A work done by an isotropic vector force field along an isotropic curve

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Abstract. We consider a 4-dimensional Riemannian manifold M with a metric g and an endomorphism Q, whose fourth power is the identity and Q acts as an isometry on g. An associated metric $\tilde{g}$ on (M, g, Q) is determined by both structures g and Q. The metric $\tilde{g}$ is necessary indefinite and it induces isotropic vectors in the tangent space at an arbitrary point on M. The physical forces are represented by vector fields. We study forces whose vectors are in a single tangent space on (M, g, Q). We calculate the corresponding physical work done by arbitrary forces along arbitrary curves with respect to $\tilde{g}$. Mainly, we suppose that the vector force fields are isotropic and they act along isotropic curves. We calculate the physical work done by such forces.

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1 Introduction

The theory of differentiable manifolds with additional structures have many applications in mathematics and physics. The physical forces on curves of differentiable manifolds are associated with vector fields on the manifolds and then one could find the physical work done by such forces. If a vector force field acts along a curve from one point to another point, the work done by the force is the product of the force and the displacement. For example, with vector fields are modeled forces such as the magnetic and gravitational forces. We consider vector force fields along curves of a Riemannian manifold equipped with an additional indefinite metric. In particular, the forces and curves are described by isotropic (null, light-like) vectors. The properties of null curves of semi-Riemannian manifolds and their important applications in general relativity are studied in detail ([8, 9]). There are some papers concerning physical results on light-like objects of differentiable manifolds (see [1, 2, 11, 12, 13, 15]).

In the present paper we consider a 4-dimensional Riemannian manifold M with a metric g and a tensor Q of type $(1, 1)$, whose fourth power is the identity. Moreover,

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Q acts as an isometry on g. Such a manifold (M, g, Q) is defined in [16] and it is studied also in [3, 4, 5, 7, 10]. We consider an associated metric $\tilde{g}$, introduced in [4], which is necessary indefinite. Therefore, $\tilde{g}$ determines isotropic (null) vectors in every tangent space T_pM at a point p on (M, g, Q). We investigate the physical work done by arbitrary vector force fields along arbitrary curves in T_pM and in subspaces of T_pM, with respect to $\tilde{g}$. Especially, we obtain the physical work done by isotropic vector force fields along isotropic curves.

2 Preliminaries

Let M be a 4-dimensional Riemannian manifold with a metric g and a tensor Q of type $(1,1)$. Let the local coordinates of Q with respect to some coordinate system form the following circulant matrix:

$$(Q_i^j) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.1}$$

Then Q has the property

$$Q^4 = \mathrm{id}. \tag{2.2}$$

We assume that g is a positive definite metric on M, which satisfies the equality

$$g(Ql, Qv) = g(l, v). \tag{2.3}$$

Remark 2.1. The notations l, v in (2.3) and furthermore r, e_1, e_2, e_3 will stand for arbitrary vector fields or arbitrary vectors in T_pM.

The manifold (M, g, Q), determined by (2.1) and (2.3), is introduced in [16].

Bearing in mind (2.3), for the angles $\varphi = \angle(v, Qv)$ and $\phi = \angle(v, Q^2v)$ we have

$$\cos\varphi = \frac{g(v, Qv)}{g(v, v)} \qquad \cos\phi = \frac{g(v, Q^2v)}{g(v, v)} \tag{2.4}$$

In [16], for (M, g, Q) and for an arbitrary nonzero vector v in T_pM, it is verified that if v induces a Q-basis of T_pM, that is a basis of the type $\{v, Qv, Q^2v, Q^3v\}$, then the angles ϕ and φ satisfy inequalities

$$0 < \phi < \pi, \quad \frac{\phi}{2} < \varphi < \pi - \frac{\phi}{2}. \tag{2.5}$$

The associated metric $\tilde{g}$ on (M, g, Q), determined by

$$\tilde{g}(l, v) = g(l, Q^2v) \tag{2.6}$$

is necessary indefinite [4].

A nonzero vector v in T_pM is isotropic with respect to $\tilde{g}$ if

$$\tilde{g}(v,v)=0. \tag{2.7}$$

An isotropic curve $c: r=r(t)$, where $t\in[\alpha,\beta]\subset\mathbb{R}$, is this one whose tangent vector field dr is isotropic, i.e.

$$\tilde{g}(dr,dr)=0. \tag{2.8}$$

If F is a vector force field, then the work A done by a force F, with respect to $\tilde{g}$, moving along a curve c is given by

$$A=\int_c \tilde{g}(F,dr), \tag{2.9}$$

where dr is the elementary displacement on the curve c.

Theorem 2.1. *Let F be an isotropic vector force field and let c be an isotropic curve. If F and the tangent vector field dr of c are linearly dependent, then the work A done by a force F, with respect to $\tilde{g}$, along the curve c is zero.*

Proof. We have that $F=kdr$, where k is a function. Then, using (2.8) and (2.9), we get $dA=\tilde{g}(kdr,dr)=k\tilde{g}(dr,dr)=0$, which implies $A=0$. □

Due to Theorem 2.1 and having in mind that every vector field and the tangent vector field to his trajectory are linearly dependent we state

Remark 2.2. If c is a trajectory of isotropic force F, then c is also isotropic and the corresponding work A is zero.

3 The work in T_pM

An orthonormal Q-basis $\{v,Qv,Q^2v,Q^3v\}$ exists in every tangent space T_pM on (M,g,Q) ([16]).

We assume that p_{xyzu} is a coordinate system such that the vectors v, Qv, Q^2v and Q^3v lie on the axes p_x, p_y, p_z and p_u, respectively. So p_{xyzu} is an orthonormal coordinate system.

A vector force field F is determined by

$$\begin{aligned}F(x,y,z,u)=&P(x,y,z,u)v+R(x,y,z,u)Qv\\&+S(x,y,z,u)Q^2v+L(x,y,z,u)Q^3v,\end{aligned} \tag{3.1}$$

where $P=P(x,y,z,u)$, $R=R(x,y,z,u)$, $S=S(x,y,z,u)$, $L=L(x,y,z,u)$ are smooth functions.

A smooth curve c is determined by

$$c: r(t)=x(t)v+y(t)Qv+z(t)Q^2v+u(t)Q^3v, \tag{3.2}$$

where $t\in[\alpha,\beta]\subset\mathbb{R}$.

Theorem 3.1. *Let F be an arbitrary vector force field and let c be an arbitrary smooth curve in T_pM, determined by* (3.1) *and* (3.2)*. Then the work A done by a force F, with respect to $\tilde{g}$, along c is*

$$A = \int_\alpha^\beta \big(Pz'(t) + Ru'(t) + Sx'(t) + Ly'(t)\big)dt. \tag{3.3}$$

Proof. From (2.6), (2.9), (3.1) and (3.2) it follows (3.3). □

We find conditions for the components of the vectors in T_pM, so that they are isotropic with respect to $\tilde{g}$.

Lemma 3.2. *Let $\{v, Qv, Q^2v, Q^3v\}$ be an orthonormal Q-basis of T_pM. If $r = xv + yQv + zQ^2v + uQ^3v$ is an isotropic vector, then its coordinates satisfy*

$$xz + yu = 0. \tag{3.4}$$

Proof. Taking into account (2.2), (2.6) and (2.7) we get (3.4). □

Furthermore, due to Lemma 3.2, we obtain

Proposition 3.3. *If the vector force field* (3.1) *is isotropic, then its components satisfy*

$$PS + RL = 0. \tag{3.5}$$

Proposition 3.4. *If the curve* (3.2) *is isotropic, then the components of its tangent vector satisfy*

$$dxdz + dydu = 0. \tag{3.6}$$

Now, we suppose that F and c are both isotropic. Hence, having in mind (3.5) and (3.6), we consider the following cases.

Case (i) We assume that $dy = 0$ and $dx = 0$. Therefore, using (3.3), we obtain

$$A = \int_\alpha^\beta \big(P(c_1, c_2, z, u)z' + R(c_1, c_2, z, u)u'\big)dt, \tag{3.7}$$

where c_1 and c_2 are specific constants.

Case (ii) If $dy = 0$ and $dz = 0$, then (3.3) implies

$$A = \int_\alpha^\beta \big(S(x, c_3, c_4, u)x' + R(x, c_3, c_4, u)u'\big)dt, \tag{3.8}$$

where c_3 and c_4 are specific constants.

Case (iii) We suppose that $R = 0$ and $P = 0$. In this case equality (3.3) yields

$$A = \int_\alpha^\beta \big(Sx' + Ly'\big)dt. \tag{3.9}$$

Case (iv) If $R = 0$ and $S = 0$, then (3.3) implies

$$A = \int_\alpha^\beta \big(Ly' + Pz'\big)dt. \tag{3.10}$$

Case (v) Let $R \neq 0$, $dy \neq 0$ be valid. From (3.5) and (3.6) we find respectively

$$L = -\frac{PS}{R}, \quad du = -\frac{dxdz}{dy}. \tag{3.11}$$

By virtue of (3.3) and (3.11) we have

$$A = \int_{\alpha}^{\beta} \frac{\left(RSx'y' - PSy'^2 + PRy'z' - R^2x'z'\right)}{Ry'^2} dt. \tag{3.12}$$

Cases (i) – (v) we summarize in the next statement.

Theorem 3.5. *Let F be an isotropic vector force field and let c be an isotropic smooth curve. Therefore the work A, determined by* (3.3)*, takes one of the following forms:*

(i) A *is* (3.7)*, if* $dx = dy = 0$*;*

(ii) A *is* (3.8)*, if* $dy = dz = 0$*;*

(iii) A *is* (3.9)*, if* $P = R = 0$*;*

(iv) A *is* (3.10)*, if* $R = S = 0$*;*

(v) A *is* (3.12)*, if* $dy \neq 0$ *and* $R \neq 0$*.*

4 The work in a 3-dimensional subspace of T_pM

Let the unit vector v induce a Q-*basis* of T_pM. Then v induces four different Q-bases of three vectors, which are $\{v, Qv, Q^2v\}$, $\{Qv, Q^2v, Q^3v\}$, $\{v, Q^2v, Q^3v\}$ and $\{v, Qv, Q^3v\}$. According to (2.3) and (2.4) all pyramids constructed on these bases are equal. Thus we will consider only one of them, the 3-dimensional subspace ε of T_pM, spanned by vectors v, Qv and Q^2v.

Lemma 4.1. *Let ε be a subspace of T_pM with a normalized basis $\{v, Qv, Q^2v\}$. The system of vectors $\{e_1, e_2, e_3\}$, determined by the equalities*

$$\begin{aligned} &e_1 = \frac{v + Q^2v}{2\cos\frac{\phi}{2}}, \qquad e_3 = \frac{v - Q^2v}{2\sin\frac{\phi}{2}}, \\ &e_2 = \frac{(-\cos\varphi)v + (1+\cos\phi)Qv - (\cos\varphi)Q^2v}{\sqrt{(1+\cos\phi)(1+\cos\phi - 2\cos^2\varphi)}}, \end{aligned} \tag{4.1}$$

is an orthonormal basis of ε.

Proof. Taking into account (2.1), (2.3), (2.4) and (4.1) we obtain

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \ g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0.$$

□

Lemma 4.2. *Let $\tilde{g}$ be the associated metric with g on (M, g, Q). Let ε be a subspace of T_pM with a basis $\{v, Qv, Q^2v\}$. Then the system of vectors $\{e_1, e_2, e_3\}$, determined by the equalities* (4.1), *satisfy*

$$\tag{4.2} \begin{gathered} \tilde{g}(e_1, e_1) = -\tilde{g}(e_3, e_3) = 1, \quad \tilde{g}(e_2, e_2) = b, \\ \tilde{g}(e_1, e_2) = \tilde{g}(e_2, e_3) = \tilde{g}(e_1, e_3) = 0, \end{gathered}$$

where

$$\tag{4.3} b = \frac{\cos\phi + \cos^2\phi - 2\cos^2\varphi}{1 + \cos\phi - 2\cos^2\varphi}.$$

Proof. Taking into account (2.1), (2.3), (2.4), (2.6) and (4.1) we find (4.2) and (4.3). □

We consider a coordinate system p_{xyz} such that the vectors e_1, e_2 and e_3 lie on the axes p_x, p_y and p_z, respectively. Hereof p_{xyz} is an orthonormal coordinate system in ε. In the subspace ε of T_pM, a vector force field F and a smooth curve c are determined as follows:

$$\tag{4.4} F(x, y, z) = P(x, y, z)e_1 + R(x, y, z)e_2 + S(x, y, z)e_3,$$

$$\tag{4.5} c : r(t) = x(t)e_1 + y(t)e_2 + z(t)e_3, \quad t \in [\alpha, \beta] \subset \mathbb{R}.$$

Theorem 4.3. *Let F be an arbitrary vector force field and let c be an arbitrary smooth curve in ε, determined by* (4.4) *and* (4.5). *Then the work A done by a force F, with respect to $\tilde{g}$, along the curve c is*

$$\tag{4.6} A = \int_{\alpha}^{\beta} \big(Px'(t) + bRy'(t) - Sz'(t)\big)dt.$$

Proof. From (2.6) and (4.2) it follows

$$\begin{aligned} \tilde{g}(F, dr) =& \tilde{g}(Pe_1 + Re_2 + Se_3, dxe_1 + dye_2 + dze_3) \\ =& P\tilde{g}(e_1, e_1)dx + R\tilde{g}(e_2, e_2)dy + S\tilde{g}(e_3, e_3)dz \\ =& Pdx + bRdy - Sdz \end{aligned}$$

Then, using (2.9), we get (4.6). □

In the following statements we find conditions for the components of the vectors in ε, such that they are isotropic.

Lemma 4.4. *Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of ε, determined by* (4.1). *If $r = xe_1 + ye_2 + ze_3$ is an isotropic vector, then its coordinates satisfy*

$$\tag{4.7} by^2 = z^2 - x^2.$$

Proof. From (2.6), (2.7) and (4.2) the proof follows. □

Remark 4.1. Bearing in mind (4.7) we will consider only the isotropic vectors determined by conditions: $b > 0$ and $|z| > |x|$, or $b = 0$ and $|z| = |x|$, or $b < 0$ and $|z| < |x|$.

Furthermore, due to Lemma 4.4 and Remark 4.1, we immediately have

Proposition 4.5. *Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of ε, determined by* (4.1)*. The vector force field* (4.4) *is isotropic if its components satisfy*

$$bR^2 = S^2 - P^2. \tag{4.8}$$

In particular, if $b > 0$ then $|P| < |S|$, if $b = 0$ then $|P| = |S|$, if $b < 0$ then $|P| > |S|$.

Proof. The equalities (4.4) and (4.7) imply (4.8). □

Proposition 4.6. *Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of ε, determined by* (4.1)*. The curve* (4.5) *is isotropic if*

$$b(dy)^2 = (dz)^2 - (dx)^2. \tag{4.9}$$

In particular, if $b > 0$ then $|dz| > |dx|$, if $b = 0$ then $|dx| = |dz|$, if $b < 0$ then $|dz| < |dx|$.

Evidently, taking into account (4.8) and (4.9) we find

$$b^2R^2(dy)^2 = (S^2 - P^2)(dz^2 - dx^2). \tag{4.10}$$

Let us denote

$$\delta = \arccos\sqrt{\frac{1}{2}(\cos\phi + \cos^2\phi)}, \tag{4.11}$$

where $0 < \phi < \frac{\pi}{2}$.

Therefore, having in mind (2.5), (4.3) and (4.11), we state the following

Theorem 4.7. *Let F be an isotropic vector force field and let c be an isotropic smooth curve in ε. If $\varphi = \delta$ or $\varphi = \pi - \delta$, then the work A, determined by* (4.6)*, takes one of the following forms:*

(i) $A = 2\int_\alpha^\beta Px'(t)dt$, *if* $Sdz = -Pdx$;

(ii) $A = 0$, *if* $Sdz = Pdx$.

Proof. From (4.3) we get $b = 0$. Consequently (4.8) and (4.9) imply $|P| = |S|$ and $|dz| = |dx|$. Therefore, using (4.6) we find conditions (i) and (ii). □

In a similar way, with the help of (2.5), (4.10) and (4.11), we obtain the next statements.

Theorem 4.8. *Let F be an isotropic vector force field and let c be an isotropic smooth curve in ε. If $\varphi \in (\frac{\phi}{2}, \delta) \cup (\pi - \delta, \pi - \frac{\phi}{2})$, $|S| < |P|$, $R > 0$ and $|dz| < |dx|$, then the work A, determined by* (4.6)*, takes the form*

$$A = \int_\alpha^\beta \left(Px'(t) - \sqrt{(S^2 - P^2)(z'^2 - x'^2)} - Sz'\right)dt.$$

Theorem 4.9. *Let F be an isotropic vector force field and let c be an isotropic smooth curve in ε. If $\varphi \in (\delta, \pi - \delta)$, $|S| > |P|$, $R > 0$ and $|dz| > |dx|$, then the work A, determined by* (4.6)*, takes the form*

$$A = \int_{\alpha}^{\beta} \big(Px'(t) + \sqrt{(S^2 - P^2)(z'^2 - x'^2)} - Sz'\big)dt.$$

5 Work in a 2-plane spanned on $\{v, Q^2v\}$

Now, we consider an arbitrary 2-plane $\epsilon = \{v, Q^2v\}$ in T_pM. Because of (2.1) we have that the structure Q^2 is almost product. Every 2-plane of the type $\{v, Q^2v\}$ in an arbitrary tangent space of an almost product manifold admits an orthonormal basis of the type $\{l, Q^2l\}$ (see [14] and [17]).

We suppose that $\{l, Q^2l\}$ is an orthonormal basis of ϵ. On ϵ we construct a coordinate system p_{xy} such that l is on the axis p_x and Q^2l is on the axis p_y. Hence p_{xy} is an orthonormal coordinate system. An arbitrary vector force field F and an arbitrary smooth curve c in ϵ are given as follows:

$$F(x,y,z) = P(x,y)l + R(x,y)Q^2l, \tag{5.1}$$

$$c : r(t) = x(t)l + y(t)Q^2l, \quad t \in [\alpha, \beta] \subset \mathbb{R}. \tag{5.2}$$

Theorem 5.1. *Let F be an arbitrary vector force field and let c be an arbitrary smooth curve in ϵ, determined by* (5.1) *and* (5.2)*. Then the work A done by a force F, with respect to $\tilde{g}$, along the curve c is*

$$A = \int_{\alpha}^{\beta} \big(P(x,y)y'(t) + R(x,y)x'(t)\big)dt, \tag{5.3}$$

Proof. Equalities (2.3), (2.6), (2.9), (5.1) and (5.2) imply (5.3). □

In the following statements we find conditions for the components of the vectors in ϵ, such that they are isotropic.

Lemma 5.2. *Let $\{l, Q^2l\}$ be an orthonormal basis of the 2-plane ϵ. If $r = xl + yQ^2l$ is an isotropic vector, then its coordinates satisfy*

$$xy = 0.$$

Proof. With the help of (2.3), (2.6) and (2.7) we get the proof. □

Immediately, by virtue of Lemma 5.2, we obtain the following propositions.

Proposition 5.3. *The vector force field* (5.1) *is isotropic if its components satisfy*

$$PR = 0. \tag{5.4}$$

Proposition 5.4. *The curve* (5.2) *is isotropic if*

$$dxdy = 0. \tag{5.5}$$

Due to Theorem 5.1 and the above propositions we obtain the next statement.

Theorem 5.5. *Let F be an isotropic vector force field and let c be an isotropic smooth curve in the 2-plane ϵ. Hence the work A, determined by* (5.3)*, takes one of the following forms:*

(i) $A = 0$, *if* $P = 0$ *and* $dx = 0$,

(ii) $A = 0$, *if* $R = 0$ *and* $dy = 0$,

(iii) $A = \int_\alpha^\beta \big(P(c_1, y)y'(t)\big)dt$, *if* $R = 0$ *and* $dx = 0$,

(iv) $A = \int_\alpha^\beta \big(R(x, c_2)x'(t)\big)dt$, *if* $P = 0$ *and* $dy = 0$,

where c_1, c_2 are specific constants.

Proof. The proof follows by virtue of (5.3), (5.4) and (5.5). □

6 Work in a 2-plane spanned on $\{v, Qv\}$

We note that the investigations in this section are based mainly on the results in [6].

We suppose that the unit vector v induces a *Q-basis* of T_pM. We consider the 2-plane $\eta = \{v, Qv\}$ in T_pM.

Lemma 6.1. *Let η be a 2-plane of T_pM with a normalized basis $\{v, Qv\}$. The system of vectors $\{e_1, e_2\}$, determined by the equalities*

$$e_1 = \frac{1}{2\cos\frac{\varphi}{2}}(v + Qv), \quad e_2 = -\frac{1}{2\sin\frac{\varphi}{2}}(v - Qv) \tag{6.1}$$

is an orthonormal basis of η.

Proof. From (2.3), (2.4) and (6.1) we have immediately $g(e_1, e_1) = g(e_2, e_2) = 1$ and $g(e_1, e_2) = 0$. □

On η we construct a coordinate system p_{xy} with basis vectors e_1 on the axis p_x and e_2 on the axis p_y, determined by (6.1). Hence p_{xy} is an orthonormal coordinate system.

With the help of (2.6) and (6.1) we state the following

Lemma 6.2. *Let $\tilde{g}$ be the associated metric with g on (M, g, Q). Let η be a 2-plane of T_pM with a basis $\{e_1, e_2\}$, determined by* (6.1)*. Then the vectors e_1 and e_2 satisfy*

$$\tilde{g}(e_1, e_1) = k_1, \quad \tilde{g}(e_2, e_2) = k_2, \quad \tilde{g}(e_1, e_2) = 0, \tag{6.2}$$

where

$$k_1 = \frac{\cos\phi + \cos\varphi}{1 + \cos\varphi}, \quad k_2 = \frac{\cos\phi - \cos\varphi}{1 - \cos\varphi}. \tag{6.3}$$

In the 2-plane η, a vector force field F and a smooth curve c are determined as follows:

$$F(x,y) = P(x,y)e_1 + R(x,y)e_2, \tag{6.4}$$

$$c : r(t) = x(t)e_1 + y(t)e_2, \quad t \in [\alpha, \beta] \subset \mathbb{R}. \tag{6.5}$$

Theorem 6.3. *Let F be an arbitrary vector force field and let c be an arbitrary smooth curve in η, determined by* (6.4) *and* (6.5). *Then the work A done by a force F, with respect to $\tilde{g}$, along the curve c is*

$$A = \int_\alpha^\beta \big(k_1 P(x,y)x'(t) + k_2 R(x,y)y'(t)\big)dt. \tag{6.6}$$

Proof. From (2.6) and (6.2) it successively follows

$$\begin{aligned}\tilde{g}(F, dr) =& \tilde{g}(Pe_1 + Re_2, dxe_1 + dye_2)\\ =& k_1 P\tilde{g}(e_1, e_1)dx + k_2 R\tilde{g}(e_2, e_2)dy + (Pdy + Rdx)\tilde{g}(e_1, e_2)\\ =& k_1 Pdx + k_2 Rdy,\end{aligned}$$

and using (2.9) we obtain (6.6). □

Farther, we find conditions for the components of the vectors in η, so that they are isotropic with respect to $\tilde{g}$.

Lemma 6.4. *Let $\{e_1, e_2\}$ be an orthonormal basis of the 2-plane η, determined by* (6.1). *If $r = xe_1 + ye_2$ is an isotropic vector, then its coordinates satisfy*

$$k_1 x^2 + k_2 y^2 = 0, \quad k_1 k_2 \leq 0.$$

Proof. By equalities (2.5), (2.6), (2.7), (6.2) and (6.3) we get the proof. □

Immediately, by virtue of Lemma 6.4, we obtain the following propositions.

Proposition 6.5. *The vector force field* (6.4) *is isotropic if its components satisfy*

$$k_1 P^2 + k_2 R^2 = 0, \quad k_1 k_2 \leq 0. \tag{6.7}$$

Proposition 6.6. *The curve* (6.5) *is isotropic if*

$$k_1 (dx)^2 + k_2 (dy)^2 = 0, \quad k_1 k_2 \leq 0. \tag{6.8}$$

Let F be an isotropic vector force field and let c be an isotropic smooth curve in η. According to the values of the coefficients k_1 and k_2 we have two cases.

First we consider the case when $k_1 k_2 = 0$. If $\varphi = \phi \neq \frac{\pi}{2}$, then (6.3) implies $k_1 \neq 0$, $k_2 = 0$. Consequently, from (6.7) and (6.8) we have $P = 0$, $dx = 0$, which implies $A = 0$. Similarly, if we suppose that $\varphi = \pi - \phi$, then $A = 0$.

Therefore we obtain the next statement.

Theorem 6.7. *Let F be an isotropic vector force field and let c be an isotropic smooth curve in the 2-plane ϵ. If $\varphi = \phi$ or $\varphi = \pi - \phi$, then the work determined by* (6.6) *is* $A = 0$.

Now, we consider the case when $k_1k_2 < 0$. Bearing in mind inequalities (2.5), we have that both angles ϕ and φ satisfy one of the following conditions:

$$\phi \in (0, \frac{\pi}{2}), \quad \varphi \in (\frac{\phi}{2}, \phi) \cup (\pi - \phi, \pi - \frac{\phi}{2}); \tag{6.9}$$

$$\phi \in (\frac{\pi}{2}, \frac{2\pi}{3}), \quad \varphi \in (\frac{\phi}{2}, \pi - \phi) \cup (\phi, \pi - \frac{\phi}{2}); \tag{6.10}$$

$$\phi = \frac{\pi}{2}, \quad \varphi \neq \frac{\phi}{2}. \tag{6.11}$$

In this case, the curve $c: \ k_1x^2 + k_2y^2 = 0$ degenerates into two lines with equations:

$$y = \pm\sqrt{\frac{-k_1}{k_2}}x, \quad k_1k_2 < 0. \tag{6.12}$$

The above considerations imply the next statement.

Theorem 6.8. *Let F be an isotropic vector force field and let c be an isotropic smooth curve in the 2-plane ϵ. If one of the conditions* (6.9) – (6.11) *is valid, then the work determined by* (6.6) *takes one of the following forms:*

(i) $A = 0$, *if* $R = \pm\sqrt{\frac{-k_1}{k_2}}P$, $dy = \mp\sqrt{\frac{-k_1}{k_2}}dx$;

(ii) $A = 2\int_\alpha^\beta P\big(t, \pm\sqrt{\frac{-k_1}{k_2}}t\big)dt$, *if* $R = \pm\sqrt{\frac{-k_1}{k_2}}P$ *and* $dy = \pm\sqrt{\frac{-k_1}{k_2}}dx$.

Proof. Substituting (6.12) into (6.6) we get conditions (i) and (ii). □

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