# Prescribed Ricci tensor in Finslerian conformal class

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**Abstract.** In this paper, a new geometric quantity, the trace-free horizontal Ricci tensor of a Finsler manifold is proposed. The conformal changes of this tensor, relatively to the conformal deformations of Finsler metrics when an Ehresmann form varies, are studied. As an application, it is shown that on a closed manifold two conformal Finsler metrics with the same horizontal Ricci tensor must be homothetic.

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**Key words**: Ehresmann connection; pulled-back tangent bundle; trace-free Ricci tensor; conformal deformation of Finsler metrics.

## 1 Introduction

A prescribed Ricci tensor is a very interesting and important problem in Geometry and in General Relativity. For Riemannian Geometry this problem has been studied by D. DeTurck [2] and by X. Xu [9]. Furthermore, an Ehresmann connection on the slit tangent bundle of a Finsler manifold plays a fundamental role and is a powerfull tool in studying Finsler Geometry (for example see Aikou-Kozma [1] and Benjancu-Farran [4]).

In this paper, using the Chern's curvature tensors, we introduce canonically the notion of trace-free Ricci tensor in Finsler geometry. We investigate the behavior of trace-free horizontal Ricci $(1,1;0)$-tensor $\mathbf{B}_F^H$ under the conformal change of Finsler metrics when the Ehresmann form varies. Using this tensor, we prove mainly the Theorem 3.1 and the Theorem 4.3.

The rest of this work is organized as follows. In Section 2, we briefly describe the notation and convention used, but for a more detailed description we refer to the book of Bao-Chern-Shen [3] and to the reference [8]. In Section 3, we study the conformal change of $\mathbf{B}_F^H$ associated with the Chern connection $\nabla$. The main result of this Section is the Theorem 3.1, that generalizes the A. L. Besse's result [5]. The Section 4 prescribes the horizontal Ricci tensor in the conformal class of $F$. The main result in this Section is the Theorem 4.3 that generalizes the X. Xu's Theorem [9] given in the book of E. Hebey.

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# 2 Preliminaries

Throughout this paper, $M$ is an $n-$dimensional $C^\infty$ manifold. We denote by $T_xM$ the tangent space at $x \in M$, by $TM := \bigcup_{x\in M} T_xM$ the tangent bundle of $M$ and by $SM := \{(x,[y])\}$ the sphere bundle of $M$. Set $\overset{\circ}{T}M = TM\backslash\{0\}$ and the natural projection $\pi : TM \longrightarrow M : \pi(x,y) \longmapsto x$. Let $(x^1,...,x^n)$ be a local coordinate on an open subset $U$ of $M$ and $(x^1,...,x^n,y^1,...,y^n)$ be the local coordinate on the open $\pi^{-1}(U) \subset TM$. The local coordinate system $(x^i)_{i=1,...,n}$ produces the bases sections $\{\frac{\partial}{\partial x^i}\}_{i=1,...,n}$ and $\{dx^i\}_{i=1,...,n}$ respectively, for the tangent bundle $TM$ and cotangent bundle $T^*M$. We use Einstein summation convention.

**Definition 2.1.** A function $F : TM \longrightarrow [0,\infty)$ is called a *Finsler metric* on $M$ if :

(1) $F$ is $C^\infty$ on the entire slit tangent bundle $\overset{\circ}{T}M$,

(2) $F$ is positively 1-homogeneous on the fibers of $TM$, that is for all $c > 0$, $F(x,cy) = cF(x,y)$,

(3) the Hessian matrix $(g_{ij}(x,y))_{1\leq i,j\leq n}$ with elements

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j} \tag{2.1}$$

is positive definite at every point $(x,y)$ of $\overset{\circ}{T}M$.

Given a manifold $M$ and a Finsler metric $F$ on $TM$, the pair $(M,F)$ is called a *Finsler manifold.*

**Remark 2.2.** $F(x,y) \neq 0$ *for all* $x \in M$ *and for every* $y \in T_xM\backslash\{0\}$.

## 2.1 Finsler-Ehresmann connection and Chern connection

Consider the tangent mapping $\pi_*$ of the restricted projection $\pi : \overset{\circ}{T}M \longrightarrow M : \pi(x,y) \longmapsto x$. The vertical subspace of $T\overset{\circ}{T}M$ is defined by $\mathcal{V} := ker(\pi_*)$ and is locally spanned by the set $\{F\frac{\partial}{\partial y^i}, 1 \leq i \leq n\}$, on each $\pi^{-1}(U) \subset \overset{\circ}{T}M$.

An horizontal subspace $\mathcal{H}$ of $T\overset{\circ}{T}M$ is by definition any complementary to $\mathcal{V}$. The bundles $\mathcal{H}$ and $\mathcal{V}$ give a smooth splitting [4, 10]

$$T\overset{\circ}{T}M = \mathcal{H} \oplus \mathcal{V}. \tag{2.2}$$

An Ehresmann connection is a selection of horizontal subspace $\mathcal{H}$ of $T\overset{\circ}{T}M$.

As explained in [6], all Finsler metric $F$ on $M$ induces a vector field on $\overset{\circ}{T}M$ in the form

$$G(x,y) = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $x = (x^i) \in M$, $y = y^i\frac{\partial}{\partial x^i} \in \overset{\circ}{T}M$ and the elements

$$G^i(x,y) := \frac{1}{4}g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)y^jy^k$$

are $y$-homogeneous of degree two. The vector field $G$ is called *spray* on $M$ and the $G^i, i = 1, .., n$ are called *spray coefficients* of $G$. Let consider the functions

$$N^i_j(x,y) := \frac{\partial G^i(x,y)}{\partial y^j}, \quad 1 \leq i, j \leq n.$$

One says that $N^i_j(x,y)$ are *Ehresmann connection coefficients* on $\mathring{T}M$. Otherwise

$$N^i_j = -\frac{1}{F}\mathcal{A}_{jkl}g^{il}\gamma^k_{rs}y^r y^s + \gamma^i_{jk}y^k, \quad i, j, k, r, s = 1, ..., n; \tag{2.3}$$

where $\gamma^i_{jk} := \frac{1}{2}g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)$ are formal Christoffel symbols of second kind. One defines a global $\pi^*TM$-valued $C^\infty$ form on $\mathring{T}M$ by

$$\theta = \frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^i_j dx^j). \tag{2.4}$$

**Remark 2.3.**

1. For objects invariant under $y \mapsto cy$, we can consider $N^i_j$ as Ehresmann connection coefficients on $SM$. In that case (we must work with)

$$\frac{N^i_j}{F} := -\mathcal{A}_{jkl}g^{il}\gamma^k_{rs}\frac{y^r}{F}\frac{y^s}{F} + \gamma^i_{jk}\frac{y^k}{F}. \tag{2.5}$$

2. On a Finsler manifold, the local tensor $\frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^i_j dx^j)$ defines a global tensor on $\mathring{T}M$ (see [3]).

In the sequel, we consider the following definitions.

**Definition 2.4.** Let $\pi : \mathring{T}M \longrightarrow M$ be the restricted projection.

1. A Finsler-Ehresmann connection of $\pi$ is the subbundle $\mathcal{H}$ of $T\mathring{T}M$ given by

$$\mathcal{H} := ker\theta, \tag{2.6}$$

where $\theta : T\mathring{T}M \longrightarrow \pi^*TM$ is the bundle morphism defined in (2.4).

2. The form $\theta : T\mathring{T}M \longrightarrow \pi^*TM$ induces a linear map

$$\theta|_{(x,y)} : T_{(x,y)}\mathring{T}M \longrightarrow T_xM, \quad for\ each\ point \quad (x,y) \in \mathring{T}M; \tag{2.7}$$

where $x = \pi(x,y)$.
The vertical lift of a section $\xi$ of $\pi^*TM$ is a unique section $\mathbf{v}(\xi)$ of $T\mathring{T}M$ such that for every $(x,y) \in \mathring{T}M$,

$$\pi_*(\mathbf{v}(\xi))|_{(x,y)} = 0_{(x,y)} \quad and \quad \theta(\mathbf{v}(\xi))_{(x,y)} = \xi_{(x,y)}. \tag{2.8}$$

3. The differential projection $\pi_* : T\mathring{T}M \longrightarrow \pi^*TM$ induces a linear map

$$\pi_*|_{(x,y)} : T_{(x,y)}\mathring{T}M \longrightarrow T_xM, \quad for\ each\ point \quad (x,y) \in \mathring{T}M; \tag{2.9}$$

where $x = \pi(x,y)$.
The horizontal lift of a section $\xi$ of $\pi^*TM$ is a unique section $\mathbf{h}(\xi)$ of $T\mathring{T}M$ such that for every $(x,y) \in \mathring{T}M$,

$$\pi_*(\mathbf{h}(\xi))|_{(x,y)} = \xi_{(x,y)} \quad and \quad \theta(\mathbf{h}(\xi))_{(x,y)} = 0_{(x,y)}. \tag{2.10}$$

**Remark 2.5.**

1. The vector bundle $\pi^*TM$ can be naturally identified with the horizontal sub-bundle $\mathcal{H}$ of $T\mathring{T}M$ or with the vertical $\mathcal{V}$ (see [1], [4]). Thus any section $\xi$ of $\pi^*TM$ is considered as a section of $\mathcal{H}$ or a section of $\mathcal{V}$. In fact

$$\xi = \xi^i \frac{\partial}{\partial x^i} \in \Gamma(\pi^*TM) \Longleftrightarrow \mathbf{h}(\xi) = \xi^i \frac{\delta}{\delta x^i} \in \Gamma(\mathcal{H}), \tag{2.11}$$

and

$$\xi = \xi^i \frac{\partial}{\partial x^i} \in \Gamma(\pi^*TM) \Longleftrightarrow \mathbf{v}(\xi) = \xi^i F \frac{\partial}{\partial y^i} \in \Gamma(\mathcal{V}), \tag{2.12}$$

where

$$\left\{ \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} = \mathbf{h}(\frac{\partial}{\partial x^i}) \right\}_{i=1,\dots,n} \quad \textit{and} \quad \left\{ F \frac{\partial}{\partial y^i} := \mathbf{v}(\frac{\partial}{\partial x^i}) \right\}_{i=1,\dots,n}$$

are respectively horizontal and vertical lifts of the natural local frame field $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\}$ with respect to the Finsler-Ehresmann connection $\mathcal{H}$.

2. In the sequel, for $g$-orthonormal basis sections $\{e_a\}_{a=1,\dots,n}$ of $\pi^*TM$, we denote respectively $\mathbf{h}(e_a)$ and $\mathbf{v}(e_a)$ by $\hat{e}_a$ and $\hat{e}_{a+n}$.

The following theorem defines the Chern connection on $\pi^*TM$.

**Theorem 2.1.** *[8] Let $(M, F)$ be a Finsler manifold, $g$ the fundamental tensor associated with $F$ and $\pi_*$ the tangent mapping of the submersion $\pi : \mathring{T}M \longrightarrow M$. There exist a unique linear connection $\nabla$ on the pulled-back tangent bundle $\pi^*TM$ such that, for all $X, Y \in \Gamma(T\mathring{T}M)$ and for every $\xi, \eta \in \Gamma(\pi^*TM)$, one has the following properties:*

*(i) Symmetry:*

$$\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_*[X, Y],$$

*(ii) Almost $g$-compatibility:*

$$X(g(\xi, \eta)) = g(\nabla_X \xi, \eta) + g(\xi, \nabla_X \eta) + 2\mathcal{A}(\theta(X), \xi, \eta),$$

*where $\mathcal{A}$ is the Cartan tensor.*

## 2.2 Tensor formalism on Chern's curvatures

The tensors that will be considered are defined as follows:

**Definition 2.6.** Let $(M, F)$ be a Finsler manifold. A tensor field $T$ of type $(p_1, p_2; q)$ on $(M, F)$ is a mapping

$$T : \underbrace{\pi^*TM \oplus ... \oplus \pi^*TM}_{p_1-times} \oplus \underbrace{T\mathring{T}M \oplus ... \oplus T\mathring{T}M}_{p_2-times} \longrightarrow \otimes^q \pi^*TM, \quad p_1, p_2 \quad \textit{and} \quad q \in \mathbb{N}$$

which is $C^\infty(\mathring{T}M)$ or $C^\infty(SM)$-linear in each arguments.

The full curvature $\phi$, of Chern connection $\nabla$, is the $(1,2;1)$-tensor defined by

$$\phi(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi, \tag{2.13}$$

where $X, Y \in \Gamma(T\mathring{T}M)$ and $\xi \in \Gamma(\pi^*TM)$. Using the decomposition (2.2), we have

$$\nabla_X = \nabla_{X^H} + \nabla_{X^V}, \tag{2.14}$$

where $X = X^H + X^V$ with $X^H \in \Gamma(\mathcal{H})$ and $X^V \in \Gamma(\mathcal{V})$.

The full curvature $\phi$ can be written as

$$\phi(X,Y)\xi \;=\; \phi^{HH}(X,Y)\xi + \phi^{HV}(X,Y)\xi + \phi^{VH}(X,Y)\xi + \phi^{VV}(X,Y)\xi,$$

where

$$\begin{aligned}
\phi^{HH}(X,Y)\xi &= \phi(X^H,Y^H)\xi = \nabla_{X^H}\nabla_{Y^H}\xi - \nabla_{Y^H}\nabla_{X^H}\xi - \nabla_{[X,Y]^H}\xi,\\
\phi^{HV}(X,Y)\xi &= \phi(X^H,Y^V)\xi = \nabla_{X^H}\nabla_{Y^V}\xi - \nabla_{Y^V}\nabla_{X^H}\xi - \nabla_{[X^H,Y^V]}\xi,\\
\phi^{VH}(X,Y)\xi &= \phi(X^V,Y^H)\xi = \nabla_{X^V}\nabla_{Y^H}\xi - \nabla_{Y^H}\nabla_{X^V}\xi - \nabla_{[X^V,Y^H]}\xi,\\
\phi^{VV}(X,Y)\xi &= \phi(X^V,Y^V)\xi = \nabla_{X^V}\nabla_{Y^V}\xi - \nabla_{Y^V}\nabla_{X^V}\xi - \nabla_{[X^V,Y^V]}\xi.
\end{aligned}$$

As in the Riemannian case, one can define a $(2,2;0)$ version of $\phi$ by the following formula:

$$\begin{aligned}
\Phi(\xi,\eta,X,Y) &= g(\phi(X,Y)\xi,\eta)\\
&= g(R(X,Y)\xi,\eta) + g(P(X,Y)\xi,\eta)\\
&= \mathbf{R}(\xi,\eta,X,Y) + \mathbf{P}(\xi,\eta,X,Y),
\end{aligned}$$

where $\mathbf{R}$ is the *first Chern's curvature tensor* and $\mathbf{P}$ is the *second Chern's curvature tensor*. One has

$$\mathbf{R}(\xi,\eta,X,Y) = \Phi(\xi,\eta,X^H,Y^H) \tag{2.15}$$

and

$$\mathbf{P}(\xi,\eta,X,Y) = \Phi(\xi,\eta,X^H,Y^V) + \Phi(\xi,\eta,X^V,Y^H). \tag{2.16}$$

**Definition 2.7.** Let $(M,F)$ be a Finsler manifold, $R$ the horizontal part of the full curvature tensor associated with the Chern connection. We define

1. the horizontal Ricci tensor $\mathbf{Ric}_F^H$ of $(M,F)$ by

   $$\mathbf{Ric}_F^H(\xi,X) := trace_g(\eta \longmapsto R(X,\mathbf{h}(\eta))\xi),$$

   for any $X \in \Gamma(T\mathring{T}M)$ and for every $\xi,\eta \in \Gamma(\pi^*TM)$. In $g$-orthonormal basis sections $\{e_a\}_{a=1,\ldots,n}$ of $\pi^*TM$, we have

   $$\mathbf{Ric}_F^H(\xi,X) := \sum_{a=1}^{n}\mathbf{R}(\xi,e_a,X,\hat{e}_a). \tag{2.17}$$

2. the horizontal scalar curvature $\mathbf{Scal}_F^H$ of $(M, F)$ is the trace of the horizontal Ricci tensor. $\mathbf{Scal}_F^H$ is a function on $\mathring{T}M$ or on $SM$. In $g$-orthonormal basis sections $\{e_a\}_{a=1,\ldots,n}$ of $\pi^*TM$,

$$\mathbf{Scal}_F^H := \sum_{a=1}^{n} \mathbf{Ric}_F^H(e_a, \hat{e}_a) = \sum_{a,b=1}^{n} \mathbf{R}(e_a, e_b, \hat{e}_a, \hat{e}_b). \tag{2.18}$$

Now, we introduce the trace-free horizontal Ricci tensor on $(M, F)$ as follows.

**Definition 2.8.** Let $(M, F)$ be an $n$-dimensional Finsler manifold and $g$ its fundamental tensor. The trace-free horizontal Ricci tensor of $(M, F)$ is a $(1, 1; 0)$-tensor on $(M, F)$ given by

$$\mathbf{B}_F^H = \mathbf{Ric}_F^H - \frac{1}{n}\mathbf{Scal}_F^H \underline{g}, \tag{2.19}$$

where $\underline{g} := \pi^* g$ that is the pullback of $g$ by the submersion $\pi : \mathring{T}M \longrightarrow M$; and for every $\xi \in \Gamma(\pi^*TM)$ and for any $X \in \Gamma(T\mathring{T}M)$, $\underline{g}(\xi, X) = g(\xi, \pi_* X)$, $\mathbf{Ric}_F^H$ is the horizontal Ricci tensor and $\mathbf{Scal}_F^H$ is the horizontal scalar curvature of $(M, F)$.

## 2.3 Differential operators on Finsler manifolds

In this paragraph, we give some fundamental differential operators on $(M, F)$.

**Definition 2.9.**

1. Let $\tau : \pi^*TM \longrightarrow TM$ be the canonical mapping defined by $\tau(x, y, v) = v$. For a smooth function $u$ on $M$, the gradient of $u$, noted by $\triangledown u$, is the section of $\pi^*TM$, given by

$$g_{(x,y)}(\triangledown u_{(x,y)}, \xi_{(x,y)}) = du_{\pi(x,y)}(\tau\xi), \tag{2.20}$$

for every section $\xi \in \Gamma(\pi^*TM)$ and for any $(x, y) \in \mathring{T}M$. Locally, one has

$$\triangledown u_{(x,y)} = g^{ij}(x, y)\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}. \tag{2.21}$$

2. For a $C^\infty$ section $\xi \in \Gamma(\pi^*TM)$, we define the horizontal divergence by

$$div^H \xi = trace_g\left(\eta \longmapsto \nabla_{\mathbf{h}(\eta)}\xi\right) \tag{2.22}$$

and the vertical divergence by

$$div^V \xi = trace_g\left(\eta \longmapsto \nabla_{\mathbf{v}(\eta)}\xi\right), \tag{2.23}$$

where $g$ is the fundamental tensor associated with $F$ and $\nabla$ is the Chern connection.

**Remark 2.10.** In the local basis sections $\{\frac{\partial}{\partial x^i}\}_{i=1,\dots,n}$ of the bundle $\pi^*TM$, we have:

$$div^H \xi = g^{ij} g\left(\nabla_{\frac{\delta}{\delta x^i}} \xi, \frac{\partial}{\partial x^j}\right) \tag{2.24}$$

and

$$div^V \xi = g^{ij} g\left(\nabla_{F\frac{\partial}{\partial y^i}} \xi, \frac{\partial}{\partial x^j}\right). \tag{2.25}$$

**Definition 2.11.** Let $(M, F)$ be a $C^\infty$ Finsler manifold and $u$ a $C^\infty$ function on $M$.

1. The Hessian of $u$ is the mapping

$$\begin{aligned} &H_u : \Gamma(\pi^*TM) \times \Gamma(T\overset{\circ}{T}M) \longrightarrow C^\infty(\overset{\circ}{T}M) \ \ such \ \ that \\ &H_u(\xi, X) = g\left(\xi, \nabla_X(\nabla u)\right) \quad \forall \xi \in \pi^*TM, \forall X \in T\overset{\circ}{T}M. \end{aligned} \tag{2.26}$$

2. The horizontal Laplacian and the vertical Laplacian of $u$ are respectively defined by the following relations:

$$\Delta^H u = -div^H \nabla u \tag{2.27}$$

and

$$\Delta^V u = -div^V \nabla u. \tag{2.28}$$

**Lemma 2.2.** *Let $u \in C^\infty(M)$. The Laplacians of $u$, $\Delta^H u$ and $\Delta^V u$, can be given in term of the Hessian of $u$ by*

$$\Delta^H u = -trace_g\left((\xi, \eta) \mapsto H_u(\xi, \mathbf{h}(\eta))\right), \qquad \xi, \eta \in \Gamma(\pi^*TM), \tag{2.29}$$

*and*

$$\Delta^V u = -trace_g\left((\xi, \eta) \mapsto H_u(\xi, \mathbf{v}(\eta))\right), \qquad \xi, \eta \in \Gamma(\pi^*TM). \tag{2.30}$$

*Furthermore, in $g-$orthonormal basis sections $\{e_a\}_{a=1,\dots,n}$, one has*

$$\Delta^H u = -\sum_{a=1}^{n} H_u(e_a, \hat{e}_a) \tag{2.31}$$

*and*

$$\Delta^V u = -\sum_{a=1}^{n} H_u(e_a, \hat{e}_{a+n}). \tag{2.32}$$

*Proof.* By definition of horizontal Laplacian and vertical Laplacian. □

# 3 Conformal deformations of Finsler metrics and Besse-type result

In this section, we first review some concepts concerning the conformal changes of Finsler metrics [11]. Then, using these concepts, we study the conformal changes of the trace-free horizontal Ricci tensor.

**Definition 3.1.** Let $F$ and $\widetilde{F}$ be two Finsler metrics on manifold $M$. Two fundamental tensors $g$ and $\widetilde{g}$, associated with $F$ and $\widetilde{F}$ respectively, are said to be conformal if there exists a $C^\infty$ function $u$ on $M$ such that $\widetilde{g} = e^{2u}g$. Equivalently, $g$ and $\widetilde{g}$ are conformal if and only if $\widetilde{F} = e^u F$. In that case, $F$ and $\widetilde{F}$ are said to be conformal or conformally related.

We obtain the following result.

**Theorem 3.1.** *Let $(M, F)$ be an $n$-dimensional Finsler manifold and $g$ the fundamental tensor of $F$. If $\widetilde{F} = e^u F$ is a Finsler metric conformal to $F$ then the trace-free horizontal Ricci tensors $\mathbf{B}^H_F$ and $\widetilde{\mathbf{B}}^H_{\widetilde{F}}$, associated with $F$ and $\widetilde{F}$ respectively, are conformally related by*

$$\widetilde{\mathbf{B}}^H_{\widetilde{F}} = \mathbf{B}^H_F - (n-2)\left(H_u - du \circ du\right) - \frac{(n-2)}{n}\left(\Delta^H u + ||\nabla u||^2_g\right) \underline{g} + Z_u \tag{3.1}$$

*where $Z_u$ is the $(1,1;0)$-tensor on $(M, F)$ given by*

$$\begin{aligned} Z_u(\xi, X) \quad := \quad & -ng(\nabla u, \Theta(X, \mathbf{h}(\xi))) \\ & + \sum_{a=1}^n \left[\frac{2}{n} g(e_a, \Theta(\hat{e}_a, \mathbf{h}(\nabla u))) + \frac{(n-2)}{n} g(\Theta(\hat{e}_a, \hat{e}_a), \nabla u)\right] g(\xi, \pi_* X) \\ & + \sum_{a=1}^n \left[g(\Theta(X, \mathbf{h}(\Theta(\hat{e}_a, \mathbf{h}(\xi)))), e_a) - g(\Theta(\hat{e}_a, \mathbf{h}(\Theta(X, \mathbf{h}(\xi))), e_a)\right] \\ & + \sum_{a=1}^n \left[g((\nabla_X \Theta)(\hat{e}_a, \mathbf{h}(\xi)), e_a) - g((\nabla_{\hat{e}_a}\Theta)(X, \mathbf{h}(\xi)), e_a)\right] \\ & - \frac{1}{n} \sum_{a,b=1}^n \left[g(\Theta(\hat{e}_b, \mathbf{h}(\Theta(\hat{e}_a, \hat{e}_b)), e_a) - g(\Theta(\hat{e}_a, \mathbf{h}(\Theta(\hat{e}_b, \hat{e}_b))), e_a)\right] g(\xi, \pi_* X) \\ & - \frac{1}{n} \sum_{a,b=1}^n \left[g((\nabla_{\hat{e}_b}\Theta)(\hat{e}_a, \hat{e}_b), e_a) - g((\nabla_{\hat{e}_a}\Theta)(\hat{e}_b, \hat{e}_b), e_a)\right] g(\xi, \pi_* X), \end{aligned} \tag{3.2}$$

*for every $\xi \in \Gamma(\pi^* TM)$ and $X \in \Gamma(T\mathring{T}M)$ with $(0,2;1)-$tensor $\Theta$ defined by*

$$\Theta(X, Y) := (\mathcal{A}(\mathcal{B}(X), \pi_* Y, \bullet))^\sharp + (\mathcal{A}(\mathcal{B}(Y), \pi_* X, \bullet))^\sharp - (\mathcal{A}(\pi_* X, \pi_* Y, \bullet)) \circ \mathcal{B})^\sharp$$

*and the $(0,1;1)-$tensor $\mathcal{B} := \frac{1}{F}\frac{\partial}{\partial y^j}(\frac{F^2}{2}g^{ir} - y^i y^r)\frac{\partial u}{\partial x^r}\frac{\partial}{\partial x^i} \otimes dx^j$.*

*Proof.* The equation (3.1) follows from the relationship between the two Chern connections $\nabla$, $\widetilde{\nabla}$ associated with $F$ and $\widetilde{F}$ respectively:

$$\widetilde{\nabla}_X \pi_* Y = \nabla_X \pi_* Y + du(\pi_* X)\pi_* Y + du(\pi_* Y)\pi_* X - g(\pi_* X, \pi_* Y)\nabla u + \Theta(X, Y)$$

where $\Theta$ is the $(0,2;1)$-tensor on $(M,F)$ given by

$$\Theta(X,Y) = (\mathcal{A}(\mathcal{B}(X),\pi_*Y,\bullet))^\sharp + (\mathcal{A}(\mathcal{B}(Y),\pi_*X,\bullet))^\sharp - (\mathcal{A}(\pi_*X,\pi_*Y,\mathcal{B}(\mathbf{h}(\bullet))))^\sharp$$

with $(\ \ )^\sharp$ the dual section of $\pi^*TM$ to the Cartan tensor $\mathcal{A}$ and with the $(0,1;1)-$tensor

$$\mathcal{B} := \frac{1}{F}\frac{\partial}{\partial y^j}\left(\frac{F^2}{2}g^{ir} - y^i y^r\right)\frac{\partial u}{\partial x^r}\frac{\partial}{\partial x^i}\otimes dx^j. \tag{3.3}$$

For every section $\xi$ of $\pi^*TM$ and for every section $X$ of $T\mathring{T}M$ we have, from Definition (2.8)

$$\widetilde{\mathbf{B}}^H_{\widetilde{F}}(\xi,X) = \widetilde{\mathbf{Ric}}^H_{\widetilde{F}}(\xi,X) - \frac{1}{n}(\widetilde{\mathbf{Scal}}^H_{\widetilde{F}}\underline{\widetilde{g}})(\xi,X).$$

We get, from the conformal changes of curvatures associated with $\nabla$ and $\widetilde{\nabla}$, that

$$\begin{aligned}
\widetilde{\mathbf{B}}^H_{\widetilde{F}}(\xi,X) = {} & (\mathbf{Ric}^H_F - \frac{1}{n}\mathbf{Scal}^H_F\underline{g})(\xi,X) \\
& -\frac{(n-2)}{n}\left(\Delta^H u + ||\nabla u||^2_g\right) g(\xi,\pi_*X) \\
& -(n-2)\left(H_u - du\circ du\right)(\xi,X) \\
& +\sum_{a=1}^n \left[g(\nabla u,\Theta(\hat{e}_a,\mathbf{h}(\xi)))g(e_a,\pi_*X) - g(\nabla u,\Theta(X,\mathbf{h}(\xi)))g_{aa}\right] \\
& +\sum_{a=1}^n \left[g(\nabla u,e_a)g(\Theta(X,\mathbf{h}(\xi)),e_a) - g(\nabla u,e_a)g(\Theta(\hat{e}_a,\mathbf{h}(\xi)),\pi_*X)\right] \\
& +\sum_{a=1}^n \left[g(\Theta(\hat{e}_a,\mathbf{h}(\nabla u)),e_a)g(\xi,\pi_*X) - g(\Theta(X,\mathbf{h}(\nabla u)),e_a)g(e_a,\xi)\right] \\
& +\frac{(n-2)}{n}\sum_{b=1}^n \left[g(\nabla u,\Theta_{bb}) + g(\Theta(\hat{e}_b,\mathbf{h}(\nabla u)),e_b)\right] g(\xi,\pi_*X) \\
& +\sum_{a=1}^n \left[g(\Theta(X,\mathbf{h}(\Theta(\hat{e}_a,\mathbf{h}(\xi)))),e_a) - g(\Theta(\hat{e}_a,\mathbf{h}(\Theta(X,\mathbf{h}(\xi)))),e_a))\right] \\
& +\sum_{a=1}^n \left[g((\nabla_X\Theta)(\hat{e}_a,\mathbf{h}(\xi)),e_a) - g((\nabla_{\hat{e}_a}\Theta)(X,\mathbf{h}(\xi)),e_a)\right] \\
& -\frac{1}{n}\sum_{a,b=1}^n \left[g(\Theta(\hat{e}_b,\mathbf{h}(\Theta_{ab})),e_a) - g(\Theta(\hat{e}_a,\mathbf{h}(\Theta_{bb})),e_a)\right] g(\xi,\pi_*X) \\
& -\frac{1}{n}\sum_{a,b=1}^n \left[g((\nabla_{\hat{e}_b}\Theta)_{ab},e_a) - g((\nabla_{\hat{e}_a}\Theta)_{bb},e_a)\right] g(\xi,\pi_*X)
\end{aligned} \tag{3.4}$$

where $\Theta_{ab} := \Theta(\hat{e}_a,\hat{e}_b)$. Putting the equation (2.19) in the right hand side of the equation (3.4) we obtain the relation in (3.1) and (3.2). □

**Remark 3.2.** If $F$ is a Riemannian metric, we obtain the results in [5] for the Riemannian case.

**Corollary 3.2.** *If the Finsler-Ehresmann form $\theta$ is invariant under the conformal change of Finsler metric, that means $\Theta \equiv 0$, the trace-free horizontal Ricci tensor and the horizontal Laplacian behave like in Riemannian case.*

# 4 Finslerian prescribed horizontal Ricci tensor

In this section with prove main results of this paper. We have

**Lemma 4.1.** *Let $(M, F)$ be a Finsler manifold, $\{\frac{\partial}{\partial x^i}\}_{1\leq i\leq n}$ the natural and $\{e_a\}_{1\leq a\leq n}$ the special $g$-orthonormal bases sections for the pulled-back bundle $\pi^*TM$. Then*

*(i)* $u_a^i g_{ij} u_b^j = \delta_{ab}$, *for* $i, j, a, b = 1, ..., n$.

*(ii)* $u_\alpha^i u_\beta^j F \frac{\partial^2 F}{\partial y^i \partial y^j} = \delta_{\alpha\beta}$, *for* $\alpha, \beta = 1, ..., n-1$.

*Proof.* The bases $\{\frac{\partial}{\partial x^i}\}_{1\leq i\leq n}$ and $\{e_a\}_{1\leq a\leq n}$ can be expressed in term of each other via the $n\times n$-matrix $\left(u_a^i\right)$ or its inverse $(u_i^a)$ as follows [3]

$$e_a = u_a^i \frac{\partial}{\partial x^i} \qquad \textit{and} \qquad \frac{\partial}{\partial x^i} = u_i^a e_a.$$

(i) By (4.1), we have

$$\delta_{ab} \quad = \quad g(u_a^i \frac{\partial}{\partial x^i}, u_b^j \frac{\partial}{\partial x^j}) = u_a^i g_{ij} u_b^j.$$

(ii) By (4.1) and (2.1) we have

$$\delta_{\alpha\beta} \quad = \quad u_\alpha^i g_{ij} u_\beta^j = u_\alpha^i \left( \frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j} + F\frac{\partial^2 F}{\partial y^i \partial y^j} \right) u_\beta^j.$$

□

The following Lemma gives the relationship between the Hessian and the horizontal Laplacian both associated with $F$ when the horizontal Ricci tensors $\mathbf{Ric}_F^H$ and $\widetilde{\mathbf{Ric}}_{\tilde{F}}^H$ are equal.

**Lemma 4.2.** *Let $F$ and $\widetilde{F}$ be two conformal Finsler metrics on a manifold $M$ and, $\mathbf{Ric}^H_F$ and $\widetilde{\mathbf{Ric}}^H_{\widetilde{F}}$ their respective associated horizontal Ricci tensors. If $\widetilde{F} = \varphi^{-1}F$ and $\widetilde{\mathbf{Ric}}^H_{\widetilde{F}} = \mathbf{Ric}^H_F$, then*

$$
\begin{aligned}
H_\varphi(e_b, \hat{e}_b) \quad = \quad & -\frac{1}{n}\left[\left(\Delta^H \varphi\right) \underline{g}\right](e_b, \hat{e}_b) \\
& + \frac{n}{n-2}\frac{1}{\varphi}\frac{\partial \varphi}{\partial x^i}\frac{\partial \varphi}{\partial x^r}\psi^r u^i_c \mathcal{A}_{bbc} \\
& - \frac{1}{n-2}\frac{1}{\varphi}\sum_{b=1}^{n}\frac{\partial \varphi}{\partial x^i}\frac{\partial \varphi}{\partial x^r}\psi^r u^i_c \mathcal{A}_{bbc} \\
& - \frac{\varphi}{n-2}\sum_{a=1}^{n}\left[g(\Theta(\hat{e}_b, \mathbf{h}(\Theta_{ab})), \hat{e}_a) - g(\Theta(\hat{e}_a, \mathbf{h}(\Theta_{bb})), \hat{e}_a)\right] \\
& + \frac{\varphi}{n(n-2)}\sum_{a,b=1}^{n}\left[g(\Theta(\hat{e}_b, \mathbf{h}(\Theta_{ab})), \hat{e}_a) - g(\Theta(\hat{e}_a, \mathbf{h}(\Theta_{bb})), \hat{e}_a)\right] \\
& - \frac{\varphi}{(n-2)}\sum_{a=1}^{n}\left[g((\nabla_{\hat{e}_b}\Theta)_{ab}, e_a) - g((\nabla_{\hat{e}_a}\Theta)_{bb}, e_a)\right] \\
& + \frac{\varphi}{n(n-2)}\sum_{a,b=1}^{n}\left[g((\nabla_{\hat{e}_b}\Theta)_{ab}, e_a) - g((\nabla_{\hat{e}_a}\Theta)_{bb}, e_a)\right],
\end{aligned} \tag{4.1}
$$

*where $\{e_a\}_{a=1,\ldots,n}$ is the special $g-$orthonormal basis sections for the vector bundle $\pi^*TM$ and where the quantities $\psi^r$ and $\mathcal{A}_{abc}$ are given by $\psi^r := \frac{1}{F}\frac{\partial}{\partial y^i}(\frac{F^2}{2}g^{ir} - y^i y^r)$ and $\mathcal{A}_{abc} := u^i_a u^j_b u^k_c \mathcal{A}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})$ with $u^i_a, u^j_b, u^k_c$ the $C^\infty$-functions on $\mathring{T}M$.*

*Proof.* Let $\{e_b\}_{b=1,\ldots,n}$ be the special $g-$orthonormal basis sections for the pulled-back bundle $\pi^*TM$. Then, on the one hand, if $\widetilde{\mathbf{Ric}}^H_{\widetilde{F}} = \mathbf{Ric}^H_F$ we have

$$
\begin{aligned}
\widetilde{\mathbf{B}}^H_{\widetilde{F}}(e_b, \hat{e}_b) \quad & = \quad \left[\widetilde{\mathbf{Ric}}^H_{\widetilde{F}} - \frac{1}{n}\left(\sum_{a=1}^{n}\widetilde{\mathbf{Ric}}^H_{\widetilde{F}}(\widetilde{e}_a, \mathbf{h}(\widetilde{e}_a))\right)\underline{\widetilde{g}}\right](e_b, \hat{e}_b) \\
& = \quad \mathbf{B}^H_F(e_b, \hat{e}_b).
\end{aligned} \tag{4.2}
$$

On the other hand, from Theorem 3.1 we get

$$
\begin{aligned}
\widetilde{\mathbf{B}}^H_{\widetilde{F}}(e_b, \hat{e}_b) \quad = \quad & \mathbf{B}^H_F(e_b, \hat{e}_b) \\
& - \frac{(n-2)}{n}\Delta^H u + \frac{(n-1)(n-2)}{n}||\nabla u||^2_g - (n-2)H_u(e_b, \hat{e}_b) \\
& + (2-n)g(\nabla u, \Theta_{bb}) - 2g(\Theta(\mathbf{h}(\nabla u), \hat{e}_b), e_b) \\
& + \sum_{a=1}^{n}\left[\frac{2}{n}g(e_a, \Theta(\hat{e}_a, \mathbf{h}(\nabla u))) + \frac{(n-2)}{n}g(\Theta_{aa}, \nabla u)\right] g_{bb} \\
& + K_u
\end{aligned} \tag{4.3}
$$

where

$$
\begin{aligned}
K_u \quad = \quad & \sum_{a=1}^{n}\left[g(\Theta(\hat{e}_b,\mathbf{h}(\Theta_{ab})),e_a)-g(\Theta(\hat{e}_a,\mathbf{h}(\Theta_{bb})),e_a)\right] \\
& +\sum_{a=1}^{n}\left[g((\nabla_{\hat{e}_b}\Theta)_{ab},e_a)-g((\nabla_{\hat{e}_a}\Theta)_{bb},e_a)\right] \\
& -\frac{1}{n}\sum_{a,b=1}^{n}\left[g(\Theta(\hat{e}_b,\mathbf{h}(\Theta_{ab})),e_a)-g(\Theta(\hat{e}_a,\mathbf{h}(\Theta_{bb})),e_a)\right] \\
& -\frac{1}{n}\sum_{a,b=1}^{n}\left[g((\nabla_{\hat{e}_b}\Theta)_{ab},e_a)-g((\nabla_{\hat{e}_a}\Theta)_{bb},e_a)\right]. \qquad (4.4)
\end{aligned}
$$

Combining (4.2) and (4.3), we get

$$
\begin{aligned}
(n-2)H_u(e_b,\hat{e}_b) \quad = \quad & \frac{(n-1)(n-2)}{n}||\nabla u||_g^2-\frac{(n-2)}{n}\Delta^H u \\
& +(2-n)g(\nabla u,\Theta_{bb})-2g(\Theta(\mathbf{h}(\nabla u),\hat{e}_b),e_b) \\
& +\sum_{a=1}^{n}\left[\frac{2}{n}g(e_a,\Theta(\hat{e}_a,\mathbf{h}(\nabla u)))+\frac{(n-2)}{n}g(\Theta_{aa},\nabla u)\right] \\
& +K_u. \qquad (4.5)
\end{aligned}
$$

Since $e^u=\varphi^{-1}$, we have $\nabla u=-\frac{1}{\varphi}\nabla\varphi$, $\quad H_u=-\frac{1}{\varphi}H_\varphi+\frac{1}{\varphi^2}\nabla\varphi\otimes\nabla\varphi$ *and* $\Delta^H u=-\frac{1}{\varphi}\Delta^H\varphi-\frac{1}{\varphi^2}\nabla\varphi\otimes\nabla\varphi$. Thus, by the Lemma (4.1) we have the following transformations

- $$\mathcal{B}=-\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^r}\psi^r\frac{\partial}{\partial x^i}\otimes dx^i \qquad (4.6)$$

  with $\psi^r=\frac{\partial}{\partial y^i}\left(y^iy^r-\frac{F^2}{2}g^{ir}\right)$. Then

  $$\mathcal{B}(\hat{e}_a) \quad = \quad -\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^r}\psi^r e_a. \qquad (4.7)$$

- $$
\begin{aligned}
g\left(\nabla u,\Theta_{aa}\right) \quad &= \quad -\mathcal{A}\left(\mathcal{B}\left(\mathbf{h}(\nabla u)\right),e_a,e_a\right)+2\mathcal{A}(\nabla u,\mathcal{B}(\hat{e}_a),e_a) \\
&= \quad \frac{1}{\varphi^2}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^r}\psi^r u_c^i\mathcal{A}_{aac}. \qquad (4.8)
\end{aligned}
$$

So (4.5) transforms to

$$
\begin{aligned}
H_\varphi(e_b,\hat{e}_b) \quad = \quad & -\frac{1}{n}\Delta^H\varphi+\frac{n}{n-2}\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^r}\psi^r u_c^i\mathcal{A}_{aac} \\
& -\frac{1}{n-2}\frac{1}{\varphi}\sum_{a=1}^{n}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^r}\psi^r u_c^i\mathcal{A}_{aac}-\frac{\varphi}{n(n-2)}K_u. \qquad (4.9)
\end{aligned}
$$

Hence from equation (4.9) we btain the relation in (4.1). □

Now we prove the following

**Theorem 4.3.** *Let $F$ and $\widetilde{F}$ be two conformal Finsler metrics on a closed, oriented manifold $M$ of dimension $n \geq 3$ and, $\mathbf{Ric}_F^H$ and $\widetilde{\mathbf{Ric}}_{\widetilde{F}}^H$ their respective associated horizontal Ricci tensors. If $\widetilde{F} = \varphi^{-1}F$ and $\widetilde{\mathbf{Ric}}_{\widetilde{F}}^H = \mathbf{Ric}_F^H$ then $\varphi$ is constant. In other words, on a closed manifold, two conformal Finsler metrics that have the same horizontal Ricci tensor must be homothetic.*

*Proof.* Suppose $\widetilde{F} = \varphi^{-1}F$ and $\widetilde{\mathbf{Ric}}_{\widetilde{F}}^H = \mathbf{Ric}_F^H$. We have, from the relationship between the two Chern connections $\nabla$, $\widetilde{\nabla}$ associated with $F$ and $\widetilde{F}$ respectively that

$$\begin{aligned}
0 &= \left(-\frac{\Delta^H\varphi}{\varphi}\underline{g} - \frac{(n-1)}{\varphi^2}||\nabla\varphi||_g^2\underline{g} + \frac{(n-2)}{\varphi}H_\varphi\right)(e_b,\hat{e}_b) \\
&\quad -\frac{n}{\varphi^2}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^r}\psi^r u_c^i\mathcal{A}_{aac} + \frac{1}{\varphi^2}\sum_{a=1}^{n}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^r}\psi^r u_c^i\mathcal{A}_{aac} \\
&\quad +\sum_{a=1}^{n}\left[g\left(\Theta(\hat{e}_b,\mathbf{h}(\Theta_{ab})),e_a\right) - g\left(\Theta(\hat{e}_a,\mathbf{h}(\Theta_{bb})),e_a\right)\right] \\
&\quad +\sum_{a=1}^{n}\left[g\left((\nabla_{\hat{e}_b}\Theta)_{ab},e_a\right) - g\left((\nabla_{\hat{e}_a}\Theta)_{bb},e_a\right)\right].
\end{aligned} \tag{4.10}$$

Putting (4.1) in (4.10) we obtain

$$\begin{aligned}
0 &= -\frac{2}{n}(n-1)\frac{1}{\varphi}\Delta^H\varphi - \frac{(n-1)}{\varphi^2}||\nabla\varphi||_g^2 \\
&\quad +\frac{1}{n}\sum_{a,b=1}^{n}\left[g\left(\Theta(\hat{e}_b,\mathbf{h}(\Theta_{ab})),e_a\right) - g\left(\Theta(\hat{e}_a,\mathbf{h}(\Theta_{bb})),e_a\right)\right] \\
&\quad +\frac{1}{n}\sum_{a,b=1}^{n}\left[g\left((\nabla_{\hat{e}_b}\Theta)_{ab},e_a\right) - g\left((\nabla_{\hat{e}_a}\Theta)_{bb},e_a\right)\right].
\end{aligned} \tag{4.11}$$

In order to compute the equation (4.11),
we set $I_2 = \sum_{a,b=1}^{n}\left[g\left(\Theta(\hat{e}_b,\mathbf{h}(\Theta_{ab})),e_a\right) - g\left(\Theta(\hat{e}_a,\mathbf{h}(\Theta_{bb})),e_a\right)\right]$. Then by relations (4.7) , we have

$$\begin{aligned}
I_2 &= \sum_{a,b=1}^{n}\left[g\left(\Theta(\hat{e}_b,\mathbf{h}(\Theta_{ab})),e_a\right) - g\left(\Theta(\hat{e}_a,\mathbf{h}(\Theta_{bb})),e_a\right)\right]g(e_c,e_c) \\
&= \sum_{a,b=1}^{n}\left(\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^r}\psi^r\right)^2\left[\mathcal{A}_{abc}^2 - \mathcal{A}_{aac}\mathcal{A}_{bbc}\right].
\end{aligned} \tag{4.12}$$

Since $\mathcal{A}_{abc}$ is zero whenever any of its three indices has the value $n$, by Lemma 4.1, the relation (4.12) becomes

$$\begin{aligned}
I_2 &= \left(\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^r}\psi^r\right)^2\left(u_c^k\mathcal{A}_{ijk}\right)^2\left(F\frac{\partial^2F}{\partial y^i\partial y^j}\right)^{-2}\sum_{\alpha,\beta=1}^{n-1}\left[(\delta_{\alpha\beta})^2 - \delta_{\alpha\alpha}\delta_{\beta\beta}\right] \\
&= (n-1)(2-n)\left(\frac{1}{\varphi}\right)^2||\nabla\varphi||_g^2\left(\psi^k\mathcal{A}_{ijk}\right)^2\left(F\frac{\partial^2F}{\partial y^i\partial y^j}\right)^{-2}.
\end{aligned} \tag{4.13}$$

Now set $I_3 = \sum_{a,b=1}^{n} [g((\nabla_{\hat{e}_b}\Theta)_{ab}, e_a) - g((\nabla_{\hat{e}_a}\Theta)_{bb}, e_a)]$. By the symmetry of $\nabla$, we get

$$
\begin{aligned}
I_3 &= \sum_{a,b=1}^{n} [g(e_b(\Theta_{ab}) - \Theta(\nabla_{\hat{e}_b} e_a, e_b) - \Theta(\hat{e}_a, \nabla_{\hat{e}_b} e_b), e_a) \\
&\quad - g(e_a(\Theta_{bb}) + 2\Theta(\nabla_{\hat{e}_a} e_b, e_b), e_a)] \\
&= 0. \qquad (4.14)
\end{aligned}
$$

Putting relations (4.13) and (4.14) in equation (4.11), we obtain

$$
\begin{aligned}
0 &= -\frac{2}{n}(n-1)\frac{1}{\varphi}\Delta^H\varphi - \frac{(n-1)}{\varphi^2}||\nabla\varphi||_g^2 \\
&\quad - \frac{(n-1)(n-2)}{n}\frac{1}{(\varphi)^2}||\nabla\varphi||_g^2 \left(\psi^k\mathcal{A}_{ijk}\right)^2 \left(F\frac{\partial^2 F}{\partial y^i \partial y^j}\right)^{-2}. \qquad (4.15)
\end{aligned}
$$

Multiplying this last equation by $\varphi^2$ we obtained

$$
(4.16)\quad \left(2\varphi\Delta^H\varphi + n||\nabla\varphi||_g^2\right)\left(F\frac{\partial^2 F}{\partial y^i \partial y^j}\right)^2 + (n-2)||\nabla\varphi||_g^2 \left(\psi^k\mathcal{A}_{ijk}\right)^2 = 0.
$$

By integration of equation (4.16) on $\mathring{T}M$ and by using integration by parts, we obtain

$$
(4.17)\qquad (n-2)\int_{\mathring{T}M}\left[|\nabla\varphi|^2\left(\left(F\frac{\partial^2 F}{\partial y^i \partial y^j}\right)^2 + \left(\psi^k\mathcal{A}_{ijk}\right)^2\right)\right]\eta_F = 0
$$

where $\eta_F := \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \wedge^n d\omega$, is the $2n$-form on $\mathring{T}M$ called volume form of $(M, F)$ [7] with $\omega := \frac{\partial F}{\partial y^i}dx^i$ the Hilbert form.

Since $n > 2$, the integrand in the left hand side of (4.17) is nonnegative. It has to vanish. In particular $|\nabla\varphi| \equiv 0$. Hence $\varphi$ is constant. □

# 5 Conclusions and Suggestions

The trace-free horizontal Ricci tensor is very important in Finsler geometry because it measures the defaut of a Finsler manifold to be horizontally an Einstein space. In particular, every (locally) Minkowski space is (locally) horizontally Einstein. Thus when a Finsler metric is Riemannian, the tensor $\mathbf{B}_F^H$ reduces to the Riemannian traceless Ricci curvature [5].

With the trace-free horizontal Ricci tensor, it was classified conformal transformations preserving pointwise horizontally Finslerian Ricci tensor. By the Theorem 3.1, we are studing Finsler manifolds which are conformally Einstein. Furthermore, we plan to investigate in the future the second Chern's curvature tensor for Finsler non-Riemannian manifolds.

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