Duality of polarized K3 surfaces

Shigeru Mukai *

To the memory of Professor Hideyuki Matsumura

The dual $\widehat{A}$ of an abelian variety A is defined as the neutral component $\mathrm{Pic}^0 A$ of the moduli space of line bundles on A. Let $\mathcal{P}$ be a Poincaré bundle on the product $A \times \widehat{A}$ whose restriction to $0 \times \widehat{A}$ is trivial. Then the classification morphism in the opposite direction

$$A \to \mathrm{Pic}^0 \widehat{A} \quad \text{defined by} \quad x \mapsto \mathcal{P}_{|x \times \hat{A}}$$

is also an isomorphism and gives the double duality $A \cong \widehat{\widehat{A}}$. Moreover, the construction of [5] gives an equivalence $\int_A \mathcal{P} \colon \mathbf{D}(\mathrm{Coh}\, A) \to \mathbf{D}(\mathrm{Coh}\, \widehat{A})$ of the derived categories of coherent sheaves on A and $\widehat{A}$; this equivalence of derived categories depends on $\mathcal{P}$, and we call it the *integral functor* with *kernel* $\mathcal{P}$ (it is sometimes called the *Fourier–Mukai transform*). See [13] for a systematic study of (auto-)equivalences of $\mathbf{D}(\mathrm{Coh}\, A)$ using universal families of semihomogeneous vector bundles. In this article we generalize these results to polarized K3 surfaces using the moduli space of semi-rigid sheaves studied in [7].

Let r and s be coprime positive integers and (S, h) a polarized K3 surface of degree $2rs$. We denote by $\widehat{S}$ the moduli space $M_S(r, h, s)$ of rank r stable sheaves E on S with $c_1(E) = h$ and $\chi(E) = r + s$ (cf. [2]). Then by the results of [6] and [7], $\widehat{S}$ is again a K3 surface and there exists a universal family $\mathcal{E}$ on the product $S \times \widehat{S}$. Let

$$c_1(\mathcal{E}) = h + \varphi \in H^2(S) \oplus H^2(\widehat{S}) \quad \text{and} \quad c_2^{\mathrm{mid}}(\mathcal{E}) \in H^2(S) \otimes H^2(\widehat{S}) \qquad (1)$$

be the first Chern class of $\mathcal{E}$ and the middle Künneth component of the second Chern class, respectively. We define a class $\psi \in H^2(\widehat{S})$ by

$$h \cup c_2^{\mathrm{mid}}(\mathcal{E}) = p \otimes \psi \in H^4(S) \otimes H^2(\widehat{S}), \qquad (2)$$

where p is the fundamental cohomology class of S. Both φ and ψ are algebraic by the Lefschetz type (1,1) theorem.

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Proposition 1.1 *(i) $\psi - 2(r-1)s\varphi$ does not depend on the choice of the universal family $\mathcal{E}$ on $S \times \widehat{S}$. We denote it by $\widehat{h}$.*

(ii) $(\widehat{h})^2 = h^2$ and $\widehat{h} \equiv s\varphi \mod r$ in $\operatorname{Pic} \widehat{S}$.

Let v be an integer with $sv \equiv 1 \mod r$. By the proposition, there exists a divisor class λ on $\widehat{S}$ such that $v\widehat{h} = \varphi + r\lambda$. For this λ, $\mathcal{E} \otimes \pi_{\hat{S}}^* \mathcal{O}_{\hat{S}}(\lambda)$ is also a universal family and its first Chern class is $\pi_S^* h + v\pi_{\hat{S}}^* \widehat{h}$. Such a universal family is said to be *normalized*. We denote by $\mathcal{F}_{rs+1}$ the moduli space of (quasi-)polarized K3 surfaces (S, h) of degree $2rs$. This is a quasiprojective algebraic variety of dimension 19.

Theorem 1.2 *Let (S, h) be a general member of $\mathcal{F}_{rs+1}$ and $\mathcal{E}$ a normalized universal sheaf on $S \times \widehat{S}$, where $\widehat{S}$ is the moduli space $M_S(r, h, s)$. If $r \geq 2$, then*

(i) $\widehat{h}$ is primitive and ample;

(ii) $\mathcal{E}$ is locally free;

(iii) for every $x \in S$, the restriction $\mathcal{E}_{|x \times \hat{S}}$ is stable with respect to $\widehat{h}$ and belongs to $M_{\hat{S}}(r, v\widehat{h}, v^2 s)$;

(iv) the classification morphism in the opposite direction $S \to M_{\hat{S}}(r, v\widehat{h}, v^2 s)$ given by $x \mapsto \mathcal{E}_{|x \times \hat{S}}$ is also an isomorphism; and

(v) the integral functor *with* kernel $\mathcal{E}$

$$\textstyle\int_S \mathcal{E} \colon \mathbf{D}(\operatorname{Coh} S) \to \mathbf{D}(\operatorname{Coh} \widehat{S}) \quad \text{given by} \quad (?) \mapsto \mathbf{R}\pi_{T*}\big(\mathcal{E} \otimes \pi_S^*(?)\big)$$

is an equivalence of categories between the derived categories $\mathbf{D}(\operatorname{Coh} S)$ and $\mathbf{D}(\operatorname{Coh} \widehat{S})$ of coherent sheaves on S and $\widehat{S}$; here $(?)$ stands for an object or morphism of $\mathbf{D}(\operatorname{Coh} S)$ and $\mathbf{R}\pi_{T*}$ for the derived functor.

Example 1.3 ($r = 2$, $s = 3$) Let (S, h) be a general member of $\mathcal{F}_7$. Then S is a complete linear section

$$\Sigma_{12}^{10} \cap H_1 \cap \cdots \cap H_8$$

of a 10-dimensional orthogonal Grassmannian $\Sigma_{12}^{10} \subset \mathbb{P}^{15}$, and h its hyperplane section. Here $\Sigma_{12}^{10} \subset \mathbb{P}^{15}$ is the Hermitian symmetric space $\mathrm{SO}(10, \mathbb{R}) / \mathrm{U}(5)$ embedded into the projectivization of the 16-dimensional half spinor representation (cf. [8]). The discriminant variety $\check{\Sigma} \subset \check{\mathbb{P}}^{15}$ in the dual projective space is also a 10-dimensional orthogonal Grassmannian (cf. [9]). Let

$h_1, \dots, h_8 \in \check{\mathbb{P}}^{15}$ be the points corresponding to the hyperplanes $H_1, \dots, H_8$. Then the intersection

$$\check{\Sigma} \cap \langle h_1, \dots, h_8 \rangle$$

is also a K3 surface, and is isomorphic to the moduli space $\widehat{S} = M_S(2, h, 3)$. In this case, the double duality (3) below also follows from the double duality $\Sigma \cong \check{\check{\Sigma}}$ of discriminant varieties.

The theorem also holds (as a tautology) in the case $r = 1$, provided that we modify (ii). In fact, every member of $M_S(1, h, s)$ is isomorphic to $\mathcal{O}_S(h) \otimes \mathfrak{m}_x$ for a point $x \in S$, and $M_S(1, h, s)$ is canonically isomorphic to S. The ideal sheaf of the diagonal $\Delta \subset S \times S$ tensored with $\mathcal{O}_S(h) \boxtimes \mathcal{O}_S(h)$ is a normalized universal family. The integral functor $\int_S \mathcal{E}$ is essentially the reflection studied in [7], §2.

When $s \equiv \pm 1$ modulo r, we can choose a universal bundle $\mathcal{E}$ such that $\det \mathcal{E} \cong \mathcal{O}_S(h) \boxtimes \mathcal{O}_S(\pm\widehat{h})$. This $\mathcal{E}$ gives the duality isomorphism

$$(S, h) \cong (\widehat{\widehat{S}}, \widehat{\widehat{h}}). \tag{3}$$

Even when s is general, we can show that $(S, h) \mapsto (\widehat{S}, \widehat{h})$ is an involution of the moduli space $\mathcal{F}_{rs+1}$ using the computation of periods in [7]. But this will be discussed elsewhere.

The most essential part of the theorem is the stability assertion in (iii), which we prove in §4 after examining $\widehat{S} = M_S(r, h, s)$ for K3 surfaces S with Picard number 2 in §3. We first show in §2 that (iv) and (v) follow from (iii).

In the talk at the Warwick conference, we explained a non-abelian analogue of Albanese maps for general polarized K3 curves of odd genus ≥ 11, which is an application of Theorem 1.2 (for $r = 2$) and non-abelian Brill–Noether theory. Consult [11] and [12] for this.

Notations and convention For simplicity, we consider algebraic varieties over the complex number field $\mathbb{C}$. For a coherent sheaf E, we denote its dual by $E^\vee$, its rank (at the generic point) by $r(E)$, and its Euler–Poincaré characteristic by $\chi(E) = \sum_i (-)^i h^i(E)$.

2 Moduli duality and integral functors

Let $\mathcal{L}$ be a line bundle on the product $A \times B$ of two abelian varieties A and B.

Proposition 2.1 ([14], §13) *The following three conditions are equivalent:*

(a) *the classification homomorphism* $A \to \operatorname{Pic}^0 B$ *given by* $a \mapsto \mathcal{L}_{|a\times B}$ *is an isomorphism;*

(b) *the classification homomorphism* $B \to \operatorname{Pic}^0 A$ *given by* $b \mapsto \mathcal{L}_{|A\times b}$ *is an isomorphism; and*

(c) $\chi(\pi_A^* L_A^{-1} \otimes \mathcal{L} \otimes \pi_B^* L_B^{-1}) = \pm 1$, *where* $L_A = \mathcal{L}_{|A\times 0}$ *and* $L_B = \mathcal{L}_{|0\times B}$.

When these equivalent conditions hold, the integral functor with kernel $\mathcal{L}$

$$\mathbf{D}(\operatorname{Coh} A) \to \mathbf{D}(\operatorname{Coh} B) \quad \text{given by} \quad (?) \mapsto \mathbf{R}\pi_{B*}\big(\mathcal{L} \otimes \pi_A^*(?)\big),$$

essentially the Fourier functor, is an equivalence of categories ([5]), where (?) is an object or a morphism in $\mathbf{D}(\operatorname{Coh} A)$. We will generalize these results to K3 surfaces.

Let $\mathcal{E}$ be a vector bundle on the product $S \times T$ of a complete algebraic surface S and a scheme T. We consider the two integral functors

$$\textstyle\int_S \mathcal{E} \colon \mathbf{D}(\operatorname{Coh} S) \to \mathbf{D}(\operatorname{Coh} T) \quad \text{given by} \quad (?) \mapsto \mathbf{R}\pi_{T*}\big(\mathcal{E} \otimes \pi_S^*(?)\big)$$

with kernel $\mathcal{E}$ from S to T, and vice versa

$$\textstyle\int_T \mathcal{E}^\vee \colon \mathbf{D}(\operatorname{Coh} T) \to \mathbf{D}(\operatorname{Coh} S) \quad \text{given by} \quad (?) \mapsto \mathbf{R}\pi_{S*}\big(\mathcal{E}^\vee \otimes \pi_T^*(?)\big)$$

with kernel $\mathcal{E}^\vee$. By the Grothendieck Riemann–Roch theorem, the following diagram is commutative:

$$\begin{array}{ccccc}
\mathbf{D}(\operatorname{Coh} T) & \overset{\int_T \mathcal{E}^\vee}{\longrightarrow} & \mathbf{D}(\operatorname{Coh} S) & \overset{\int_S \mathcal{E}}{\longrightarrow} & \mathbf{D}(\operatorname{Coh} T) \\
\operatorname{ch}\downarrow & & \operatorname{ch}\downarrow & & \operatorname{ch}\downarrow \\
H^*(T,\mathbb{Q}) & \underset{[\mathcal{E}^\vee]_T}{\longrightarrow} & H^*(S,\mathbb{Q}) & \underset{[\mathcal{E}]_S}{\longrightarrow} & H^*(T,\mathbb{Q}),
\end{array} \tag{4}$$

where ch is the Chern character map, and $[\mathcal{E}]_S$, $[\mathcal{E}^\vee]_T$ the correspondences

$$\begin{aligned}
\alpha &\mapsto \pi_{T\bullet}(\operatorname{ch}\mathcal{E} \cup \pi_S^\bullet(\alpha \cup \operatorname{td}_S)), \\
\beta &\mapsto \pi_{S\bullet}(\operatorname{ch}\mathcal{E}^\vee \cup \pi_T^\bullet(\beta \cup \operatorname{td}_T)),
\end{aligned}$$

with kernels $\operatorname{ch}\mathcal{E} \cup \pi_S^\bullet \operatorname{td}_S$ and $\operatorname{ch}\mathcal{E}^\vee \cup \pi_T^\bullet \operatorname{td}_T$.

To generalize Proposition 2.1 to vector bundles, we need the following assumption:

$$\begin{array}{c}\text{there exists an ample line bundle } \Lambda \text{ on } S \text{ such that} \\ \mathcal{E}_{|S\times t} \text{ is stable with respect to } \Lambda \text{ for every } t \in T.\end{array} \tag{$\Sigma_{S,T}$}$$

Under this condition, we obtain a classification morphism $t \mapsto \mathcal{E}_{|S\times t}$ corresponding to $\mathcal{E}$ from T to the moduli M_S of stable vector bundles on S ([2]). We denote it by $\Phi \colon T \to M_S$

Theorem 2.2 *Let S be a K3 surface, T a scheme, and $\mathcal{E}$ a vector bundle on $S \times T$ such that $(\Sigma_{S,T})$ holds. Then the following three conditions are equivalent:*

(i) the classification morphism $\Phi\colon T \to M_S$ is an isomorphism onto a connected component;

(ii) under the projection $\pi_{23}\colon S \times T \times T \to T \times T$, the direct images of the vector bundle $\pi_{12}^*\mathcal{E}^\vee \otimes \pi_{13}^*\mathcal{E}$ satisfy the following:

$$R^q\pi_{23*}(\pi_{12}^*\mathcal{E}^\vee \otimes \pi_{13}^*\mathcal{E}) = \begin{cases} \mathcal{O}_\Delta & \text{if } q = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where Δ is the diagonal of $T \times T$; and

(iii) the composite $[\mathcal{E}]_S \circ [\mathcal{E}^\vee]_T$ in (4) is the identity map of $H^*(T, \mathbb{Q})$.

Proof (i) $\Rightarrow$ (ii) As proved in [7], Proposition 4.10, the second direct image is a line bundle, say L, on the diagonal $\Delta \subset T \times T$ and the other direct images are zero. Since the restriction $\mathcal{E}_{|S\times t}$ is a simple sheaf for every $t \in T$, the natural homomorphism

$$\mathcal{O}_T \to \pi_{T*}(\mathcal{E}^\vee \otimes \mathcal{E})$$

is an isomorphism. By Serre duality, $R^2\pi_{T*}(\mathcal{E}^\vee \otimes \mathcal{E})$ is also isomorphic to $\mathcal{O}_T$. Since $\pi_{12}^*\mathcal{E}^\vee \otimes \pi_{13}^*\mathcal{E}$ restricted to $S \times \Delta$ is $\mathcal{E}^\vee \otimes \mathcal{E}$, the line bundle L is trivial by the base change theorem.

(ii) $\Rightarrow$ (iii) On the one hand, the composite $[\mathcal{E}]_S \circ [\mathcal{E}^\vee]_T\colon H^*(T, \mathbb{Q}) \to H^*(T, \mathbb{Q})$ is the correspondence given by the following cocycle in $H^*(T \times T, \mathbb{Q})$:

$$\begin{aligned} Z &= \pi_{23\bullet}(\pi_{12}^\bullet \operatorname{ch} \mathcal{E}^\vee \cup \pi_{23}^\bullet \operatorname{ch} \mathcal{E} \cup \pi_1^\bullet \operatorname{td}_S \cup \pi_2^\bullet \operatorname{td}_T) \\ &= \pi_{23\bullet}(\pi_{12}^\bullet \operatorname{ch} \mathcal{E}^\vee \cup \pi_{23}^\bullet \operatorname{ch} \mathcal{E} \cup \pi_1^\bullet \operatorname{td}_S) \cup \pi_1^\bullet \operatorname{td}_T . \end{aligned}$$

On the other hand, by (ii) and the Grothendieck Riemann–Roch theorem, we have

$$\pi_{23\bullet}(\pi_{12}^\bullet \operatorname{ch} \mathcal{E}^\vee \cup \pi_{23}^\bullet \operatorname{ch} \mathcal{E} \cup \pi_1^\bullet \operatorname{td}_S) = \operatorname{ch} \mathcal{O}_\Delta.$$

Hence $Z = \operatorname{ch} \mathcal{O}_\Delta \cup \pi_1^\bullet \operatorname{td}_T = \Delta$ and the composite is the identity map.

(iii) $\Rightarrow$ (i) The condition (iii) implies

$$\pi_{23\bullet}(\pi_{12}^\bullet \operatorname{ch} \mathcal{E}^\vee \cup \pi_{23}^\bullet \operatorname{ch} \mathcal{E} \cup \pi_1^\bullet \operatorname{td}_S \cup \pi_2^\bullet \operatorname{td}_T) = \Delta.$$

Comparing the H^0 component of both sides, we have

$$\varkappa_4(\operatorname{ch} E^\vee \cup \operatorname{ch} E \cup \operatorname{td}_T) = 0,$$

where E is the restriction of $\mathcal{E}$ to a fibre $S \times t$ and $\varkappa_4$ is the projection of $H^*(S, \mathbb{Q})$ to the direct summand $H^4(S, \mathbb{Q}) \cong \mathbb{Q}$. By the Riemann–Roch theorem, we have $\chi(E^\vee \otimes E) = 0$. Since S is a K3 surface and E is simple, we have $h^1(E^\vee \otimes E) = 2$. Therefore, the connected component $\widehat{S}$ of M_S containing the image $\Phi(T)$ is a surface.

Let K be the fibre of $\Phi\colon T \to M_S$ over $\Phi(y)$. As a set, K consists of the points $t \in T$ such that $\mathcal{E}_{|S\times t} \cong E$. By assumption $(\Sigma_{S,T})$, there exists a line bundle L on K such that $\mathcal{E}_{|S\times K} \cong E \boxtimes L$. Now apply the functor $\int_T \mathcal{E}^\vee$ to the structure sheaf $\mathcal{O}_K$. Then we have

$$\begin{aligned} R^q\pi_{S*}(\mathcal{E}^\vee \otimes \pi_T^* \mathcal{O}_K) &= R^q\pi_{S*}(\mathcal{E}^\vee_{|S\times K}) \\ &\cong R^q\pi_{S*}(\pi_S^* E^\vee \otimes \pi_T^* L^{-1}) = E^\vee \otimes_{\mathbb{C}} H^q(S, L^{-1}) \end{aligned}$$

and

$$[\mathcal{E}^\vee]_T(\operatorname{ch} \mathcal{O}_K) = \chi(L^{-1}) \operatorname{ch} E^\vee.$$

Similarly, writing $k(t)$ for the sky scraper sheaf at $t \in T$, we have

$$\left(\textstyle\int_T \mathcal{E}^\vee\right)(k(t)) = E^\vee \quad \text{and} \quad [\mathcal{E}^\vee]_T(p) = \operatorname{ch} E^\vee,$$

where $p \in H^4(S, \mathbb{Q})$ is the fundamental cohomology class of S. Since $[\mathcal{E}^\vee]_T$ is injective by (iii), we have $\operatorname{ch} \mathcal{O}_M = \chi(L^{-1})p$. Therefore, K is finite and $T \to \widehat{S} \subset M_S$ is finite. By the same argument as in [7], Proposition 4.10, the direct image $R^q\pi_{23*}(\pi_{12}^* \mathcal{E}^\vee \otimes \pi_{13}^* \mathcal{E})$ is zero for $q \neq 2$, and a line bundle on the pullback Δ' of the diagonal of $\widehat{S} \times \widehat{S}$ by the morphism $\Phi \times \Phi$ for $q = 2$. By (iii) and the Grothendieck Riemann–Roch formula, we have

$$\operatorname{ch}\left(R^2\pi_{23*}(\pi_{12}^* \mathcal{E}^\vee \otimes \pi_{13}^* \mathcal{E})\right) = \operatorname{ch} \mathcal{O}_\Delta.$$

Therefore, Δ' coincides with Δ and $T \to \widehat{S}$ is an isomorphism. $\square$

When T is also a K3 surface, $[\mathcal{E}]_S \circ [\mathcal{E}^\vee]_T$ is the identity map if and only if $[\mathcal{E}]_T \circ [\mathcal{E}^\vee]_S$ is. Therefore, we have

Corollary 2.3 *Assume that both S and T are K3 surfaces and that a vector bundle $\mathcal{E}$ on $S \times T$ satisfies both $(\Sigma_{S,T})$ and $(\Sigma_{T,S})$. Then the classification morphism $T \to M_S$ is an isomorphism onto a connected component if and only if $S \to M_T$ is.*

Remark 2.4 (i) The Chern character $\operatorname{ch} \mathcal{E}$ is a cocycle with integral coefficients ([7], Lemma 4.1). Hence in the situation of the theorem, $[\mathcal{E}]_S$ is an isomorphism from $H^*(S, \mathbb{Z})$ to $H^*(T, \mathbb{Z})$.

(ii) We assume $\mathcal{E}$ locally free only to simplify the notation. The same holds for a coherent sheaf $\mathcal{E}$ on $S \times T$ that is both S-flat and T-flat.

The composite $\int_S \mathcal{E} \circ \int_T \mathcal{E}^\vee$ of two integral functors is also an integral functor. Moreover, its kernel is the composite $\mathbf{R}\pi_{23*}(\pi_{12}^* \mathcal{E}^\vee \otimes \pi_{13}^* \mathcal{E})$ of the two kernels, which is isomorphic to $\mathcal{O}_\Delta[2]$ under the equivalent conditions of Theorem 2.2 (where [2] denotes a shift in the derived category). Hence, we have

Theorem 2.5 *Under the same conditions as in Corollary 2.3, the integral functor with kernel $\mathcal{E}$*

$$\int_S \mathcal{E} \colon \mathbf{D}(\operatorname{Coh} S) \to \mathbf{D}(\operatorname{Coh} T)$$

is an equivalence of categories. Moreover, its inverse is $\int_T \mathcal{E}^\vee[-2]$, that is, the integral functor with kernel $\mathcal{E}^\vee$ in the opposite direction shifted by 2 *to the right.*

3 Construction of examples

Let r and s be integers with $r \geq 2$ and $s \geq 1$. In this section, we investigate the moduli space $\widehat{S} = M_S(r, h, s)$ for a polarized K3 surface (S, h) satisfying the following conditions:

$$h = e + rf, \text{ where } \operatorname{Pic} S = \mathbb{Z}e \oplus \mathbb{Z}f, \text{ and the divisor classes } e \text{ and } f \text{ have intersection numbers } e^2 = -2r,\ e \cdot f = s + 1 \text{ and } f^2 = 0. \tag{$*$}$$

By the surjectivity of the period map ([1], Chap. VIII), such polarized K3 surfaces (S, h) exist. Note that there is no -2-vector perpendicular to h. For integers m and n, we have

$$\begin{aligned} (me + nf)^2 &= 2m\{-rm + (s+1)n\} \\ \text{and} \quad h \cdot (me + nf) &= rsm + \{-rm + (s+1)n\}. \end{aligned} \tag{5}$$

Lemma 3.1 *(i) If $s + 1$ does not divide $r - 1$, then $D^2 \geq 0$ for every effective divisor D on S.*

(ii) Assume that $r - 1 = k(s + 1)$, and put $e' = e + kf$. Then the divisor class e' contains an irreducible curve E' isomorphic to $\mathbb{P}^1$. Moreover, E' is the unique irreducible curve on S with negative self-intersection.

Proof Since the canonical class K_S of S is trivial, any irreducible curve C on S satisfies

$$C^2 = C \cdot (C + K_S) = 2p_a(C) - 2 \geq -2 \tag{6}$$

If $s+1$ does not divides $r-1$, then $C^2 = (me+nf)^2 \neq -2$ by (5). Therefore, we have (i).

Assume that $s+1$ divides $r-1$ and let C be an irreducible curve with negative self-intersection. By (5) and (6), C is linearly equivalent to $\pm e'$. Since $h \cdot C$ and $h \cdot e' = rs - 1$ are positive, C is linearly equivalent to e'. By the Riemann–Roch theorem, $\chi(\mathcal{O}_S(e')) = (e'^2)/2 + 2 = 1$ and e' contains an effective divisor E'. Since $(E'^2) < 0$, E' contains a curve C' with $(C'^2) < 0$ as an irreducible component. By what we have shown, C' is linearly equivalent to E'. Since $h^0(\mathcal{O}_S(C')) = 1$, we have $E' = C'$, which proves (ii). □

By the lemma, we have

$$H^0(\mathcal{O}_S(e)) = H^0(\mathcal{O}_S(-e)) = 0. \tag{7}$$

since $(e^2) = -2r \leq -4$. By Serre duality and the Riemann–Roch theorem, we have

$$h^1(\mathcal{O}_S(\pm e)) = -\chi(\mathcal{O}_S(\pm e)) = r - 2.$$

Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_S$ at $x \in S$. By duality, we have

$$\dim \operatorname{Ext}^1\big(\mathfrak{m}_x\mathcal{O}_S(e+f), \mathcal{O}_S(f)\big) = h^1(\mathfrak{m}_x(-e)) = r - 1.$$

Hence we obtain the unique exact sequence

$$0 \to \mathcal{O}_S(f)^{\oplus(r-1)} \to E_x \to \mathfrak{m}_x(e+f) \to 0 \tag{8}$$

with the property $\operatorname{Hom}(E_x, \mathcal{O}_S(f)) = 0$. Since

$$\dim \operatorname{Ext}^1\big(\mathcal{O}_S(e+f), \mathcal{O}_S(f)\big) < \dim \operatorname{Ext}^1\big(\mathfrak{m}_x\mathcal{O}_S(e+f), \mathcal{O}_S(f)\big),$$

E_x is locally free of rank r. It is easy to verify that $\det E_x \cong \mathcal{O}_S(h)$ and

$$\chi(E_x) = (r-1)\chi(\mathcal{O}_S(f)) + \chi(\mathcal{O}_S(e+f)) - 1 = r + s. \tag{9}$$

We prove (i) and (iii) of Theorem 1.2 for this pair (S, h):

Theorem 3.2 *If (S, h) satisfies $(*)$, then*

(i) for every $x \in S$, E_x is stable with respect to h and belongs to $M_S(r, h, s)$;

(ii) if $x \neq y \in S$, then E_x and E_y are not isomorphic;

(iii) every member of $M_S(r,h,s)$ is locally free and isomorphic to E_x for some $x \in S$;

(iv) $\widehat{S} = M_S(r,h,s)$ is isomorphic to S; and

(v) there exist a vector bundle $\mathcal{E}$ on the product $S \times S$, a vector bundle G on S and an exact sequence

$$0 \to \mathcal{O}_S(f) \boxtimes G^\vee \to \mathcal{E} \to \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}_S(e+f) \to 0 \tag{10}$$

on $S \times S$ whose restriction to $S \times x$ is isomorphic to the exact sequence (8)*, where $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal $\Delta \subset S \times S$ and $\pi_1, \pi_2\colon S \times S \to S$ are the projections to the two factors. Moreover, this vector bundle $\mathcal{E}$ is a universal family for $M_S(r,h,s)$.*

In order to show the stability of E_x, we need an estimate of the degree $h \cdot D$ for an effective divisor D on S.

Lemma 3.3 *Let m and n be integers. If a divisor class $D = me + nf$ is effective, then $n \geq 0$ and $h \cdot D \geq (rs-1)m$.*

Proof We first consider the case $D^2 \geq 0$. By (5), we have

$$m\{-rm + (s+1)n\} \geq 0 \quad \text{and} \quad h \cdot D = rsm + \{-rm + (s+1)n\} > 0.$$

Hence we have $m \geq 0$ and $-rm + (s+1)n \geq 0$. Therefore, we have $(s+1)n \geq rm \geq 0$ and $h \cdot D \geq rsm \geq (rs-1)m$. This proves the lemma in the case where $s+1$ does not divide $r-1$. Assume that $s+1$ divides $r-1$. By (ii) of Lemma 3.1, D is linearly equivalent to the sum of an effective divisor D_+ with nonnegative self-intersection number and a multiple of E'. The lemma holds for both D_+ and E' since $E' = e + kf$ and $h \cdot E' = rs - 1$. Therefore, it also holds for D. □

Proposition 3.4 *E_x is μ-stable with respect to h.*

Proof Let F be a torsion free quotient sheaf of $E_x(-f)$, which is neither zero nor $E_x(-f)$ itself. It suffices to show that

$$\mu(F) > \mu(E_x(-f)) = (h \cdot e)/r = s - 1.$$

Tensoring (8) with $\mathcal{O}_S(-f)$ gives the exact sequence

$$0 \to \mathcal{O}_S^{\oplus(r-1)} \to E_x(-f) \to \mathfrak{m}_x(e) \to 0.$$

with $\mathrm{Hom}(E_x(-f), \mathcal{O}_S) = 0$. Let F_0 be the image of the composite

$$\mathcal{O}_S^{\oplus(r-1)} \hookrightarrow E_x(-f) \to F.$$

Then $\det F_0$ is effective. Since $\mathrm{Hom}(F_0, \mathcal{O}_S) \subset \mathrm{Hom}(E_x(-f), \mathcal{O}_S) = 0$, F_0 and hence $\det F_0$ is nontrivial.

If $\operatorname{rank} F_0 < \operatorname{rank} F$, then the quotient F/F_0 is isomorphic to $\mathfrak{m}_x(e)$. Hence $\det F \cong \det F_0 \otimes \mathcal{O}_S(e)$ and we have

$$\mu(F) \geq \frac{(h \cdot e)}{r(F)} > \frac{(h \cdot e)}{r} = \mu(E_x(-f)).$$

Assume that $\operatorname{rank} F_0 = \operatorname{rank} F$ and put $\det F_0 \cong \mathcal{O}_S(me + nf)$. Then, by the preceding lemma, we have

$$\mu(F) \geq \mu(F_0) = \frac{h \cdot (me + nf)}{r(F)} \geq \frac{m(rs-1)}{r(F)} \geq ms.$$

Hence, we have $\mu(F) > s - 1$ if $m \geq 1$. So assume further that $m = 0$. Since a general member of $|f|$ is an (irreducible) elliptic curve, we have $h^0(\det F_0) = h^0(\mathcal{O}_S(nf)) = n + 1$. (Even when $s + 1$ divides $r - 1$, $|f|$ is base point free since $|f - \ell e'| = \emptyset$ for every integer $\ell \geq 1$.) There exists a trivial subbundle T of rank $r(F) + 1$ such that

$$\mathcal{O}_S^{\oplus r(F)+1} \cong T \to F_0$$

is surjective outside a finite set of points. Since its kernel is isomorphic to $(\det F_0)^{-1}$, we have

$$n + 1 = h^0(\det F) \geq r(F) + 1,$$

that is, $n \geq r(F)$. Hence, in this case, we also have

$$\mu(F_0) = \frac{(h \cdot nf)}{r(F)} \geq (f.h) = s + 1 > \mu(E_x(-f)). \quad \square$$

(i) of Theorem 3.2 follows from this and the computation (9). It is easy to prove (ii):

Lemma 3.5 *If $E_x \cong E_y$, then $x = y$.*

Proof Let $f\colon E_x \to E_y$ be an isomorphism. Since $\mathrm{Hom}(\mathcal{O}_S, \mathcal{O}_S(e)) = 0$ by (7), f maps the subsheaf $\mathcal{O}_S^{\oplus r-1} \subset E_x$ to the subsheaf $\mathcal{O}_S^{\oplus r-1} \subset E_y$. Therefore, f induces an isomorphism from $\mathfrak{m}_x(e)$ onto $\mathfrak{m}_y(e)$. Hence we have $x = y$. $\square$

The sheaf $I_\Delta \otimes \pi_1^*\mathcal{O}_S(e)$ is flat over S and its restriction to the fibre $S \times x$ is isomorphic to $\mathfrak{m}_x \otimes \mathcal{O}_S(e)$ for every $x \in S$. Let G be the sheaf on S associated to the presheaf

$$\begin{aligned} U \mapsto &\mathcal{E}xt^1_{S\times U}(\mathcal{I}_\Delta \otimes \pi_1^*\mathcal{O}_S(e)_{|S\times U}, \mathcal{O}_{S\times U}) \\ &\cong \mathcal{E}xt^1_{S\times U}(\mathcal{I}_\Delta \otimes \pi_1^*\mathcal{O}_S(e+f)_{|S\times U}, \pi_1^*\mathcal{O}_S(f)) \end{aligned}$$

for U an open subset of S. Since $\mathrm{Hom}(\mathfrak{m}_x(e), \mathcal{O}_S) = \mathrm{Ext}^2(\mathfrak{m}_x(e), \mathcal{O}_S) = 0$ for every $x \in S$, G is locally free and we have

$$G \otimes k(x) \cong \mathrm{Ext}^1\big(\mathfrak{m}_x(e), \mathcal{O}_S\big).$$

Precisely stated, the exact sequence (8) is

$$0 \to \mathcal{O}_S(f) \otimes_{\mathbb{C}} \mathrm{Ext}^1\big(\mathfrak{m}_x(e), \mathcal{O}_S\big)^\vee \to E_x \to \mathfrak{m}_x(e+f) \to 0. \tag{11}$$

Hence there exists an exact sequence

$$0 \to \pi_1^*\mathcal{O}_S(f) \otimes \pi_2^*G^\vee \to \mathcal{E} \to I_\Delta \otimes \pi_1^*\mathcal{O}_S(e+f) \to 0$$

over $S \times S$ whose restriction to $S \times x$ is (11) for every $x \in S$. This is the exact sequence (10) of Theorem 3.2. We have $G^\vee \cong R^1\pi_{2*}(I_\Delta \otimes \pi_1^*\mathcal{O}(e))$ by (relative) Serre duality and obtain the exact sequence

$$0 \to \mathcal{O}_S(e) \to G^\vee \to \mathcal{O}_S \otimes_{\mathbb{C}} H^1(\mathcal{O}_S(e)) \to 0 \tag{12}$$

from $0 \to I_\Delta \to \mathcal{O}_{S\times S} \to \mathcal{O}_\Delta \to 0$.

Let $S \to M_S(r,h,s)$ be the classification morphism of $\mathcal{E}$. Since $M_S(r,h,s)$ is smooth of dimension 2, it is an open embedding by Lemma 3.5; since $M_S(r,h,s)$ is connected, it is an isomorphism. So we have completed the proof of Theorem 3.2.

Now we compute the divisor class $\widehat{h}$ on the moduli space $\widehat{S} = M_S(r,h,s)$ defined in Proposition 1.1. We have

$$\begin{aligned} c_1(\mathcal{O}_S(f) \boxtimes G^\vee) &= (r-1)\pi_1^*f + \pi_2^*e \\ c_2(\mathcal{O}_S(f) \boxtimes G^\vee) &= (r-2)\pi_1^*f \cup \pi_2^*e. \end{aligned}$$

by the exact sequence (12), and

$$\begin{aligned} c_1(\mathcal{E}) &= (r-1)\pi_1^*f + \pi_2^*e + \pi_1^*(e+f) = \pi_1^*h + \pi_2^*e \\ c_2(\mathcal{E}) &= c_1(\mathcal{O}_S(f) \boxtimes G^\vee) \cup \pi_1^*(e+f) + c_2(\mathcal{O}_S(f) \boxtimes G^\vee) + \Delta \end{aligned}$$

by the exact sequence (10). Hence the middle Künneth component of $c_2(\mathcal{E})$ is

$$c_2^{\mathrm{mid}}(\mathcal{E}) = \pi_1^*(e + (r-1)f) \cup \pi_2^* e + \Delta \in H^2(S) \otimes H^2(S).$$

Therefore, by (1), (2) of the introduction and Proposition 1.1, we have

$$\begin{aligned} &\varphi = e, \quad \psi = (h \cdot e + (r-1)f)e + h = h + (2rs - s - 1)e \quad \text{and} \\ &\widehat{h} = \psi - 2(r-1)s\varphi = h + (s-1)e = se + rf = sh + r(1-s)f \end{aligned} \tag{13}$$

By our assumption $(*)$, $\{e, f\}$ is a free $\mathbb{Z}$-basis of $\operatorname{Pic} S$. Hence we have

Lemma 3.6 *$\widehat{h}$ is primitive if and only if r and s are coprime.*

Since $(\widehat{h})^2 = 2rs > 0$ and $(\widehat{h} \cdot e') > 0$, $\widehat{h}$ is ample by Lemma 3.1. Since $2(h \cdot e)/(e^2) = s - 1$, we have

Lemma 3.7 *(i) $\widehat{h}$ is ample.*

(ii) $\widehat{h}$ is the reflection of h with respect to e, and hence $(\widehat{h})^2 = h^2$.

Let K_x be the restriction of the universal bundle $\mathcal{E}$ to a fibre $x \times S$ of the other direction. Restricting (10) to $x \times S$, we have an exact sequence

$$0 \to G^\vee \to K_x \to \mathfrak{m}_x \to 0. \tag{14}$$

Since K_x is locally free, this does not split.

Assume that r and s are coprime and fix a pair of integers (t, u) such that $ts + ur = 1$. The first Chern class of $\mathcal{E} \otimes \pi_2^* \mathcal{O}_S(-ue + tf)$ is $\pi_1^* h + t\pi_2^* \widehat{h}$. Hence this universal bundle is normalized and its restriction to $x \times S$ is $K_x(-ue + tf)$.

Lemma 3.8 *Assume that $(r, s) = 1$ and let F be a torsion free quotient sheaf of $K_x(-ue + tf)$, which is neither zero nor $K_x(-ue + tf)$ itself. If $c_1(F) = \ell\widehat{h}$ for an integer ℓ, then $\ell/r(F) > t/r$.*

Proof $F(ue - tf)$ is a quotient of K_x. By the exact sequences (12) and (14), either $c_1(F(ue - tf))$ or $c_1(F(ue - tf)) - e$ is effective. If we put $c_1(F(ue - tf)) = me + nf$, then we have $n \geq 0$ by Lemma 3.3. Since $me + nf - (ue - tf)q = \ell(se + rf)$ by our assumption, we have $n + tq = \ell r$, where we put $q = r(F)$. Since r and t are coprime and since $q < r$, $n = 0$ is impossible. Hence we have $\ell r - tq = n > 0$. $\square$

4 Proof of Proposition 1.1 and Theorem 1.2

First we prove (i) of Proposition 1.1, for which we do not need the results of the preceding section. Let $\mathcal{E}$ be a universal sheaf on $S \times \widehat{S}$ with $\widehat{S} = M_S(r, h, s)$. Any universal sheaf $\mathcal{E}'$ is isomorphic to $\mathcal{E} \otimes \pi_M^* \Lambda$ for a line bundle Λ on $\widehat{S}$. Hence we have

$$\begin{aligned} c_1(\mathcal{E}') &= c_1(\mathcal{E}) + r\pi_M^*\lambda, \\ c_2(\mathcal{E}') &= c_2(\mathcal{E}) + (r-1)c_1(\mathcal{E}) \cup \pi_M^*\lambda + \frac{r(r-1)}{2}\pi_M^*\lambda^2, \quad \text{and} \\ c_2^{\mathrm{mid}}(\mathcal{E}') &= c_2^{\mathrm{mid}}(\mathcal{E}) + (r-1)\pi_S^* h \cup \pi_M^*\lambda, \end{aligned}$$

where we put $\lambda = c_1(\Lambda)$. Therefore, we have

$$\varphi' = \varphi + r\lambda, \quad \psi' = \psi + (r-1)h^2\lambda$$

and

$$\psi' - \frac{(r-1)h^2}{r}\varphi' = \psi - \frac{(r-1)h^2}{r}\varphi.$$

Since $h^2 = 2rs$, we have (i).

Now take a family $(\pi\colon \mathcal{S} \to T, \mathcal{L})$ of polarized K3 surfaces of degree $2rs$ with the following properties:

(a) The classification morphism

$$T \to \mathcal{F}_{rs+1} \quad \text{given by} \quad t \mapsto (S_t, L_t)$$

is dominant, where S_t is the fibre $\pi^{-1}(t)$ and L_t the restriction of $\mathcal{L}$ to S_t.

(b) There exists a point $o \in T$ such that (S_o, L_o) satisfies the conditions $(*)$ at the beginning of §3.

For example, these are satisfied by an open subset of the Hilbert scheme of a suitable projective space $\mathbb{P}^N$. By virtue of [3], there exists a scheme $\mathfrak{M}$ over T whose fibre over $t \in T$ is $\widehat{S}_t = M_{S_t}(r, L_t, s)$. By [6], $\mathfrak{M}$ is smooth of dimension 2 over T. Since r and s are coprime, there exists a (coherent) sheaf $\widetilde{\mathcal{E}}$ on the fibre product $\mathcal{S} \times_T \mathfrak{M}$ whose restriction to $S_t \times \widehat{S}_t$ is a universal family $\mathcal{E}_t$ for every $t \in T$.

Define φ_t, ψ_t and $\widehat{h}_t \in H^{1,1}(\widehat{S}_t, \mathbb{Z})$ by

$$c_1(\mathcal{E}_t) = \pi_S^* c_1(L_t) + \pi_M^*\varphi_t, \quad c_1(L_t) \cup c_2^{\mathrm{mid}}(\mathcal{E}_t) = p \otimes \psi_t,$$
$$\text{and} \quad \widehat{h}_t = \psi_t - 2(r-1)s\varphi_t.$$

We have $(\widehat{h}_o^2) = 2rs$ by Lemma 3.7 and $\widehat{h}_o \equiv s\varphi_o \bmod r$ by (13). Hence we have $(\widehat{h}_t^2) = 2rs$ and $\widehat{h}_t \equiv s\varphi_t \bmod r$ for every t, which completes the proof of Proposition 1.1.

Now we prove Theorem 1.2. Let $(\mathcal{S}/T, \mathcal{L})$ and $\mathfrak{M}/T$ be as above. As we saw in the preceding section, $\widehat{h}_o$ is primitive and ample, and $\mathcal{E}_o$ is locally free. (See Lemmas 3.6, 3.7 and Theorem 3.2.) Hence, replacing T by an open subset containing o, we may assume that the following is satisfied for every $t \in T$:

1. the divisor class $\widehat{h}_t$ on $\widehat{S}_t$ is primitive and ample, and
2. $\mathcal{E}_t$ is locally free.

In particular, we have (i) and (ii) of Theorem 1.2.

Note that the isomorphism $[\mathcal{E}]_S \colon H^*(S, \mathbb{Q}) \to H^*(T, \mathbb{Q})$ in the diagram (4) maps Hodge cocycles to Hodge cocycles since $\operatorname{ch}\mathcal{E}$ is algebraic. Therefore, S_t and the moduli space $\widehat{S}_t$ have the same Picard number (see [7], Theorem 1.4 for more general result). Therefore, by our choice of $\mathcal{S}$, the set of points t for which $\widehat{S}_t$ is Picard general is dense in T. Hence we have

3. φ_t is a multiple of $\widehat{h}_t$ for every $t \in T$.

Claim 4.1 *There exists a point $t \in T$ such that $\mathcal{E}_{t|_{x\times\hat{S}}}$ is stable with respect to $\widehat{h}_t$ for every $x \in S_t$.*

Proof Assume the contrary and let Z be the set of points $x \in \mathcal{S}$ such that the restriction of $\mathcal{E}_{\pi(x)}$ to $x \times \widehat{S}_{\pi(x)}$ is unstable. This is a closed subvariety of $\mathcal{S}$ and we have $\pi(Z) = T$ by our assumption. Take a morphism $f \colon C \to Z$ from a curve C and two points p and $q \in C$ such that $\pi(f(p)) = o$ and $S_{\pi(f(q))}$ is Picard general. Let $\mathcal{E}_C$ be the pullback of $\widetilde{\mathcal{E}}$ to $C \times_T \mathfrak{M}$. Then $\mathcal{E}_{C|_{x\times\hat{S}}}$ is unstable for every $x \in C$. Let $\mathcal{F}$ be a destabilizing subsheaf of $\mathcal{E}_C$, that is, an extension to $C \times_T \mathfrak{M}$ of the destabilizing subsheaf over the generic point of C (cf. [4]). Since $c_1(\mathcal{F}_{|q\times\hat{S}})$ is an integral multiple of $\widehat{h}$, so is $c_1(\mathcal{F}_{|p\times\hat{S}})$. But this contradicts Lemma 3.8.

By the claim and Corollary 2.3, if $S = S_t$ is general, then the classification morphism $S \to M_{\hat{S}}$ is an isomorphism onto a connected component, which we denote by $M_{\hat{S}}(r, v\widehat{h}, w)$ for integers v and w. Since $M_{\hat{S}}(r, v\widehat{h}, w)$ is of dimension 2, we have $2rw = v^2(\widehat{h})^2$ and $w = v^2 s$ by the formula

$$h^1(\mathcal{E}nd\, E) - 2\dim \operatorname{End} E = -\chi(\mathcal{E}nd\, E) = c_1(E)^2 - 2r(E)\big(\chi(E) - r(E)\big)$$

(cf. [6]). So we have proved (iii) and (iv) of Theorem 1.2. (v) is a consequence of (iv) and Theorem 2.5.

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Shigeru Mukai,
Graduate School of Mathematics,
Nagoya University,
Furō-chō, Chikusa-ku,
Nagoya, 464-01, Japan
e-mail: mukai@math.nagoya-u.ac.jp