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# Workshop: Komplexe Algebraische Geometrie

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## Introduction by the Organisers

Something meaninful...

**MSC Classification:**

# Workshop: Komplexe Algebraische Geometrie

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## Abstracts

### Geometric proof of finite generation of certain rings of invariants

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Let $\rho : \mathbf{C}^n \downarrow S = \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ be the standard unipotent linear action of the $n$-dimensional additive group $\mathbf{C}^n$ on the polynomial ring $S$ of $2n$ variables, that is, $(t_1, \dots, t_n) \in \mathbf{C}^n$ acts by $\begin{cases} x_i \mapsto x_i \\ y_i \mapsto y_i + t_i x_i \end{cases}$ for $1 \le i \le n$. In 1958, Nagata[5] proved that the ring of invariants $S^G$ with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension 3 was not finitely generated for $n = 16$. We studied this example more systematically and obtained the following:

**Theorem** *The ring of invariants $S^G$ of $\rho$ with respect to a general linear subspace $G \subset \mathbf{C}^n$ of codimension $r$ is finitely generated if and only if*

$$\frac{1}{2} + \frac{1}{r} + \frac{1}{n-r} > 1.$$

This inequality is equivalent to the finiteness of the Weyl group of the Dynkin diagram $T_{2,r,n-r}$ with three legs of length 2, $r$ and $n-r$. The 'only if' part of the theorem follows from this observation and the following geometric interpretation of $S^G$. (See [2] for the details.)

**Proposition** *Let $\mathbf{P}^{r-1}$ be the projective space $\mathbf{P}_*(\mathbf{C}^n/G)$ and $\{p_1, \dots, p_n\} \subset \mathbf{P}^{r-1}$ be the image of the standard basis of $\mathbf{C}^n$. Then the ring of invariants $S^G$ is isomorphic to the total coordinate ring, or the Cox ring, $TC(X)$ of the blow-up $X = X_G$ of $\mathbf{P}^{r-1}$ at the $n$ points $p_1, \dots, p_n$.*

In this talk, as a continuation of [4], I explained the proof of 'if' part in the case $\dim G = 2$. The variety $X$ in the proposition is the blow-up of $\mathbf{P}^{n-3}$ at $n$ points in general position. The key of our proof is that $X$ is the moduli space of parabolic 2-bundles over an $n$-pointed projective line $(\mathbf{P}^1 : p_1, \dots, p_n)$ for a certain weight. This fact enables us to determine the effective cone of $X$, the movable cone $\operatorname{Mov} X$ (see [2]) and its chamber structure. For example, taking

$$\begin{cases} -K_X = (n-2)h - (n-4)\sum_1^i e_i \\ f_1 = h + e_1 - e_2 - \cdots - e_n \\ f_2 = h - e_1 + e_2 - \cdots - e_n \\ \vdots \\ f_n = h - e_1 - e_2 - \cdots + e_n \end{cases}$$

as a basis of $\operatorname{Pic} X \otimes \mathbf{Q}$, a divisor $D \sim -aK_X + \sum_1^n b_i f_i$ is movable if and only if

$$(n-4)a - \sum_{i \in I} b_i + \sum_{j \notin I} b_j \ge 0$$

holds for every $I \subset \{1. \dots, n\}$ with $|I|$ even and $|b_i| \le a$ holds for evey $1 \le i \le n$, where $h$ is the pull-back of a hyperplane and $e_1, \dots, e_n$ are the exceptional divisors.

The cone $\operatorname{Mov} X$ is divided into finitely many chambers, which are rational polyhedral cones, by the flopping walls. For every movable divisor $D$ on $X$, the graded ring $\bigoplus_{n\geq 0} H^0(X, nD)$ is finitely generated by the GIT-construction of the moduli spaces. Hence the ($\operatorname{Mov} X$)-part of $TC(X)$ is finitely generated. The total coordinate ring $TC(X)$ is finitely generated since it is generated by the equations of $2^{n-1}$ exceptional divisors, whose linear equivalence classes are

$$e_I = \frac{1}{4}(-K_X - \sum_{i\in I} f_i + \sum_{j\notin I} f_j)$$

with $I$ odd, over the ($\operatorname{Mov} X$)-part. In the case of $\dim G = 3$ (and $n \leq 8$), the finite generation is similarly proved replacing the parabolic bundles over $\mathbf{P}^1$ by vector bundles over a del Pezzo surface.

In the case $r = 3$, the minimal (finite) set of generators of $TC(X)$, $X$ itself being a del Pezzo surface, is determined in [1]. Other cases of 'if' part is easy.

## References


[1] V. Batyrev and O.N. Popov, *The Cox ring of a del Pezzo surface*, Arithmetic of higher-dimensional algebraic varieties, eds. Poonen and Tschinkel, Birkhauser, 2004, pp. 85–103.

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[5] M. Nagata, *On the fourteenth problem of Hilbert*, Proc. Int'l Cong. Math., Edingburgh, 1958, pp. 459–462, Cambridge Univ. Press, 1960.


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