Almost pseudo Ricci symmetric manifold admitting W_2- Ricci tensor

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Abstract. This paper deals with almost pseudo Ricci symmetric manifolds admitting W_2-curvature tensor. We determine several properties of these manifolds and give an example for the existence of such manifolds satisfying certain conditions.

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1 Introduction

In the late twenties, because of the important role of symmetric spaces in differential geometry, Cartan [2], who, in particular obtained a classification of those spaces established Riemannian symmetric spaces.

Let (M^n, g) be an n-dimensional Riemannian manifold with the metric g and let ∇ be the Levi-Civita connection of (M^n, g). A Riemanian manifold is called locally symmetric [2] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetric is equivalent to the fact that every point $p \epsilon M^n$, the local geodesic symmetry $F(p)$ is an isometry [18]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

During the last five decades the notion of locally symmetric manifolds have been weakended by many authors in several ways to a different extend such as conformally symmetric manifolds by Chaki and Gupta [5], recurrent manifolds introduced by Walker [27], conformally recurrent manifolds by Adati and Miyazawa [1], conformally symmetric Ricci-recurrent spaces by Roter [17], pseudo-Riemannian manifolds with recurrent concircular curvature tensor by Olszak and Olszak [23], semi-symmetric manifolds by Szabo [3], pesudo symmetric manifolds introduced by Chaki [25], weakly symmetric manifolds by Tamassy and Binh [22], projective symmetric manifolds by Soos [6], etc.

The Einstein equations [18], imply that the energy-momentum tensor is of vanishing divergence. This requirement is satisfied [6] if the energy- momentum tensor

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is covariant-constant. In the paper [6], Chaki and Ray had shown that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$ where S is the Ricci tensor of the spacetime. If however $\nabla S \neq 0$, then such a spacetime may be called pseudo Ricci symmetric. It can be said that the Ricci symmetric condition is only a special case of the pseudo Ricci symmetric condition. It is, therefore, meaningful to study the properties of pseudo Ricci symmetric spacetimes in general relativity.

Let Q be the symmetric endomorphism corresponding to the Ricci tensor as indicated below

$$S(X, Y) = g(QX, Y),$$

for all vector fields X and Y.

A non-flat Riemannian manifold is called pseudo Ricci symmetric and denoted by $(PRS)_n$ if the Ricci tensor S of type $(0, 2)$ of the manifold is non-zero and satisfies the condition

$$(\nabla_Z S)(X, Y) = 2A(Z)S(X, Y) + A(X)S(Y, Z) + A(Y)S(X, Z), \tag{1.1}$$

where ∇ denotes the Levi-Civita connection and A is a non-zero $1-$form such that

$$g(X, \rho) = A(X), \tag{1.2}$$

for all vector fields X, ρ being the vector field corresponding to the associated $1-$form A. If in (1.1), the $1-$form $A = 0$, then the manifold reduces to Ricci symmetric manifold or covariantly constant

$$(\nabla_Z S)(X, Y) = 0. \tag{1.3}$$

The notion of pseudo Ricci symmetry is different from that of R. Deszcz [12].

So, the pseudo Ricci symmetric manifolds have some importance in general theory of relativity. By this motivation, Chaki and Kawaguchi [7] generalized the pseudo Ricci symmetric manifolds and introduced the notion of almost pseudo Ricci symmetric manifold as

$$(\nabla_Z S)(X, Y) = [A(Z) + B(Z)]S(X, Y) + A(X)S(Y, Z) + A(Y)S(X, Z), \tag{1.4}$$

where A and B are two non-zero $1-$forms and ∇ denotes the operator of the covariant differentiation with respect to the metric g. In such a case, A and B are called the associated $1-$forms and an $n-$dimensional manifold of this kind is denoted by $A(PRS)_n$.

If $B = A$, then the equation (1.3) reduces to (1.1), that is, $A(PRS)_n$ reduces to a pseudo Ricci symmetric manifold [4]. Thus, pseudo Ricci symmetric manifold is a particular case of $A(PRS)_n$. In 1993, Tamassy and Binh [26] introduced the notion of weakly Ricci symmetric manifold which is the generalization of pseudo Ricci symmetric manifold in the sense of Chaki. It may be mentioned that an $A(PRS)_n$ is not a particular case of a weakly Ricci symmetric manifold introduced by Tamassy and Binh [26].

Let $g(X, \rho) = A(X)$ and $g(X, Q) = B(X)$ for all X. Then ρ, Q are called basic vector fields of the manifold corresponding to the associated $1-$forms A and B, respectively.

Almost pseudo Ricci symmetric manifolds on some structures have been studied by many authors such as De and Gazi [9], Shaikh, Hui and Bagewadi [21], De, Özgür and De [8], Hui and Özen Zengin [14], De and Mallick [10], De and Pal [11], Kırık and Özen Zengin [15], etc.

A non-flat Riemannian manifold (M^n, g), $(n > 2)$ is called generalized recurrent if the Ricci tensor S is non-zero and satisfies the condition

$$(\nabla_Z S)(X,Y) = A(Z)S(X,Y) + B(Z)g(X,Y), \tag{1.5}$$

where A and B are non-zero $1-$forms [20]. If the associated $1-$form B becomes zero, then the manifold reduces to a Ricci recurrent, i.e

$$(\nabla_Z S)(X,Y) = A(Z)S(X,Y). \tag{1.6}$$

In a Riemannian manifold, the Ricci tensor is called Codazzi type if the following condition holds:

$$(\nabla_Z S)(X,Y) = (\nabla_X S)(Y,Z). \tag{1.7}$$

2 The $\mathbf{W}_2-$ curvature tensor

In 1970, Pokhariyal and Mishra [19] introduced a new tensor, called W_2, in a Riemannian manifold and studied their properties. According to them [19], a W_2- curvature tensor on a manifold (M^n, g), $(n > 3)$ is defined by

$$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)]. \tag{2.1}$$

The W_2-curvature tensor on some special manifolds has been examined by many authors such as Taleshian and Hosseinzadeh [24], Özen Zengin [28], Hui [13], Mallick and De [16], etc. The object of the present paper is to study W_2-curvature tensor field in an almost pseudo Ricci symmetric manifold. The paper is organized as follows: Section 3 is concerned with W_2-flat $A(PRS)_n$.

In Section 4, the $A(PRS)_n$ admitting W_2-curvature tensor is studied. In this section, a necessary and sufficient condition is found for W_2-curvature tensor to be divergence-free. After that, the conditions for which the contracted W_2 tensor of type $(0,2)$ is recurrent, Codazzi type and covariantly constant are examined. In section 5, we give an example for the existence of $A(PRS)_n$ admitting W_2-curvature tensor.

3 $\mathbf{W}_2-$flat $\mathbf{A(PRS)}_n$

In this section, we denote the contracted W_2-curvature tensor which is type of $(0,2)$ as $\bar{W}_2$ and call it W_2-Ricci tensor. Now, contracting (2.1) over X and U, we obtain the contracted W_2 tensor, i.e., W_2-Ricci tensor

$$\bar{W}_2(X,Y) = \frac{n}{n-1}\left[S(X,Y) - \frac{r}{n}g(X,Y)\right]. \tag{3.1}$$

If we assume that our manifold is W_2-flat, then from (2.1)

$$S(X,Y) = \frac{r}{n}g(X,Y). \tag{3.2}$$

Assuming that our manifold (M^n, g) is $A(PRS)_n$ admitting W_2 curvature tensor, from (3.2) we get

$$(\nabla_Z S)(X,Y) = \frac{1}{n}(\nabla_Z r)g(X,Y). \tag{3.3}$$

Putting (1.4) and (3.2) in (3.3), we obtain

$$(\nabla_Z r)g(X,Y) = r[(A(Z)+B(Z))g(X,Y) + A(X)g(Y,Z) + A(Y)g(X,Z)]. \tag{3.4}$$

Contracting (3.4) over X and Y, we infer

$$n(\nabla_Z r) = r[(n+2)A(Z) + nB(Z)]. \tag{3.5}$$

Again, contracting (3.4) over X and Z, then we have

$$(\nabla_Z r) = r[(n+2)A(Z) + B(Z)]. \tag{3.6}$$

Then, comparing (3.5) and (3.6), we get

$$r[(n+2)A(Z) + nB(Z)] = nr[(n+2)A(Z) + B(Z)].$$

Since $r \neq 0$, we have from the above equation $A(Z) = 0$. Thus, we have the following theorem:

Theorem 3.1. *W_2-flat $A(PRS)_n$ reduces to a recurrent manifold with the recurrence vector field generated by the $1-$form B.*

4 A(PRS)$_n$ admitting non-zero W$_2-$Ricci tensor

Now, we assume that our manifold $A(PRS)_n$ is of non-zero W_2-curvature tensor. By taking the covariant derivative of (3.1), we get

$$(\nabla_Z \bar{W}_2)(X,Y) = \frac{n}{n-1}\left[(\nabla_Z S)(X,Y) - \frac{1}{n}(\nabla_Z r)g(X,Y)\right]. \tag{4.1}$$

Provided that our manifold is $A(PRS)_n$, if we contract (1.4) over X and Y, then we obtain

$$(\nabla_Z r) = [A(Z) + B(Z)]r + 2A(QZ). \tag{4.2}$$

By putting (1.4) and (4.2) in (4.1), we can found

(4.3)
$$\begin{aligned}(\nabla_Z \bar{W}_2)(X,Y) = \frac{n}{n-1}\{&[A(Z)+B(Z)]S(X,Y) + A(X)S(Y,Z) + A(Y)S(X,Z) \\ &- \frac{r}{n}[A(Z)+B(Z)]g(X,Y) - \frac{2}{n}A(QZ)g(X,Y)\}.\end{aligned}$$

Now, contracting (4.3) over X and Z, we get

$$(div\bar{W}_2)(Y) = \frac{n}{n-1}\left[\left(\frac{n-1}{n}\right)(2A(QY) + rA(Y)) + B(QY) - \frac{r}{n}B(Y)\right]. \tag{4.4}$$

If we assume that $\bar{W}_2$ is divergence-free, then from (4.4)

$$\frac{n}{n-1}\left[\left(\frac{n-1}{n}\right)(2A(QY)+rA(Y))+B(QY)-\frac{r}{n}B(Y)\right]=0. \tag{4.5}$$

If $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector Q where $g(X,Q)=B(X)$, then $-\frac{r}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ where $g(X,\rho)=A(X)$. Conversely, if the equation (4.5) holds then from (4.4), W_2-Ricci tensor is divergence-free. Thus, we have the following theorem:

Theorem 4.1. *For an $A(PRS)_n$, a necessary and sufficient condition the contracted W_2 curvature tensor $\bar{W}_2$ be divergence free is that $-\frac{r}{2}$ and $\frac{r}{n}$ be eigenvalues of the Ricci tensor S corresponding to the eigenvectors ρ and Q where $g(X,\rho)=A(X)$ and $g(X,Q)=B(X)$, respectively.*

Let $\bar{W}_2$ be reccurent, i.e., from (1.6)

$$(\nabla_Z\bar{W}_2)(X,Y)=\alpha(Z)\bar{W}_2(X,Y), \tag{4.6}$$

where α is a $1-$form. Using (3.1) and (4.3) in (4.6), it can be found that

(4.7)
$$\alpha(Z)[S(X,Y)-\frac{r}{n}g(X,Y)]=[A(Z)+B(Z)]S(X,Y)+A(X)S(Y,Z)+A(Y)S(X,Z)$$
$$-\frac{r}{n}[A(Z)+B(Z)]g(X,Y)-\frac{2}{n}A(QZ)g(X,Y).$$

If we contract (4.7) over X and Z, then we have

$$\alpha(QZ)-\frac{r}{n}\alpha(Z)=\frac{(n-1)}{n}[2A(QZ)+rA(Z)]+B(QZ)-\frac{r}{n}B(Z). \tag{4.8}$$

This leads to the following result:

Theorem 4.2. *In an $A(PRS)_n$, let us assume that W_2-Ricci tensor is recurrent with the recurrence vector generated by the $1-$form α. If $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvectors both Q and μ where $g(X,Q)=B(X)$, $g(X,\mu)=\alpha(X)$ then $-\frac{r}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ where $g(X,\rho)=A(X)$.*

If we take $\alpha(Z)=A(Z)$ in (4.8), we find

$$\left(\frac{2-n}{n}\right)A(QZ)-A(Z)r=B(QZ)-\frac{r}{n}B(Z).$$

If $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector Q then $\frac{n}{2-n}r$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ. Thus we have the following theorem:

Theorem 4.3. *In an $A(PRS)_n$, let us consider that W_2-Ricci tensor is recurrent with the recurrence vector generated by the $1-$form A. A necessary and sufficient condition for $\frac{r}{n}$ be an eigenvalue of the Ricci tensor S corresponding to the eigenvector Q where $g(X,Q)=B(X)$ is that $\frac{n}{2-n}r$ $(n>2)$ be an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ where $g(X,\rho)=A(X)$.*

Now, by taking $\alpha(Z) = B(Z)$ in (4.8), we obtain

$$A(QZ) = -\frac{r}{2}A(Z). \tag{4.9}$$

If we differentiate the equation (4.9), then we can find

$$(\nabla_X A)(QZ) = -\frac{1}{2}[(\nabla_X r)A(Z) + r(\nabla_X A)(Z)]. \tag{4.10}$$

By using (1.4), (4.2) and (4.9) in (4.10), we get

$$|A|A(Z)r = 0.$$

If $r \neq 0$ then from the above equation, $A(Z)$ must be 0. Thus, we have the following:

Theorem 4.4. *An $A(PRS)_n$ admitting W_2-Ricci tensor which is recurrent with the recurrence vector generated by the $1-$ form B cannot be exist.*

Let us assume that $\bar{W}_2$ is generalized recurrent. Thus, from (1.5)

$$(\nabla_Z \bar{W}_2)(X, Y) = \alpha(Z)\overline{W}_2(X, Y) + \gamma(Z)g(X, Y).$$

Using (3.1) and (4.3) in the above equation and contracting over X and Y, we get

$$n\gamma(Z) = 0.$$

Hence, $\gamma(Z) = 0$. Thus, we have the following theorem:

Theorem 4.5. *An $A(PRS)_n$ admitting W_2-Ricci tensor which is generalized recurrent cannot be exist.*

If r is a non-zero constant, then (4.1) reduces to

$$(\nabla_Z \overline{W}_2)(X, Y) = \frac{n}{n-1}(\nabla_Z S)(X, Y). \tag{4.11}$$

Using (1.4) and (4.11), we get

(4.12)

$$(\nabla_Z \bar{W}_2)(X, Y) = \frac{n}{n-1}[(A(Z) + B(Z))S(X, Y) + A(X)S(Y, Z) + A(Z)S(X, Y)].$$

If $\bar{W}_2$ is Codazzi type, from (1.7) and (4.12)

$$B(Z)S(X, Y) - B(Y)S(X, Z) = 0.$$

By contracting the above equation over X and Y, we have

$$B(Z)r = B(QZ). \tag{4.13}$$

In this case, we infer:

Theorem 4.6. *In an $A(PRS)_n$ admitting constant scalar curvature, if W_2-Ricci tensor is Codazzi type then r is an eigenvalue of the Ricci tensor S corresponding to the eigenvector Q where $g(X, Q) = B(X)$.*

If we assume that r is a non-zero constant and $\bar{W}_2$ is covariantly constant, then

$$(\nabla_Z S)(X,Y) = 0. \tag{4.14}$$

Hence from (1.4) and (4.14), we get

$$[A(Z) + B(Z)]r + 2A(QZ) = 0. \tag{4.15}$$

Finally, we obtain from (4.15)

$$A(QZ) = -\frac{r}{2}[A(Z) + B(Z)], \tag{4.16}$$

and further obtain the following:

Theorem 4.7. *If an $A(PRS)_n$ admits constant scalar curvature and covariantly constant W_2-Ricci tensor then the Ricci tensor is covariantly constant and $A(Z)$ and $B(Z)$ are related by*

$$A(QZ) = -\frac{r}{2}[A(Z) + B(Z)].$$

Let us assume that r is a non-zero constant and $\bar{W}_2$ is Codazzi type. If we take the covariant derivative of (4.13), then we get

$$(\nabla_X B)(QZ) = r(\nabla_X B)(Z). \tag{4.17}$$

By using (1.4) and contracting over X and Z in (4.17), we find

$$r|B|(3\langle A, B\rangle + |B|) = 0, \tag{4.18}$$

where $\langle\ ,\ \rangle$ is the inner product. If $r \neq 0$ and $|B| \neq 0$ in (4.18), we obtain

$$\langle A, B\rangle = -\frac{|B|}{3}. \tag{4.19}$$

Since we know that $\langle A, B\rangle = |A||B|\cos\theta$, from (4.19), we get

$$|A|\cos\theta = -\frac{1}{3} \qquad \textit{where} \qquad \frac{\pi}{2} < \theta \leq \pi.$$

Therefore, we can state the following:

Theorem 4.8. *If an $A(PRS)_n$ with non-zero constant scalar curvature is of Codazzi type W_2-Ricci tensor then the angle between the vector fields generated by the 1-forms A and B is*

$$\theta = \arccos\left(-\frac{1}{3|A|}\right).$$

Thus, θ is in $(\frac{\pi}{2}, \pi]$ where $|A|$ is the length of the vector field generated by the $1-$form A.

5 An example for the existence $A(PRS)_n$ admitting $\mathbf{W}_2-$ Curvature tensor

In this section we want to construct an example of an four-dimensional almost pseudo Ricci symmetric manifold with constant scalar curvature and $\bar{W}_2$ tensor. In local coordinates, let us consider a Riemannian metric g on $\mathbb{R}^4$ with coordinates (x^1, x^2, x^3, x^4) by

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1}(dx^1)^2 + e^{x^2}(dx^2)^2 + (dx^3)^2 + (\sin x^3)^2(dx^4)^2. \tag{5.1}$$

Then the only non vanishing components of the Christoffel symbols and curvature tensors are, respectively,

$$\Gamma^1_{11} = \Gamma^2_{22} = \frac{1}{2}, \quad \Gamma^3_{44} = -\sin x^3 \cos x^3, \quad \Gamma^4_{43} = \cot x^3,$$

and

$$R^3_{443} = -(\sin x^3)^2, \quad R^3_{443} = -1$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors are

$$S_{33} = -1, \quad S_{44} = -(\sin x^3)^2. \tag{5.2}$$

It can be shown that the scalar curvature r of $(\mathbb{R}^4, g)$ is -2. By using (3.1), (5.1) and (5.2), we get the only non-vanishing components of $\bar{W}_2$ are

$$\bar{W}_{11} = \frac{2}{3}e^{x^1}, \quad \bar{W}_{22} = \frac{2}{3}e^{x^2}, \quad \bar{W}_{33} = -\frac{2}{3}, \quad \bar{W}_{44} = -\frac{2}{3}(\sin x^3)^2. \tag{5.3}$$

By taking the covariant derivatives of each of $\overline{W}_{ij}$ tensors in (5.3), we find that $\overline{W}_{ij,k} = 0$ for all i, j, k. This shows that $\overline{W}_{ij}$ are covariantly constant. In this case, by taking the covariant derivatives of S_{33} and S_{44} and by using (5.2), we obtain that $S_{ij,k} = 0$ for all i, j, k. Let us choose the $1-$forms A and B as

$$A = e^{x^1+x^2} + e^{x^3+x^4} \tag{5.4}$$

and

$$B = -e^{x^1+x^2} + 2e^{x^3+x^4}. \tag{5.5}$$

Now, by taking the derivatives of (5.4) and (5.5), we get

$$A_i = \begin{cases} e^{x^1+x^2}, & i = 1, 2 \\ e^{x^3+x^4}, & i = 3, 4 \end{cases} \tag{5.6}$$

and

$$B_i = \begin{cases} -e^{x^1+x^2}, & i = 1, 2 \\ -2e^{x^3+x^4}, & i = 3, 4 \ . \end{cases} \tag{5.7}$$

With the help of (5.1), (5.2), (5.6) and (5.7), the equations

$$A^1R_{11} = -\tfrac{r}{2}(A_1+B_1), \quad A^2R_{22} = -\tfrac{r}{2}(A_2+B_2)$$
$$A^3R_{33} = -\tfrac{r}{2}(A_3+B_3), \quad A^4R_{44} = -\tfrac{r}{2}(A_4+B_4)$$

are satisfied. From these results, it is clear that $(\mathbb{R}^4, g)$ given by (5.1) is an $A(PRS)_n$ satisfying Theorem 4.7. Thus, we can state the following:

Theorem 5.1. *Let us consider a Riemannian metric g on $\mathbb{R}^4$ given by*

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1}(dx^1)^2 + e^{x^2}(dx^2)^2 + (dx^3)^2 + (sinx^3)^2(dx^4)^2.$$

Assume that this manifold is an $A(PRS)_n$ with the constant scalar curvature and covariantly constant $\overline{W}_{ij}$ tensor. If we choose the $1-$ forms A and B as follows:

$$A = e^{x^1+x^2} + e^{x^3+x^4} \quad and \quad B = -e^{x^1+x^2} + 2e^{x^3+x^4},$$

then Theorem 4.7 holds.

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