Divergence Lemma and Hopf's Theorem on Finslerian slit tangent bundle

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Abstract. Let F be a Finslerian metric on a C^∞ manifold M. We define horizontal and vertical Laplace type operators for C^∞ functions on the slit tangent bundle $\mathring{T}M$ of the Finslerian manifold (M, F). The expression of the vertical Laplacian Δ^v, in local coordinates, is obtained by using the Chern connection on the pulled-back tangent bundle. Furthermore, we prove the vertical divergence lemma and we show that if (M, F) is an oriented Finslerian manifold then every globally defined function u on $\mathring{T}M$, with $\Delta^v u \geq 0$ everywhere or $\Delta^v u \leq 0$ everywhere, must be independant of directional arguments.

M.S.C. 2010: 31A05, 53C60, 55R25.
Key words: Finslerian manifolds; pulled-back tangent bundle; Chern connection; vertical Laplacian.

1 Introduction

The geometry of the vector bundles began in the 1950s with Sasaki, who built a metric on the tangent bundle of a Riemannian manifold [1]. Let consider a Finslerian manifold (M, F) and $\mathring{T}M \equiv TM\backslash\{0\}$ its slit tangent bundle. The Finslerian metric F induces on $\mathring{T}M$ a fundamental tensor $g = g_{ij}(x, y)dx^i \otimes dx^j$, with $(x, y) \in \mathring{T}M$ whose (x^i, y^i) is its representation in local coordinate. As explain in [3], the tensor g defines at every point $(x, y) \in \mathring{T}M$ an inner product on each tangent space T_xM of the manifold M.

The Laplace operator plays an important role in the theory of harmonic maps. This last subject has found many applications. More informations on harmonic maps can be found: in [7] for Euclidean spaces particularly for harmonic functions and in [2] for Riemannian manifolds. Hence, questions about aspects of harmonic (subharmonic, superharmonic) functions for Finslerian manifolds arise. In Finslerian geometry, the Laplace type operator can be defined in different ways and is made either on M or on $\mathring{T}M$. See [6] for Laplacians on $\mathring{T}M$ in terms of Cartan connection and [10, 11] for Laplacians on M in terms of Chern connection. The Chern connection is, for

Balkan Journal of Geometry and Its Applications, Vol.25, No.1, 2020, pp. 93-103.

us, the most important Finslerian connection. Note that, as introduced in the first paragraph of this Section, the components g_{ij} of g, associated with F, depend not only on the base point $x \in M$, as in the Riemannian case, but also on the direction $y \in T_xM$. Then, it is very important to define a Laplacian operator for functions on $\overset{\circ}{T}M$ of (M, F).

In this paper, we use the pulled-back tangent bundle approach [3, 12] to define vertical Laplacian operator, denoted by Δ^v, on $\overset{\circ}{T}M$. Our main goal is to generalize the classical Riemannian E. Hopf's theorem, given in [3] for harmonic (subharmonic, superharmonic) functions. With respect to the horizontal Laplacian, the analogous of this theorem refers to [6] where it is proved by applying Cartan ideas and by decomposition of the Laplace-Beltrami operator on $(\overset{\circ}{T}M, g^s)$ where g^s is the Sasakian metric.

The rest is organised as follows: in the second Section, we recall some basic notions on Finslerian manifolds which are used throught this paper. In Section 3, with the Chern connection on the pulled-back tangent bundle π^*TM by the submersion $\pi : \overset{\circ}{T}M \longrightarrow M$, we define Δ^v for C^∞-functions on $\overset{\circ}{T}M$ and give some of its properties. In Section 4, we give a vertical divergence formula for sections of π^*TM. Thus, we prove our main results.

2 Some basic notions

Throughout this paper, M is an $n-$dimensional connected C^∞ manifold. We denote by T_xM the tangent space at $x \in M$, by $TM := \bigcup_{x\in M} T_xM$ the tangent bundle of M and by $SM := \{(x, [y])\}$ the sphere bundle of M. Set $\overset{\circ}{T}M = TM\backslash\{0\}$. The natural projection $\pi : TM \longrightarrow M$ is given by $\pi(x, y) = x$. Let $(x^1, ..., x^n)$ be a local coordinate on an open subset U of M and $(x^1, ..., x^n, y^1, ..., y^n)$ be the local coordinate on $\pi^{-1}(U) \subset TM$. The said (x^i), with $i = 1, ..., n$, produce the basis sections $\{\frac{\partial}{\partial x^i}\}$ and $\{dx^i\}$, respectively, for TM and cotangent bundle T^*M. We use Einstein summation convention.

2.1 Finslerian manifolds

Definition 2.1. A *Finslerian metric* on M is a function $F : TM \longrightarrow [0, \infty)$ with the following properties :

(1) F is C^∞ on the entire slit tangent bundle $\overset{\circ}{T}M$.

(2) F is positively 1-homogeneous on the fibers of TM, that is

$$\forall c > 0, \quad F(x, cy) = cF(x, y).$$

(3) the Hessian matrix $(g_{ij}(x, y))_{1\le i,j\le n}$ with elements

$$g_{ij}(x, y) := \frac{\partial^2 \left[\frac{1}{2}F^2(x, y)\right]}{\partial y^i \partial y^j} \tag{2.1}$$

is positive-definite at every point (x, y) of $\overset{\circ}{T}M$.

Given a manifold M and a Finslerian metric F on TM, the pair (M, F) is called a *Finslerian manifold*.

2.2 The vector bundle π^*TM and related objects

Let M be a connected C^∞ manifold. The restricted projection $\pi : \mathring{T}M \longrightarrow M$ pulles back TM to a bundle π^*TM over $\mathring{T}M$ called the *pulled-back tangent bundle*. π^*TM is a vector bundle over the slit tangent bundle $\mathring{T}M$ whose fiber $(\pi^*TM)|_{(x,y)}$ at $(x, y) \in \mathring{T}M$ is just a copy of the tangent space T_xM. Then

$$(\pi^*TM)|_{(x,y)} := \{(x, y, \xi) : \xi \in T_xM\} \cong T_xM. \tag{2.2}$$

Set

$$g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \tag{2.3}$$

and

$$\mathcal{A}_{ijk}(x, y) := \frac{F}{2}\frac{\partial g_{ij}(x, y)}{\partial y^k} = \frac{F}{4}\frac{\partial^3 F^2(x, y)}{\partial y^i \partial y^j \partial y^k}. \tag{2.4}$$

One obtains two tensors $g = g_{ij}dx^i \otimes dx^j$ and $\mathcal{A} = \mathcal{A}_{ijk}dx^i \otimes dx^j \otimes dx^k$ called respectively *fundamental tensor* and *Cartan tensor*. g is a symmetric section of $\pi^*T^*M \otimes \pi^*T^*M$ while $\mathcal{A}$ is a symmetric section of $\pi^*T^*M \otimes \pi^*T^*M \otimes \pi^*T^*M$.

Lemma 2.1. *Let (M, F) be an n-dimensional Finslerian manifold and $\mathcal{A}$ the Cartan tensor. Then, for all $i, j, k \in \{1, ..., n\}$ and $(x, y) \in \mathring{T}M$,*

(i) $\mathcal{A}_{ijk} = \mathcal{A}_{ikj} = \mathcal{A}_{jik}$.

(ii) $y^i\mathcal{A}_{ijk}(x, y) = y^j\mathcal{A}_{ijk}(x, y) = y^k\mathcal{A}_{ijk}(x, y) = 0$.

(iii) *(M, F) is Riemannian if and only if $\mathcal{A}_{ijk}(x, y) = 0$.*

Now, consider all points in $\mathring{T}M$ of the form (x, cy), with x, y fixed and c an arbitrary positive real number. Over each such point, one erects the same vector space T_xM. Because the components $g_{ij}(x, y)$ of g are invariant under the rescaling $y \mapsto cy$ then the inner products, assigned to the copies $(\pi^*TM)|_{(x,cy)}$ of T_xM, are also identic. One set $SM := \{(x, [y]) : (x, y) \in \mathring{T}M\}$, where $[y] := \{cy : c > 0, y \in T_xM\backslash\{0\}\}$, and call the *sphere bundle* which is the $(2n-1)$-dimensional manifold over M.

Remark 2.2. We work on $\mathring{T}M$. But, in order to use objects that have the same sense following that one works on $\mathring{T}M$ or SM, we focus on objects which are invariant by transformation $y \longmapsto cy, c > 0$. For example, we use $\mathcal{A}_{ikj}$ and $\frac{\delta y^i}{F}$ instead of $\frac{1}{F}\mathcal{A}_{ikj}$ and δy^i respectively. Since the objects are invariant by rescaling in y, they can be seen as carried out on SM using homogeneous coordinates.

2.3 Finslerian Ehresmann connection

Consider the tangent mapping π_* of the restricted projection $\pi : \mathring{T}M \longrightarrow M : \pi(x, y) \longmapsto x$. The vertical subspace of $T\mathring{T}M$ is defined by $\mathcal{V} := ker(\pi_*)$ which is locally spanned by the set $\{F\frac{\partial}{\partial y^i}, 1 \leq i \leq n\}$, on each $\pi^{-1}(U) \subset \mathring{T}M$.

An horizontal subspace $\mathcal{H}$ of $T\mathring{T}M$ is by definition any complementary to $\mathcal{V}$. The bundles $\mathcal{H}$ and $\mathcal{V}$ give a smooth splitting [9]

$$T\mathring{T}M = \mathcal{H} \oplus \mathcal{V}. \tag{2.5}$$

An Ehresmann connection is a selection of horizontal subspace $\mathcal{H}$ of $T\mathring{T}M$.

All Finslerian metric F on M induces a vector fields on $\mathring{T}M$ [4] in the form $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ where the elements

$$G^i(x,y) := \frac{1}{4} g^{il} \left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right) y^j y^k$$

are y-homogeneous of degree two. The vector field G is called *spray* on M and the G^i are called *spray coefficients* of G. Consider the functions

$$N^i_j(x,y) := \frac{\partial G^i(x,y)}{\partial y^j}.$$

One says that $N^i_j(x,y)$ are *Ehresmann connection coefficients* on manifold $\mathring{T}M$. One has

$$N^i_j = -\frac{1}{F} \mathcal{A}_{jkl} g^{il} \gamma^k_{rs} y^r y^s + \gamma^i_{jk} y^k, \quad i,j,k,r,s = 1, ..., n \tag{2.6}$$

where $\gamma^i_{jk} := \frac{1}{2} g^{il} \left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right)$ are formal Christoffel symbols of the second kind.

Remark 2.3. According to the remark 2.2, for objects invariant under $y \mapsto cy$, we can consider N^i_j as Ehresmann connection coefficients on SM. In that case, we must work with $\frac{N^i_j}{F} := -\mathcal{A}_{jkl} g^{il} \gamma^k_{rs} \frac{y^r}{F} \frac{y^s}{F} + \gamma^i_{jk} \frac{y^k}{F}$.

Consider the local coordinate (x^i, y^i) in TM. One has, respectively, the local coordinates bases $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ for the tangent bundle of TM and $\{dx^i, dy^i\}$ for the cotangent bundle of TM. For the tangent bundle of $\mathring{T}M$, a local coordinate basis adapted to the above decomposition is consists of the $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$ and $\frac{\partial}{\partial y^i}$. The dual of the two last basis elements are respectively dx^i and $\delta y^i := dy^i + N^i_j dx^j$.

In the sequel, we consider the Ehresmann connection (called Finslerian Ehresmann connection) defined as follows

Definition 2.4. A Finslerian Ehresmann connection of the restricted projection π is a subspace $\mathcal{H}$ of $T\mathring{T}M$, which is complementary to the vertical subspace $\mathcal{V}$ of $T\mathring{T}M$, given by $\mathcal{H} := ker\theta$ where θ is, globally, a C^∞ function from $T\mathring{T}M$ to π^*TM called the *Ehresmann form.* Locally, θ is given by

$$\theta = \frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^i_j dx^j). \tag{2.7}$$

Proposition 2.2. *[12] Given a Finslerian manifold (M,F) and g the fundamental tensor associated with Finslerian metric F, let π_* be the tangent mapping of the* *restricted projection* $\pi : \overset{\circ}{T}M \longrightarrow M$. *There exist a unique linear connection* ∇ *on the pulled-back tangent bundle* π^*TM,

$$\begin{array}{rcl} \nabla : \Gamma(T\overset{\circ}{T}M) \times \Gamma(\pi^*TM) & \longrightarrow & \Gamma(\pi^*TM) \\ (X \quad , \quad \xi) & \longmapsto & \nabla_X \xi \end{array}$$

such that, for all $X, Y \in \Gamma(T\overset{\circ}{T}M)$ *and* $\xi, \eta \in \Gamma(\pi^*TM)$, *one has the following properties:*

(i) *Symmetry:* $\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_*[X, Y]$,

(ii) *Almost g-compatibility:* $X(g(\xi, \eta)) = g(\nabla_X \xi, \eta) + g(\xi, \nabla_X \eta) + 2\mathcal{A}(\theta(X), \xi, \eta)$,

where $\mathcal{A}$ *is the Cartan tensor and* θ *is the Ehresmann form given by (2.7).*

2.4 Riemannian metric on the slit tangent bundle $\overset{\circ}{T}M$

The slit tangent bundle $\overset{\circ}{T}M$ has a natural Riemannian metric

$$g^s = g_{ij} dx^i \otimes dx^j + g_{ij} \frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}, \tag{2.8}$$

known (see [3], Section 2.1) as a Sasaki (type) metric, where g_{ij} are the components of the fundamental tensor g. With respect to this metric, we have $span\{\frac{\delta}{\delta x^i}\} \perp span\{F\frac{\partial}{\partial y^i}\}$, that is

$$\begin{aligned} T\overset{\circ}{T}M &= \mathcal{H} \oplus \mathcal{V} \\ &= span\left\{\frac{\delta}{\delta x^i}\right\} \oplus span\left\{F\frac{\partial}{\partial y^i}\right\}. \end{aligned} \tag{2.9}$$

Remark 2.5. The y-homogeneous of degree zero of g induces the invariance of g^s under the positive rescaling of y and g^s can be considered as a Riemannian metric on SM.

3 Fundamental differential operators on $\overset{\circ}{T}M$

Let (M, F) be an n-dimensional Finslerian manifold, g be the fundamental tensor of F and g^s be the Sasakian (type) metric on the slit tangent bundle $\overset{\circ}{T}M$. In this Section, we define fundamental differential operators on $\overset{\circ}{T}M$ which are used in the following.

3.1 Gradient section of the vector bundle π^*TM

For the Riemannian manifold $(\overset{\circ}{T}M, g^s)$, the gradient of a function $u \in C^\infty(\overset{\circ}{T}M)$ is given by:

$$g^s(\nabla u, X) = X(u), \quad \forall X \in \Gamma(T\overset{\circ}{T}M). \tag{3.1}$$

Let $\left\{\frac{\delta}{\delta x^i}, F\frac{\partial}{\partial y^i}\right\}$ be the corresponding basis to the direct sum of the horizontal $\mathcal{H}$ and the vertical $\mathcal{V}$ subspaces of $T\overset{\circ}{T}M$, where $F\frac{\partial}{\partial y^i}$ is the homogenized usual partial derivative. Using a local coordinate (x^i, y^i) in $\mathring{T}M$, every section X of $T\mathring{T}M$ is given by $X = (X^h)^i \frac{\delta}{\delta x^i} + (X^v)^i F \frac{\partial}{\partial y^i}$ where the $(X^h)^i$ and the $(X^v)^i$ are C^∞ functions on $\mathring{T}M$. Then, locally, we have

$$\nabla u = g^{ij} \frac{\delta u}{\delta x^i} \frac{\delta}{\delta x^j} + g^{ij} F^2 \frac{\partial u}{\partial y^i} \frac{\partial}{\partial y^j}. \tag{3.2}$$

It is known, the vertical space $\mathcal{V}$, as well as the horizontal $\mathcal{H}$, can be naturally identified with π^*TM (see [11]). Then, we have the following

Proposition 3.1. *Let $\xi \in \Gamma(\pi^*TM)$ and $X \in \Gamma(T\mathring{T}M)$. If $\mathcal{H} \cong \pi^*TM$ then, using*

$$\pi_*\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial x^i}, \qquad \pi_*\left(F\frac{\partial}{\partial y^i}\right) = 0, \tag{3.3}$$

one get locally $\xi = \xi^i \frac{\partial}{\partial x^i}$ with $\xi^i = (X^h)^i$ and if $\mathcal{V} \cong \pi^*TM$ and, using

$$\theta\left(\frac{\delta}{\delta x^i}\right) = 0, \qquad \theta\left(F\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial x^i}, \tag{3.4}$$

one get locally $\xi = \xi^i \frac{\partial}{\partial x^i}$ with $\xi^i = (X^v)^i$.

Definition 3.1. Let (M, F) be C^∞ Finslerian manifold and $\mathring{T}M$ its slit tangent bundle. Let u be a C^∞ function on $\mathring{T}M$. The h-gradient of u is a section of the pulled-back tangent bundle π^*TM given by

$$\nabla^h u = g^{ij} \frac{\delta u}{\delta x^i} \frac{\partial}{\partial x^j}. \tag{3.5}$$

The v-gradient of u is a section of the pulled-back tangent bundle π^*TM given by

$$\nabla^v u = g^{ij} F \frac{\partial u}{\partial y^i} \frac{\partial}{\partial x^j}. \tag{3.6}$$

3.2 Divergence of a section of the bundle π^*TM

Definition 3.2. For a C^∞ section $\xi \in \Gamma(\pi^*TM)$, one defines the horizontal divergence by

$$div^h \xi = trace_g(\eta \longmapsto \nabla_{\eta^h} \xi) \tag{3.7}$$

and the vertical divergence by

$$div^v \xi = trace_g(\eta \longmapsto \nabla_{\eta^v} \xi) \tag{3.8}$$

where g is the fundamental tensor associated with F and ∇ is the Chern connection.

In the basis sections $\{\frac{\partial}{\partial x^i}\}_{i=1,\dots,n}$ of the vector bundle π^*TM, one has:

$$div^h \xi = g^{ij} g\left(\nabla_{\frac{\delta}{\delta x^i}} \xi, \frac{\partial}{\partial x^j}\right) \quad \text{and} \quad div^v \xi = g^{ij} g\left(\nabla_{F\frac{\partial}{\partial y^i}} \xi, \frac{\partial}{\partial x^j}\right). \tag{3.9}$$

The vertical divergence of ξ is a C^∞ function on $\mathring{T}M$. We get

$$div^v \xi = \frac{F \partial \xi^i}{\partial y^i}. \tag{3.10}$$

3.3 Laplacians of C^∞ functions on $\mathring{T}M$

Definition 3.3. Let (M,F) be a C^∞ Finslerian manifold and $u \in C^\infty(\mathring{T}M)$. The horizontal Laplacian $\Delta^h u$ of u is given by

$$\Delta^h u = div^h(\triangledown^h u) \tag{3.11}$$

and the vertical Laplacian $\Delta^v u$ of u is defined by

$$\Delta^v u = div^v(\triangledown^v u). \tag{3.12}$$

Lemma 3.2. *Let (M,F) be a Finslerian manifold. For every function $u \in C^\infty(\mathring{T}M)$, we have*

$$\Delta^v u = g^{ij}\left[F\frac{\partial}{\partial y^i}\left(F\frac{\partial u}{\partial y^j}\right) - 2g^{mk}\mathcal{A}_{ijk}F\frac{\partial u}{\partial y^m}\right].$$

Then Δ^v is a second-order differential operator. Furthermore Δ^v is $\mathbb{R}$-linear.

Proof. The vertical Laplacian $\Delta^v u$ is locally obtained by

$$\Delta^v u = g^{ij} g\left(\nabla_{F\frac{\partial}{\partial y^i}} \triangledown^v u, \frac{\partial}{\partial x^j}\right). \tag{3.13}$$

By the almost g-compatibility of the Chern connection (Proposition 2.2) in the vertical directions, we have

$$\begin{aligned}
\Delta^v u &= g^{ij}\Big\{F\frac{\partial}{\partial y^i}\left[g\left(\triangledown^v u, \frac{\partial}{\partial x^j}\right)\right] - g\left(\triangledown^v u, \nabla_{F\frac{\partial}{\partial y^i}}\frac{\partial}{\partial x^j}\right) \\
&\quad -2\mathcal{A}\left(\theta\left(F\frac{\partial}{\partial y^i}\right), \triangledown^v u, \frac{\partial}{\partial x^j}\right)\Big\} \\
&= g^{ij}\Big\{F\frac{\partial}{\partial y^i}\left[g\left(g^{mk}F\frac{\partial u}{\partial y^m}\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right)\right] \\
&\quad -2\mathcal{A}\left(\theta\left(F\frac{\partial}{\partial y^i}\right), g^{mk}F\frac{\partial u}{\partial y^m}\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right)\Big\} \\
&= g^{ij}\left\{F\frac{\partial}{\partial y^i}\left[F\frac{\partial u}{\partial y^m}\delta^m_j\right] - 2g^{mk}F\frac{\partial u}{\partial y^m}\mathcal{A}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right)\right\} \\
&= g^{ij}\nabla_{F\frac{\partial}{\partial y^i}}\nabla_{F\frac{\partial}{\partial y^i}} u - g^{ij}g^{mk}\nabla_{F\frac{\partial}{\partial y^m}} g_{ij}\nabla_{F\frac{\partial}{\partial y^k}} u. \qquad (3.14)
\end{aligned}$$

One can show that for any $f, g \in C^\infty(\mathring{T}M)$ and for every $c \in \mathbb{R}$

$$\Delta^v(f+g) = \Delta^v f + \Delta^v g \quad \textit{and} \quad \Delta^v(cf) = c\Delta^v f.$$

That is, Δ^v is $\mathbb{R}$-linear. □

Remark 3.4. As explain in Remark 2.2, the operator Δ^v, defined above, acts on $C^\infty(\mathring{T}M)$ or on $C^\infty(SM)$. For homogeneous C^∞ functions of degree $r \neq 0$ on $\mathring{T}M$, we use the non-homogeneous coordinates $\{\frac{\partial}{\partial y^i}\}$ to define Δ^v. We get

$$\triangledown^v u = g^{ij}\frac{\partial u}{\partial y^i}\frac{\partial}{\partial x^j} \quad \textit{and} \quad div^v \xi = \frac{\partial \xi^i}{\partial y^i} \quad \forall u \in C^\infty(\mathring{T}M), \quad \forall \xi \in \Gamma(\pi^* TM).$$

Then, in the following we will use the homogeneous coordinates $\{F\frac{\partial}{\partial y^i}\}$ for objects on SM only.

We have the following.

Corollary 3.3. *Let (M, F) be a Finslerian manifold. For every function $u \in C^\infty(\mathring{T}M)$, we have*

$$\Delta^v u = g^{ij}\frac{\partial^2 u}{\partial y^i \partial y^j} - g^{ij} g^{mk}\frac{\partial g_{ij}}{\partial y^m}\frac{\partial u}{\partial y^k}. \tag{3.15}$$

Then Δ^v is a second-order differential operator and $\mathbb{R}$-linear.

Proof. The proof is similar to the proof of the Lemma 3.2. □

4 Main results

4.1 Vertical divergence lemma

Let (M, F) be an n-dimensional C^∞ Finslerian manifold. Let $\xi := \xi^i \frac{\partial}{\partial x^i}$ be an arbitrary C^∞ local section of the pulled-back tangent bundle π^*TM. ξ is a tensor field of rank $(1, 0)$ on the manifold $\mathring{T}M$. In order to find the divergence formula for a section of π^*TM, we need a volume form on $\mathring{T}M$. We consider the $2n$-form

$$\eta_F := \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \wedge^n d\omega, \tag{4.1}$$

with $\omega := \frac{\partial F}{\partial y^i} dx^i$ the Hilbert form. The η_F is a volume form on $\mathring{T}M$, called the Dazord volume form of (M, F) [5].

Remark 4.1. On the sphere bundle SM, we have the $(2n-1)$-form, defined by

$$\eta_F^o := \frac{(-1)^{\frac{n(n-1)}{2}}}{(n-1)!}\omega \wedge (d\omega)^{n-1}, \tag{4.2}$$

known also as the Dazord volume form of (M, F).

We recall the following:

Theorem 4.1. *[8] Let M be an oriented compact manifold with a fixed volume element dv. For every vector field X on M, we have*

$$\int_M divX dv = 0. \tag{4.3}$$

Remark 4.2. (1) The above formula is valid for a non-compact manifold M as long as X has a compact support [8].

(2) Every section ξ of the vector bundle π^*TM, with identification $\pi^*TM \cong \mathcal{V} := ker(\pi_*)$, can be considered as a vector field on $\mathring{T}M$.

We have the following vertical divergence lemma.

Lemma 4.2. *Let (M,F) be an oriented Finslerian manifold, $\mathring{T}M$ the slit tangent bundle of M and π^*TM the pulled-back tangent bundle by the projection $\pi : \mathring{T}M \longrightarrow M$ given by $\pi(x,y) = x$. Then, for every section ξ of π^*TM which has a compact support in $\mathring{T}M$, with identification $\pi^*TM \cong \mathcal{V} := ker(\pi_*)$, we have*

$$\int_{\mathring{T}M} (div^v \xi)\eta_F = 0. \tag{4.4}$$

Proof. The proof is obtained by using the Theorem 4.1 and the Remark 4.2. □

4.2 Hopf-type theorems for Finslerian manifolds

We have the following results.

Theorem 4.3. *Let (M,F) be an oriented Finslerian manifold and $\mathring{T}M$ its slit tangent bundle. Then every globally defined C^∞ function u on $\mathring{T}M$, whose v-gradient has compact support with $\Delta^v u \geq 0$ everywhere or $\Delta^v u \leq 0$ everywhere, must be independant of directional arguments. In particular, the slit tangent bundle of a Finslerian manifold has no vertically harmonic function except for functions independant of directional arguments.*

Proof. For $u, f \in C^\infty(\mathring{T}M)$ and $\xi \in \Gamma(\pi^*TM)$ one easily show that

$$div^v(f\xi) \quad = \quad f div^v \xi + g(\nabla^v f, \xi) \tag{4.5}$$

where g is the fundamental tensor. By the Remark 3.4, it follows that

$$div^v(f\nabla^v u) \quad = \quad f\Delta^v u + g(\nabla^v u, \nabla^v f) \tag{4.6}$$

Applying the Lemma 4.2 in (4.5), we get

$$\int_{\mathring{T}M} (f\Delta^v u)\eta_F = -\int_{\mathring{T}M} g(\nabla^v u, \nabla^v f)\eta_F. \tag{4.7}$$

In particular

$$\int_{\mathring{T}M} (u\Delta^v u)\eta_F = -\int_{\mathring{T}M} |\nabla^v u|_g^2 \eta_F \tag{4.8}$$

where $|\nabla^v u|_g$ denote the norm of $\nabla^v u$ induced by the fundamental tensor g of F, given by

$$|\nabla^v u|_g^2 = g^{ij}(\nabla_{\frac{\partial}{\partial y^i}} u)(\nabla_{\frac{\partial}{\partial y^j}} u) = g^{ij} u_{;i} u_{;j}. \tag{4.9}$$

By the Lemma 4.2 again we obtain, from the expression (3.12), that

$$\int_{\mathring{T}M} (\Delta^v u)\eta_F = 0. \tag{4.10}$$

Since Δ^v is $\mathbb{R}$-linear, by replacing u by $-u$ if necessary, we may assume without loss of generality that $\Delta^v u \geq 0$ everywhere. Since $u \in C^2(\mathring{T}M)$ and $\Delta^v u \geq 0$ by assumption, we conclude that $\Delta^v u = 0$ everywhere on $\mathring{T}M$, that is u is v-harmonic. It follows that

$$\int_{\mathring{T}M} (u\Delta^v u)\eta_F = 0. \tag{4.11}$$

The expression (4.8) becomes

$$-\int_{\mathring{T}M} g^{ij}(\nabla_{\frac{\partial}{\partial y^i}} u)(\nabla_{\frac{\partial}{\partial y^j}} u)\eta_F = 0. \tag{4.12}$$

Hence $u_{;i} = \frac{\partial u}{\partial y^i}$ vanishes identically on $\mathring{T}M$ and u must be independant of the direction y. □

Theorem 4.4. *Let (M, F) be a closed Finslerian manifold and SM the sphere bundle of M. Then every globally defined function $u \in C^\infty(SM)$, with $\Delta^v u \geq 0$ everywhere or $\Delta^v u \leq 0$ everywhere, must be independant of direction arguments. In particular, the sphere bundle of a Finslerian manifold has no v-harmonic function except for functions depending on the base point only.*

Proof. A straightforward computation of the vertical divergence of the section ξ of π^*TM over SM shows that $div^v\xi = F\frac{\partial \xi^i}{\partial y^i}$ and the proof is obtained by the Theorem 4.1 and the Lemma 4.2. □

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