Characterization of real hypersurfaces in a nonflat complex space form having a special shape operator

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Abstract. Let M be a real hypersurfaces in a complex space form $M_n(c)$, $c \neq 0$, whose Lie derivative of shape operator in the direction of the Reeb vector field coincides with the covariant derivative of it in the same direction. In this paper, we characterize such real hypersurfaces of $M_n(c)$.

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1 Introduction

A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n\mathbb{C}$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The Reeb vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant ([3]) and that M is called a *Hopf hypersurface*.

Typical examples of Hopf hypersurfaces in $P_n\mathbb{C}$ are homogeneous ones, namely those real hypersurfaces are given as orbits under subgroup of the projective unitary groups $PU(n+1)$. R.Takagi [12] completely classified homogeneous real hypersurfaces in such hypersurfaces as six model spaces A_1, A_2, B, C, D and E. On the other hand, real hypersurfaces in $H_n\mathbb{C}$ have been investigated by Berndt [1], Montiel and Romero [7] and so on. Berndt [1] classified all homogeneous Hopf hyersurfaces in $H_n\mathbb{C}$ as four model spaces which are said to be A_0, A_1, A_2 and B. A real hypersurface of A_1 or A_2 in $P_n\mathbb{C}$ or A_0, A_1, A_2 in $H_n\mathbb{C}$, then M is said to be a type A for simplicity.

As a typical characterization of real hypersurfaces of type A, the following is due to Okumura [11] for $c > 0$ and Montiel and Romero [7] for $c < 0$.

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Theorem A ([7,11]) *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on M if and only if M is locally congruent to one of the model spaces of type A.*

For the shape operator A on M, we define the Lie derivative $\mathcal{L}_\xi$ by $(\mathcal{L}_\xi A)X = [\xi, AX] - A[\xi, X]$ for any vector field X on M. With regard to Lie derivative, the study of real hypersurfaces in the complex space form is one of the very interesting and important problems that are being studied by many geometricians (see [6],[8],[9], etc). The Lie derivative and covariant derivative of the structure Jacobi operator with respect to ξ was investigated by Perez and Santos (see [10]). More precisely, they classified real hypersurfaces in $P_n\mathbb{C}$, whose structure Jacobi operator satisfies $\mathcal{L}_\xi R_\xi = \nabla_\xi R_\xi$. Panagiotidou and Xenos(see [9]) classified real hypersurfaces satisfying the same condition in $P_2\mathbb{C}$ and $H_2\mathbb{C}$. As for the Lie derivative, Ki, Kim and Lim (see [5]) have proved the following.

Theorem B ([7,12]) *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Then it satisfies $R_\xi \mathcal{L}_\xi g = 0$ if and only if M is locally congruent to one of the model spaces of type A.*

In this paper, we shall study a real hypersurface in a nonflat complex space form $M_n(c)$ with Lie derivative and covariant derivative of shape operator A and give some characterizations of such real hypersurface in $M_n(c)$.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

2 Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\widetilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\widetilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\widetilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{aligned} \tag{2.1}$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify from the Gauss and Weingarten formulas the followings :

$$\nabla_X \xi = \phi AX, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi. \tag{2.2}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss, Codazzi equations and operator of Lie derivative respectively :

$$\begin{aligned} R(X,Y)Z = & \tfrac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ & -2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.3}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\}, \tag{2.4}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

Let Ω be the open subset of M defined by

$$\Omega = \{p \in M \mid A\xi - \alpha\xi \neq 0\}. \tag{2.5}$$

where $\alpha = \eta(A\xi)$. We put

$$A\xi = \alpha\xi + \mu W, \tag{2.6}$$

where W be the unit vector field orthogonal to ξ and μ does not vanish on Ω.

3 Real hypersurface satisfying $\mathcal{L}_\xi A = \nabla_\xi A$.

Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, satisfying $\mathcal{L}_\xi A = \nabla_\xi A$. In this section, we assume that the open set Ω given in (2.5) is not empty. Then the above the condition together with (2.2) and Lie derivative in the ξ implies that

$$(\phi A^2 - A\phi A)X = 0 \tag{3.1}$$

or equivalently

$$(A\phi A - A^2\phi)X = 0. \tag{3.2}$$

for any vector field X on M. Now, we prove the following Lemma.

Lemma 3.1*Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$ satisfying $\mathcal{L}_\xi A = \nabla_\xi A$. If the open set Ω is not empty, then we have*

$$AW = \mu\xi - \alpha W, \qquad A\phi W = 0, \tag{3.3}$$

$$\alpha^2 + \mu^2 = 0 \tag{3.4}$$

on Ω.

Proof. If we put $X = \xi$ into (3.1) and make use of (2.5)and (2.6), Then we have

$$(A\phi - \phi A)W = \alpha\phi W. \tag{3.5}$$

Putting $X = \xi$ into (3.2) and using (2.5) and (2.6), we get $A\phi W = 0$ and hence the second equation of (3.3) on Ω. If we substitute the second equation of (3.3) into (3.5) then we get $\phi AW = -\alpha\phi W$, or equivalentiy, the first equation of (3,3). It follows immediately from $X = W$ into (3.1) or $X = \phi W$ into (3.2) and using (2.6) and the first equation (3.3) that the equation (3.4) is given. □

Differentiating the smooth function $\mu = g(A\xi, W)$ along any vector field X on M using (2.2),(2.4),(2.6) and (3.3), we have

$$X\mu = g((\nabla_\xi A)W + \frac{c}{4}\phi W, X).$$

Since we have $(\nabla_\xi A)W = \nabla_\xi(\mu\xi - \alpha W) - A\nabla_\xi W$, we see from the above equation that the gradient vector field $\nabla\mu$ of μ is given by

$$\nabla\mu = -(A + \alpha I)\nabla_\xi W + (\xi\mu)\xi - (\xi\alpha)W + (\mu^2 + \frac{c}{4})\phi W. \tag{3.6}$$

where I indicates the identity transformation on M. If we differentiate $\alpha = g(A\xi, \xi)$ along vector field X and take account of (2.2),(2.4),(2.6) and the second equation (3.3), then we obtain $\nabla\alpha = (\nabla_\xi A)\xi$ and hence

$$\nabla\alpha = \mu\nabla_\xi W + (\xi\alpha)\xi + (\xi\mu)W + (\alpha\mu)\phi W. \tag{3.7}$$

As a similar argument above, we can see that the gradient vector field of $-\alpha = g(AW, W)$

$$\nabla\alpha = (A = \alpha I)\nabla_W W - (W\mu)\xi + (W\alpha)W + \alpha\mu\phi W. \tag{3.8}$$

Comparing (3.7) and (3.8), we can verify that

$$\mu\nabla_\xi W - (A + \alpha I)\nabla_W W = -\{(\xi\alpha) + W\mu)\}\xi + \{(W\alpha) - (\xi\mu)\}W.$$

If we take inner product of ξ and W respectively, we find

$$\xi\alpha = -W\mu \qquad \textit{and} \qquad W\alpha = \xi\mu \tag{3.9}$$

by the virtue of the equation (3.3) on Ω, and hence the initial equation is reduced to

$$\mu\nabla_\xi W - (A + \alpha I)\nabla_W W = 0. \tag{3.10}$$

By means of (2.2),(2.6) and (3.3), we can verify that

$$(\nabla_W A)\xi = \nabla_W(A\xi) - A\nabla_W\xi = \mu\nabla_W W + (W\alpha)\xi + (W\mu)W - \alpha^2\phi W.$$

and

$$(\nabla_\xi A)W = \nabla_\xi(AW) - A\nabla_\xi W = -(A+\alpha I)\nabla_\xi W + (\xi\mu)\xi - (\xi\alpha)W + \mu^2\phi W.$$

Therefore it follows from the equation (2.4) of Codazzi and making use of (3.9) that

$$\mu\nabla_W W + (A+\alpha I)\nabla_\xi W = \{al^2 + \mu^2 - \frac{c}{4}\}\phi W. \tag{3.11}$$

If we compare (3.10) and (3.11), we can verify that

$$\{A^2 + 2\alpha A + (\alpha^2 - \mu^2)I\}\nabla_\xi W = \alpha\mu(\alpha^2 + \mu^2 - \frac{c}{4})\phi W. \tag{3.12}$$

4 Some Lemmas.

In this section we assume that M is a real hypersurface satisfying $\mathcal{L}_\xi A = \nabla_\xi A$ in a complex space form $M_n(c)$, $c \neq 0$, and the open set Ω given in (2.5) is not empty. Then we may consider from (3.4) in Lemma 3.1 that we have $\alpha^2 + \mu^2 = 0$ on Ω. We shall prove some Lemmas, which will be used later.

Lemma 4.1*If M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$ satisfying $\mathcal{L}_\xi A = \nabla_\xi A$. If the open set Ω is not empty, then the vector field $\nabla\alpha$, $\nabla\mu$, $\nabla_\xi W$ and $\nabla_W W$ are expressed in terms of the vector fields ξ, W and ϕW only on Ω.*

Proof. Let D be the distribution spanned by the unit vector field ξ, W and ϕW on Ω, that is, $D_p = span\{\xi, W, \phi W\}_P$ for any point p on Ω. Then we see from (2.6) and (3.3) that D is invariant under the shape operator A and the structure tensor field ϕ. Also, since A is symmetric we can choose a local orthogonal frame field $\{\xi, W, \phi W, X_4, \cdots, X_{2n-1}\}$ on Ω such that $AX_i = \lambda X_i$ for $4 \leq i \leq 2n-1$. The vector field $\nabla_\xi W$ can be expressed as

$$\nabla_\xi W - \sum_{i=4}^{2n-1} a_i X_i \equiv 0 \qquad (modD). \tag{4.1}$$

It follows from (3.6) and (4.1) that

$$\nabla\mu + \sum_{i=4}^{2n-1} a_i(\lambda_i + \alpha)X_i \equiv 0 \qquad (modD). \tag{4.2}$$

and from (3.7) and (4.1) that

$$\nabla\alpha - \sum_{i=4}^{2n-1} a_i X_i \equiv 0 \qquad (modD). \tag{4.3}$$

We can verify from (3.11) and (4.1) that

$$\mu \nabla_W W + \sum_{i=4}^{2n-1} a_i(\lambda_i + \alpha)X_i \equiv 0 \qquad (mod D).$$

Using this equation, (3.8) is reduced to

$$\mu \nabla \alpha + \sum_{i=4}^{2n-1} a_i(\lambda_i + \alpha)^2 X_i \equiv 0 \qquad (mod D). \tag{4.4}$$

Since, by (3.4), the scalar function $\alpha^2 + \mu^2 = 0$ on Ω, we can substitute (4.2) and (4.4) into $\alpha\nabla\alpha + \mu\nabla\mu = 0$ yields

$$\sum_{i=4}^{2n-1} a_i\{\mu^2(\lambda_i + \alpha) + \alpha(\lambda_i + \alpha)^2\}X_i \equiv 0 \qquad (mod D). \tag{4.5}$$

for $4 \leq i \leq 2n-1$ on Ω. Next, substituting (4.2) and (4.3) into $\alpha\nabla\alpha + \mu\nabla\mu = 0$, we can get

$$\sum_{i=4}^{2n-1} a_i\{\alpha - \mu(\lambda_i + \alpha)X_i \equiv 0 \qquad (mod D). \tag{4.6}$$

If we compare (4.5) with (4.6), Then we have

$$\alpha \sum_{i=4}^{2n-1} a_i\{\mu + (\lambda_i + \alpha)^2\}X_i \equiv 0 \qquad (mod D).$$

If α is zero in the above equation and make use of (3.4), Then we get $\mu = 0$ and hence it is a contradiction. Therefore, the above equation is rewritten as

$$\sum_{i=4}^{2n-1} a_i\{\mu + (\lambda_i + \alpha)^2\}X_i \equiv 0 \qquad (mod D). \tag{4.7}$$

If we substitute (4.3) into (3.12) and using (4.7), then we obtain

$$(1 + \mu) \sum_{i=4}^{2n-1} a_i X_i \equiv 0 \qquad (mod D). \tag{4.8}$$

If we take inner product of this equation with X_i, then we have $a_i = 0$ for $4 \leq i \leq 2n-1$ provided $\mu \neq -1$.

In the case where $\mu = -1$ and using the covariant differential of equation (3.4), we get $\nabla\alpha = 0$. By the equation (4.1), we can verify that

$$\sum_{i=4}^{2n-1} a_i X_i \equiv 0 \qquad (mod D).$$

If we take inner product of this equation with X_i, then we have $a_i = 0$ for any $4 \leq i \leq 2n-1$ and $\mu = -1$ or $\mu \neq -1$. It follows (3.11), (4.1), (4.2) and (4.3)that the vector fields $\nabla_\xi W$, $\nabla\alpha$, $\nabla\mu$ and $\nabla_W W$ are expressed in terms of ξ, W and ϕW only. $\square$

Lemma 4.2 *Under the assumption of Lemma 4.1. if $\alpha^2+\mu^2=0$ holds on Ω, then we have*

$$A\nabla_\xi W = \frac{c}{4}\phi W. \tag{4.9}$$

Proof. From the derivative of a given assumption, we get

$$\mu\nabla\mu + \alpha\nabla\alpha = 0. \tag{4.10}$$

If we take inner product of (4.10) with ξ and W, respectively, Then we have

$$\mu\xi\mu + \alpha\xi\alpha = 0 \qquad \textit{and} \qquad \mu W\mu + \alpha W\alpha = 0. \tag{4.11}$$

If we substitute (3.6) and (3.7) into (4.10) and using (3.9), (4.1) and the assumption, Then we get the equation of (4.9) on Ω. $\square$

5 Characterizations of real hypersurfaces in a non-flat complex space form.

In this section, we shall prove the following Theorems.

Lemma 5.1 *Let M be a real hypersurface satisfying $\mathcal{L}_\xi A = \nabla_\xi A$ in a nonflat complex space form $M_n(c)$, $c \neq 0$. Then M is a Hopf hypersurface in $M_n(c)$.*

Proof. Assume that the open set Ω given in (2.5) is not empty. Then we can consider from (3.4) that $\alpha^2+\mu^2=0$ holds on Ω. If substitute (4.9) into (3.12) and using the second equation of (3.3) then we get

$$(\alpha^2-\mu^2)\nabla_\xi W = \alpha\mu(-\frac{c}{4})\phi W.$$

Since we have $\alpha^2+\mu^2=0$, the above equation is rewritten as

$$\nabla_\xi W = -\frac{c}{8\alpha}\mu\phi W. \tag{5.1}$$

Because the shape operator A is invariant under D, if we apply A to (5.1) and by using the second equation of (3.3) and (4.9), then we obtain $c=0$ and it is a contradiction. Thus, the set Ω is empty, and hence M is a Hopf hypersurface. $\square$

Lemma 5.2 *Let M be a real hypersurface in a complex space form $M_n(c)$, $c\neq 0$. Then we have $\mathcal{L}_\xi A = \nabla_\xi A$ on M if and only if M is locally congruent to one of the model space of type A.*

Proof. By Theorem 5.1, M is a Hopf hypersurface in $M_n(c)$, that is $A\xi=\alpha\xi$. Therefore the assumption $\mathcal{L}_\xi A = \nabla_\xi A$ is equivalent to

$$(\phi A^2 - A\phi A)X = 0. \tag{5.2}$$

$$(A\phi A - A^2\phi)X = 0. \tag{5.3}$$

by use of (2.2) and (2.6). On the other hand, if we differentiate $A\xi = \alpha\xi$ covariantly and make use of the Codazzi equation, then we have

$$A\phi A - \frac{\alpha}{2}(\phi A + A\phi) - \frac{c}{4}\phi = 0. \tag{5.4}$$

For any vector field X on M such that $AX = \lambda X$, it follows from (5.4) that

$$(\lambda - \frac{\alpha}{2})A\phi X = \frac{1}{2}(\alpha\lambda + \frac{c}{2})\phi X. \tag{5.5}$$

If $\lambda \neq \frac{\alpha}{2}$, then we see from (5,5) that $A\phi X = \frac{2\alpha\lambda + c}{2(2\lambda - \alpha)}\phi X$ so that ϕX is also principal direction and we can write $A\phi X = \mu\phi X$. Assume that there is a point p on M such that $\lambda(\lambda - \mu) = 0$. If $\lambda = 0$, then we see from (5.3) that $\mu = 0$ However, since $\mu = \frac{2\alpha\lambda + c}{2(2\lambda - \alpha)}$, we obtain $c = 0$ at p and it is a contradiction. Therefore we see that $\lambda = \mu$ on M and from this result we obtain

$$\phi A = A\phi \tag{5.6}$$

on the whole M.

Conversely if it satisfies (5.6), then it is easily seen that (5.2) or (5.3) holds, that is $\mathcal{L}_\xi A = \nabla_\xi A$ is satisfies on M. Thus, Theorem 5.2 follows from Theorem A. □

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