonal complement L of a negative root lattice $\simeq A_1 + A_2$ in $U + \Lambda(-1)$ has an action of McL. Since the Picard lattice of Fer_4 has a primitive embedding into L and its orthogonal complement is generated by an element of norm -6, it is natural to seek the possibility above, namely $\langle v^2 \rangle = 6$ and 8-dimensional.

The group McL contains the simple groups $U_4(3)$ and M_{11} as maximal subgroups, and hence it is generated by these two subgroups. The action of the former on the moduli space is not surprising since its $\mathbb{Q}$-twisted expression is $\overline{M}_{Fer}(3, 0, -1)$ (Proposition 18). Seeking after an action of the latter, we pose the following:

Question 15. Is $\overline{M}_{Fer}(3, \alpha, -3)$ birational to the conjectural LLSvS 8-fold associated with the M_{11}-cubic 4-fold V (constructed in §2)?

Remark 16. The LLSvS 8-fold in the question is triply conjectural since it is constructed in [13] only for smooth cubic 4-folds over $\mathbb{C}$ which does not contain a plane. Our M_{11}-cubic 4-fold has 11 cusps, defined in characteristic 3 and the author does not know whether it contains a plane or not.

3.2. **Segre's hemisystem and the McLaughlin graph in a Picard lattice.** The Fermat quartic surface Fer_4 has 280 $\mathbb{F}_9$-(rational) points, with weight distribution 2: 24, 3: 64 and 4: 192. For each $\mathbb{F}_9$-point p, the tangent plane T_p cuts out the union of 4 lines passing through p from Fer_4. Since every line has ten $\mathbb{F}_9$-points, the number of lines in Fer_4 is $280 \times 4/10 = 112$. The Picard lattice is generated by these line classes. Its discriminant group $\mathrm{Disc}(Fer_4)$ is isomorphic to $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ (see e.g. [11]).

Segre's hemisystem is a set H of 56 lines, among the 112, which covers $Fer_4(\mathbb{F}_9)$ doubly, that is, every $\mathbb{F}_9$-point is contained in exactly two members of H. There are 648 hemisystems and they are divided into 4 orbits of length 162 by the action of $U_4(3)$. These 4 orbits corresponds to the four elements of norm $2/3$ modulo $2\mathbb{Z}$ in the discriminant group $\mathrm{Disc}(Fer_4)$ as we will see below. We choose one of them. Then the intersection size $|H \cap H'|$ of two among our 162 hemisystems are either 20 or 32 ([3, §10.34]).

Proposition 17. ([3, §10.61]) *The graph with the following three types of vertices and a suitable adjacency is a strongly regular graph $srg(275, 112, 30, 56)$, isomorphic to the McLaughlin graph: (i) ∞, (ii) the 112 lines in Fer_4 and (iii) the 162 hemisystems. (See* [7, §7] *for the adjacency.)*

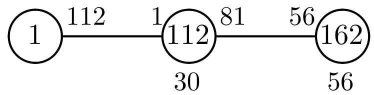

FIGURE 2. McLaughlin graph

We realize this graph inside the extended Picard lattice $U(-1) \oplus \mathrm{Pic}\, Fer_4$ of the Fermat quartic surface, or more precisely, in the sublattice $(3, \alpha, -3)^{\perp}$, which is expected to be the Picard lattice of the conjectural moduli symplectic 8-fold ([14], [17], [18] but only over $\mathbb{C}$). Here U denotes the standard hyperbolic lattice of rank 2. The intersection pairing $(D.D')$ on the Picard lattice extends to the orthogonal sum $\mathbb{Z} \oplus \mathrm{Pic} \oplus \mathbb{Z}$ obviously but with changing the sign of U, namely,

$$\langle (r, D, s), (r', D', s') \rangle = -rs' + (D.\, D') - sr', \quad (r, s), (r', s') \in U(-1). \tag{13}$$

Now we define a divisor class for a hemisystem H. Consider the sum $\sum_{m \in H} m$ of its all members in the Picard grup $\mathrm{Pic}\, Fer_4$. Then we have

$$(\sum_{m \in H} m.\, l) = \begin{cases} 8 & \text{if } l \in H, \\ 20 & \text{otherwise.} \end{cases} \tag{14}$$

In particular, $\sum_{m \in H} m$ is divisible by 4 in the Picard group. So we define

$$\alpha_H := 2h - \frac{1}{4} \sum_{m \in H} m \in \mathrm{Pic}\, Fer_4,$$

where h is the hyperplane section class of Fer_4. Since $(\alpha_H.l)$ is divisible by 3 for all lines l, $\alpha_H/3$ defines an element in the discriminant group, whose norm is $-4/3$ since $(\alpha_H^2) = -12$.

Proposition 18. *The graph on the following three types of (-2)-vectors in $(3, 0, -1)^{\perp} \otimes \mathbb{Q}$, adjacent when non-orthogonal, is isomorphic to the McLaughlin graph:*

(i) $(3, h, 1)$,
(ii) $(0, l, 0)$ *for the 112 lines l in Fer_4 and*
(iii) $(1, -\frac{\alpha_H}{3}, \frac{1}{3})$ *for the 162 hemisystems H chosen as above.*

Proof. We just check adjacencies here and that in [7, §7] are the same. For example, $(3, h, 1)$ has inner product 1 with $(0, l, 0)$ and hence they are adjacent in $(3, 0, -1)^{\perp} \otimes \mathbb{Q}$. The corresponding ∞ and all 112 lines are adjacent in [7, §7] by definition. Other cases are similar but tedious and we omit it. (The MOG computation in [19, §5.5.2] may be useful for a better proof.) □

Geometrically, these are the Mukai vectors of (i) the rank 3 bundle $T_{\mathbb{P}^3}(-1)$ restricted to Fer_4, (ii) torsion sheaves supported on lines and (iii) $\mathbb{Q}$-line bundles on Fer_4, respectively.

Now we fix a hemisystem F among our 162, put $\alpha = \alpha_F$ and take twist by tensor product of the $\mathbb{Q}$-line bundle $\mathcal{O}_{Fer}(\frac{\alpha}{3})$. Then all the vertices in Proposition 18 become integral. Since the the tensor of a line bundle preserves the inner product (13), we have

Corollary 19. *The graph on the following three types of (-2) Mukai vectors in $(3, \alpha, -3)^{\perp}$, adjacent when non-orthogonal, is isomorphic to the McLaughlin graph:*

(i) $(3, h + \alpha, -3)$,

(ii) $(0, l, *)$ *for the 112 lines* l *in* Fer_4 *and*
(iii) $(1, \frac{\alpha-\alpha_H}{3}, **)$ *for the 162 hemisystems* H*,*

where $*$ *is equal to 0 if* $l \in F$ *and* 1 *otherwise, and* $**$ *is equal to* $1, -1, -2$ *according as* $H = F, |H \cap F| = 20$ *and* $|H \cap F| = 32$*.*

Proof. $\alpha - \alpha_H$ is divisible by 3 since both $\alpha/3$ and $\alpha_H/3$ defines the same element in $\mathrm{Disc}(Fer_4)$. Hence the vertices are Mukai vectors of a rank 3 bundles, torsion sheaves and the 162 line bundles $\mathcal{O}_{Fer}(\frac{\alpha-\alpha_H}{3})$. □

Remark 20. Two more strongly regular graphs are similarly realized by taking (-2) Mukai vectors as their vertices in characteristic 2 and 5, which will be discussed elsewhere.

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