Abstracts

Finite and infinite generation of Nagata invariant ring

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An m-dimensional linear representation of an (algebraic) group G induces an action on the polynomial ring $\mathbf{C}[z_1, \dots, z_m]$ of m variables. This is called a *linear action* on the polynomial ring. In 1890, Hilbert showed that the invariant ring was finitely generated for classical representations of the general and special linear groups. The following is known as his (original) fourteenth problem ([1]):

Question. *Is the invariant ring $\mathbf{C}[z_1, \dots, z_m]^G$ of a linear action of an algebraic group finitely generated?*

The answer is affirmative for the (1-dimensional) additive algebraic group $\mathbf{G}_a$ ([3]). In 1958, Nagata considered the standard unipotent linear action

$$(1) \qquad (t_1, \dots, t_n) \in \mathbf{C}^n \ \downarrow \ \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_n] =: S$$

$$\begin{cases} x_i \mapsto x_i \\ y_i \mapsto y_i + t_i x_i \end{cases}, \quad 1 \le i \le n,$$

of $\mathbf{C}^n$ on the polynomial ring S of $2n$ variables and showed that the invariant ring S^G with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension 3 was not finitely generated for $n = 16$. I studied this example systematically and obtained the following:

Theorem 1. *The invariant ring S^G of* (1) *with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension r is finitely generated if and only if*

$$\frac{1}{2} + \frac{1}{r} + \frac{1}{n-r} > 1.$$

This inequality is equivalent to the finiteness of the Weyl group $W(T_{2,n-r,r})$ of the Dynkin diagram $T_{2,n-r,r}$ with three legs of length 2, $n-r$ and r. There are four infinite series [I]–[IV] and five exceptonal cases [V]–[IX] where this holds:

	[I]	[II]	[III]	[IV]	[V]	[VI]	[VII]	[VIII]	[IV]
r	1		2		3	3	4	3	5
$n-r$		1		2	3	4	3	5	3
diagram	A_n	A_n	D_n	D_n	E_6	E_7	E_7	E_8	E_8

The ‘if’ part of the theorem is proved case by case. In the cases [I] and [III], the invariant ring is very explicit and the proof is immediate. The case [II] is classical and the invariant ring S^G is the homogeneous coordinate ring of a Grassmannian variety. In the case [IV], that is, $\dim G = 2$, the invariant ring is the total coordinate ring, or the Cox ring, of the moduli space of parabolic 2-bundles on an n-pointed projective line. Note that the following part of the 14th problem seems still open:

Question. *Is the invariant ring* $\mathbf{C}[z_1, \dots, z_m]^G$ *of a linear action of the 2-dimensional additive group* $G = \mathbf{G}_a \times \mathbf{G}_a$ *finitely generated?*

See [2] for the 'only if' part.

References

1. Nagata, M.: On the fourteenth problem of Hilbert, Proc. Int'l Cong. Math., Edingburgh, 1958, pp. 459–462, Cambridge Univ. Press, 1960.
2. Mukai, S.: Counterexample to Hilbert's fourteenth problem for three dimensional additive groups, RIMS preprint, #1343, 2001.
3. Seshadri, C.S.: On a theorem of Weitzenböck in invariant theory, J. Math. Kyoto Univ., **1**(1962), 403–409.