The Einstein-Hilbert type action on almost multi-product manifolds

V. Rovenski

Abstract. A Riemannian manifold endowed with $k > 2$ complementary orthogonal distributions (called a Riemannian almost multi-product structure) appears in such topics as multiply twisted or warped products, the webs or nets composed of several foliations, Ricci curvature and Einstein equations, multi-time geometric dynamics and Dupin hypersurfaces. In the paper we consider the mixed scalar curvature of such structure, derive Euler-Lagrange equations for the Einstein-Hilbert type action with respect to adapted variations of metric, and present them in a nice form of Einstein equation.

M.S.C. 2010: 53C20, 53C21, 53C24.
Key words: Almost multi-product manifold; mixed scalar curvature; integral formula; Einstein-Hilbert action.

1 Introduction

Many examples of Riemannian metrics come (as critical points) from variational problems, a particularly famous of which is the *Einstein-Hilbert action*, e.g., [5]. The Euler-Lagrange equation for this action (called the *Einstein equation*) is

$$\mathrm{Ric} - (1/2)\,\mathrm{S}\cdot g + \Lambda\, g = \mathfrak{a}\cdot\Xi \tag{1.1}$$

where g is a pseudo-Riemannian metric on a smooth manifold M, Ric – the Ricci curvature, S – the scalar curvature, Λ – a constant (the "cosmological constant"), $\mathcal{L}$ – Lagrangian describing the matter contents, $\mathfrak{a}$ – the coupling constant involving the gravitational constant and the speed of light and Ξ – the energy-momentum tensor. The solution of (1.1) is a metric, satisfying this equation, where the tensor Ξ is given. The classification of solutions of (1.1) is a deep and largely unsolved problem [5].

Distributions on a manifold (i.e., subbundles of the tangent bundle) appear in various situations, e.g., [4, 8] and are used to build up notions of integrability, and specifically of a foliated manifold. On a manifold equipped with an additional structure, e.g., almost product or contact, one can consider an analogue of the Einstein-Hilbert

Balkan Journal of Geometry and Its Applications, Vol.26, No.1, 2021, pp. 81-92.

action adjusted to that structure. This approach was taken in [2, 3, 11, 14, 15], for M endowed with a distribution $\mathcal{D}$ or a foliation.

In this article, continuing our study [2, 3, 11, 14, 14], a similar change in the classical action is considered on an almost multi-product structure $(M,g;\mathcal{D}_1,\ldots,\mathcal{D}_k)$, see [13], i.e., a connected smooth n-dimensional manifold endowed with $k>2$ pairwise orthogonal n_i-dimensional distributions with $\sum n_i=n$. The notion of a multiply warped product, e.g., [6], is a special case of this structure, which can be also viewed in the theory of webs and nets composed of different foliations, see [1], in studies of the curvature and Einstein equations, see [7], multi-time geometric dynamics and Dupin hypersurfaces. The *mixed Einstein-Hilbert action* on $(M,\mathcal{D}_1,\ldots,\mathcal{D}_k)$, defined by

$$J_{\mathcal{D}}: g\mapsto \int_M \Big\{\frac{1}{2\mathfrak{a}}\,(\mathrm{S}_{\,\mathcal{D}_1,\ldots,\mathcal{D}_k}-2\,\Lambda)+\mathcal{L}\Big\}\,\mathrm{d}\,\mathrm{vol}_g, \tag{1.2}$$

is an analog of the Einstein-Hilbert action, where S is replaced by the mixed scalar curvature $\mathrm{S}_{\,\mathcal{D}_1,\ldots,\mathcal{D}_k}$, see (2.1). To deal also with non-compact manifolds ("spacetimes"), it is assumed that the integral above is taken over M if it converges; otherwise, one integrates over arbitrarily large, relatively compact domain $\Omega\subset M$, which also contains supports of variations of g. The geometrical part of (1.2) is the *total mixed scalar curvature* of $(M,g;\mathcal{D}_1,\ldots,\mathcal{D}_k)$

$$J^g_{\mathcal{D}}: g\mapsto \int_M \mathrm{S}_{\,\mathcal{D}_1,\ldots,\mathcal{D}_k}\,\mathrm{d}\,\mathrm{vol}_g\,. \tag{1.3}$$

The mixed scalar curvature is the simplest curvature invariant of a pseudo-Riemannian almost product structure, which can be defined as an averaged sum of sectional curvatures of planes that non-trivially intersect with both of the distributions. Investigation of $\mathrm{S}_{\,\mathcal{D}_1,\mathcal{D}_1^\perp}$ led to multiple results regarding the existence of foliations and submersions with interesting geometry, e.g., integral formulas and splitting results, curvature prescribing and variational problems, see [12, 16, 18]. Varying (1.2) as a functional of adapted metric g, we obtain the Euler-Lagrange equations in the beautiful form of Einstein equation (1.1), i.e.,

$$\mathcal{R}ic_{\,\mathcal{D}}-\,(1/2)\,\mathcal{S}_{\mathcal{D}}\cdot g+\,\Lambda\,g=\mathfrak{a}\cdot\Xi, \tag{1.4}$$

where the Ricci tensor and the scalar curvature are replaced by the Ricci type tensor $\mathcal{R}ic_{\,\mathcal{D}}$, see (3.16), and its trace $\mathcal{S}_{\mathcal{D}}$, and Ξ is given by $\Xi_{\mu\nu}=-2\,\partial\mathcal{L}/\partial g^{\mu\nu}+g_{\mu\nu}\mathcal{L}$.

Using the equality

$$\mathrm{S}=2\,\mathrm{S}_{\,\mathcal{D}_1,\ldots,\mathcal{D}_k}+\sum\nolimits_i \mathrm{S}(\mathcal{D}_i),$$

where $\mathrm{S}(\mathcal{D}_i)$ is the scalar curvature of the distribution $\mathcal{D}_i$, one can combine the Einstein-Hilbert action on $(M,\mathcal{D}_1,\ldots,\mathcal{D}_k)$ (e.g., [9] for multiply warped products) with our action (1.2). The result is the perturbed Einstein-Hilbert action, whose critical points describe the "space-times" in an extended theory of gravity. The geometrical part of this action is $J_{\mathcal{D},\varepsilon}: g\mapsto\int_M(\mathrm{S}+\varepsilon\,\mathrm{S}_{\,\mathcal{D}_1,\ldots,\mathcal{D}_k})\,\mathrm{d}\,\mathrm{vol}_g,\ \varepsilon\in\mathbb{R}$.

Our action (1.2) can also be useful in studying the interaction of several m-flows (m-dimensional distributions) in multi-time geometric dynamics, e.g., [10, 17].

We delegate the following questions for further study:

a) generalize our results for arbitrary variations of metrics;

b) extend our results for metric-affine manifolds (as in Einstein-Cartan theory);

c) find applications of our results in geometry, dynamics and physics.

2 The mixed scalar curvature

Here, we recall the properties of the mixed scalar curvature of a Riemannian multi-product manifold $(M, g; \mathcal{D}_1, \dots, \mathcal{D}_k)$, see [13]. A pseudo-Riemannian metric $g = \langle\cdot,\cdot\rangle$ of index q on a smooth manifold M is an element $g \in \mathrm{Sym}^2(M)$ (of the space of symmetric $(0,2)$-tensors) such that each g_x $(x \in M)$ is a non-degenerate bilinear form of index q on the tangent space T_xM. For $q = 0$ (i.e., g_x is positive definite) g is a Riemannian metric and for $q = 1$ it is called a Lorentz metric. A distribution $\mathcal{D}$ on (M, g) is *non-degenerate*, if g_x is non-degenerate on $\mathcal{D}_x \subset T_xM$ for all $x \in M$; in this case, the orthogonal complement of $\mathcal{D}^\perp$ is also non-degenerate. Denote by $\mathrm{Riem}(M; \mathcal{D}_1, \dots \mathcal{D}_k)$ the subspace of *adapted pseudo-Riemannian metrics*, that is making $\{\mathcal{D}_i\}$ pairwise orthogonal and non-degenerate. Let $P_i : TM \to \mathcal{D}_i$ be the orthoprojector, then $P_i^\perp = \mathrm{id}_{\,TM} - P_i$ is the orthoprojector onto $\mathcal{D}_i^\perp$. The second fundamental form $h_i : \mathcal{D}_i \times \mathcal{D}_i \to \mathcal{D}_i^\perp$ and the skew-symmetric integrability tensor $T_i : \mathcal{D}_i \times \mathcal{D}_i \to \mathcal{D}_i^\perp$ of $\mathcal{D}_i$ are defined by

$$h_i(X, Y) = \frac{1}{2}\, P_i^\perp(\nabla_X Y + \nabla_Y X),$$
$$T_i(X, Y) = \frac{1}{2}\, P_i^\perp(\nabla_X Y - \nabla_Y X) = \frac{1}{2}\, P_i^\perp\, [X, Y].$$

Similarly, $h_i^\perp$, $H_i^\perp = \mathrm{Tr}_g\, h_i^\perp$, $T_i^\perp$ are the second fundamental forms, mean curvature vector fields and the integrability tensors of distributions $\mathcal{D}_i^\perp$ in M. Note that $H_i = \sum_{j\ne i} P_j H_i$, etc. Recall that a distribution $\mathcal{D}_i$ is called integrable if $T_i = 0$, and $\mathcal{D}_i$ is called totally umbilical, harmonic, or totally geodesic, if $h_i = (H_i/n_i)\, g$, $H_i = 0$, or $h_i = 0$, respectively.

Given $g \in \mathrm{Riem}(M; \mathcal{D}_1, \dots \mathcal{D}_k)$, there is a local orthonormal frame $\{E_1, \dots, E_n\}$ on M, where $\{E_1, \dots, E_{n_1}\} \subset \mathcal{D}_1$ and $\{E_{n_{i-1}+1}, \dots, E_{n_i}\} \subset \mathcal{D}_i$ for $2 \le i \le k$, and $\varepsilon_a = \langle E_a, E_a\rangle \in \{-1, 1\}$. All quantities defined below using such frame do not depend on the choice of this frame.

A plane $X \wedge Y$ in TM spanned by two vectors belonging to different distributions $\mathcal{D}_i$ and $\mathcal{D}_j$ will be called *mixed*, and the sectional curvature $K(X, Y) = R(X, Y\, X,\, Y)\rangle/(\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2)$ is said to be mixed. The mixed scalar curvature of $(M, g; \mathcal{D}_1, \dots, \mathcal{D}_k)$ is defined as an averaged mixed sectional curvature.

Definition 2.1 (see [13]). Given $g \in \mathrm{Riem}(M; \mathcal{D}_1, \dots, \mathcal{D}_k)$ with $k \ge 2$, the following function on M will be called the *mixed scalar curvature*:

$$\mathrm{S}_{\,\mathcal{D}_1,\dots,\mathcal{D}_k} = \sum_{i<j} \sum_{n_{i-1}<a\le n_i,\ n_{j-1}<b\le n_j} K(E_a, E_b), \tag{2.1}$$

where $K(E_a, E_b) = \varepsilon_a\, \varepsilon_b\, \langle R(E_a, E_b)\, E_a,\, E_b\rangle$ is the mixed sectional curvature of the plane $E_a \wedge E_b$. The following symmetric $(0,2)$-tensor r is called the *partial Ricci tensor*:

$$r(X, Y) = \frac{1}{2} \sum\nolimits_i r_{\mathcal{D}_i}(X, Y), \quad X, Y \in \mathfrak{X}_M,$$

where the partial Ricci tensor related to $\mathcal{D}_i$ is

$$r_{\mathcal{D}_i}(X, Y) = \sum\nolimits_{n_{i-1}<a\le n_i} \varepsilon_a\, \langle R_{\,E_a,\, P_i^\perp X}\, E_a,\, P_i^\perp Y\rangle, \quad X, Y \in \mathfrak{X}_M. \tag{2.2}$$

Proposition 2.1. *We have*

$$\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} = \frac{1}{2}\sum\nolimits_i \mathrm{S}_{\mathcal{D}_i,\mathcal{D}_i^\perp} = \mathrm{Tr}_g\, r.$$

Proof. This follows from definitions (2.1)–(2.2) and the equality $\mathrm{Tr}_g\, r_{\mathcal{D}_i} = \mathrm{S}_{\mathcal{D}_i,\mathcal{D}_i^\perp}$. □

Recall that the divergence of a $(1,s)$-tensor field S on (M,g) is a $(0,s)$-tensor field

$$\mathrm{div}\, S = \mathrm{trace}(Y \to \nabla_Y S).$$

For $s=0$, we get the divergence $\mathrm{div}\, X = \mathrm{Tr}(\nabla X)$ of a vector field X, e.g., [5].

The squares of norms of tensors are obtained using

$$\langle h_i, h_i\rangle = \sum\nolimits_{n_{i-1}<a,b\le n_i} \varepsilon_a\varepsilon_b\, \langle h_i(E_a,E_b), h_i(E_a,E_b)\rangle,$$
$$\langle T_i, T_i\rangle = \sum\nolimits_{n_{i-1}<a,b\le n_i} \varepsilon_a\varepsilon_b\, \langle T_i(E_a,E_b), T_i(E_a,E_b)\rangle.$$

The following formula for a Riemannian manifold (M,g) endowed with two complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$, see [18]:

$$\begin{aligned} &\mathrm{div}(H+H^\perp) = \mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} \\ &+ \langle h,h\rangle + \langle h^\perp,h^\perp\rangle - \langle H,H\rangle - \langle H^\perp,H^\perp\rangle - \langle T,T\rangle - \langle T^\perp,T^\perp\rangle, \end{aligned} \tag{2.3}$$

has many interesting global corollaries (e.g., decomposition criteria using the sign of S, [16]). In [13], we generalized (2.3) to (M,g) with $k>2$ distributions and gave applications to splitting and isometric immersions of manifolds, in particular, multiply warped products. Set

$$Q(\mathcal{D},g) = \langle H^\perp,H^\perp\rangle + \langle H,H\rangle - \langle h,h\rangle - \langle h^\perp,h^\perp\rangle + \langle T,T\rangle + \langle T^\perp,T^\perp\rangle, \tag{2.4}$$

then (2.3) can be written as

$$\mathrm{S}_{\mathcal{D},\mathcal{D}^\perp} = Q(\mathcal{D},g) + \mathrm{div}(H+H^\perp). \tag{2.5}$$

The mixed scalar curvature of a pair of distributions $(\mathcal{D}_i,\mathcal{D}_i^\perp)$ on (M,g) is

$$\mathrm{S}_{\mathcal{D}_i,\mathcal{D}_i^\perp} = \sum\nolimits_{n_{i-1}<a\le n_i,\ b\ne(n_{i-1},n_i]} \varepsilon_a\varepsilon_b\, \langle R_{E_a,E_b}E_a, E_b\rangle.$$

If $\mathcal{D}_i$ is spanned by a unit vector field N, i.e., $\langle N,N\rangle = \varepsilon_N$, then

$$\mathrm{S}_{\mathcal{D}_i,\mathcal{D}_i^\perp} = \varepsilon_N \mathrm{Ric}_{N,N},$$

where $\mathrm{Ric}_{N,N}$ is the Ricci curvature in the N-direction. We have

$$\mathrm{S}_{\mathcal{D}_i,\mathcal{D}_i^\perp} = \mathrm{Tr}_g\, r_{\mathcal{D}_i} = \mathrm{Tr}_g\, r_{\mathcal{D}_i^\perp}.$$

If $\dim\mathcal{D}_i = 1$ then $r_{\mathcal{D}_i} = \varepsilon_N\, R_N$, where $R_N = R(N,\,\cdot\,)\,N$ is the Jacobi operator, and $r_{\mathcal{D}_i^\perp} = \mathrm{Ric}_{N,N}\, g_i^\perp$, where $g_i^\perp(X,Y) := \langle P_i^\perp X, P_i^\perp Y\rangle$ for all $X,Y\in\mathfrak{X}_M$.

The $\mathcal{D}_i$-*deformation tensor* of $Z \in \mathfrak{X}_M$ is the symmetric part of ∇Z restricted to $\mathcal{D}_i$,

$$2\,\mathrm{Def}_{\mathcal{D}_i} Z(X,Y) = \langle \nabla_X Z, Y\rangle + \langle \nabla_Y Z, X\rangle, \quad X,Y \in \mathcal{D}_i.$$

The "musical" isomorphisms $\sharp$ and $\flat$ will be used for rank one and symmetric rank 2 tensors. For example, if $\omega \in \Lambda^1(M)$ is a 1-form and $X,Y \in \mathfrak{X}_M$ then $\omega(Y) = \langle \omega^\sharp, Y\rangle$ and $X^\flat(Y) = \langle X, Y\rangle$. For arbitrary (0,2)-tensors B and C we also have

$$\langle B, C\rangle = \mathrm{Tr}_g(B^\sharp C^\sharp) = \langle B^\sharp, C^\sharp\rangle.$$

The shape operator $(A_i)_Z$ of $\mathcal{D}_i$ with $Z \in \mathcal{D}_i^\perp$, and the operator $(T_i)^\sharp_Z$ are defined by

$$\langle (A_i)_Z(X), Y\rangle = \ h_i(X,Y), Z\rangle, \quad \langle (T_i)^\sharp_Z(X), Y\rangle = \langle T_i(X,Y), Z\rangle, \quad X,Y \in \mathcal{D}_i.$$

The Casorati type operators $\mathcal{A}_i : \mathcal{D}_i \to \mathcal{D}_i$ and $\mathcal{T}_i : \mathcal{D}_i \to \mathcal{D}_i$, and the $(0,2)$-tensor Ψ_i, see [3, 14], are defined using A_i and T_i by

$$\mathcal{A}_i = \sum\nolimits_{E_a \in \mathcal{D}_i^\perp} \varepsilon_a ((A_i)_{E_a})^2, \quad \mathcal{T}_i = \sum\nolimits_{E_a \in \mathcal{D}_i^\perp} \varepsilon_a ((T_i)^\sharp_{E_a})^2,$$
$$\Psi_i(X,Y) = \mathrm{Tr}((A_i)_Y (A_i)_X + (T_i)^\sharp_Y (T_i)^\sharp_X), \quad X,Y \in \mathcal{D}_i^\perp.$$

We define a self-adjoint $(1,1)$-tensor $\mathcal{K}_i : \mathcal{D}_i \to \mathcal{D}_i$ by the formula with Lie bracket,

$$\mathcal{K}_i = \sum\nolimits_{E_a \in \mathcal{D}_i^\perp} \varepsilon_a \, [(T_i^\sharp)_{E_a}, (A_i)_{E_a}].$$

For any $(1,2)$-tensors Q_1, Q_2 and a $(0,2)$-tensor S define the $(0,2)$-tensor Υ_{Q_1,Q_2} by

$$\langle \Upsilon_{Q_1,Q_2}, S\rangle = \sum\nolimits_{\lambda,\mu} \varepsilon_\lambda \, \varepsilon_\mu \big[S(Q_1(e_\lambda, e_\mu), Q_2(e_\lambda, e_\mu)) + S(Q_2(e_\lambda, e_\mu), Q_1(e_\lambda, e_\mu))\big],$$

where on the left-hand side we have the inner product of $(0,2)$-tensors induced by g, $\{e_\lambda\}$ is a local orthonormal basis of TM and $\varepsilon_\lambda = \langle e_\lambda, e_\lambda\rangle \in \{-1, 1\}$.

Remark 2.2. If g is definite then $\Upsilon_{h_i,h_i} = 0$ if and only if $h_i = 0$. Indeed, we have

$$\langle \Upsilon_{h_i,h_i}, X^\flat \otimes X^\flat\rangle = 2 \sum\nolimits_{a,b} \langle X, h_i(E_a, E_b)\rangle^2, \quad X \in \mathcal{D}_i^\perp.$$

The above sum is equal to zero if and only if every summand vanishes. This yields $h_i = 0$. Thus, Υ_{h_i,h_i} is a "measure of non-total geodesy" of the distribution $\mathcal{D}_i$. Similarly, Υ_{T_i,T_i} can be viewed as a "measure of non-integrability" of $\mathcal{D}_i$.

The following presentation of the partial Ricci tensor in (2.2) is valid, see [3, 14]:

$$r_{\mathcal{D}_i} = \mathrm{div}\, h_i + \langle h_i,\, H_i\rangle - \mathcal{A}_i^\flat - \mathcal{T}_i^\flat - \Psi_i^\perp + \mathrm{Def}_{\mathcal{D}_i^\perp} H_i^\perp. \tag{2.6}$$

Tracing (2.6) over $\mathcal{D}_i$ and applying the equalities

$$\mathrm{Tr}_g\,(\mathrm{div}\, h_i) = \mathrm{div}\, H_i, \quad \mathrm{Tr}\langle h_i,\, H_i\rangle = \langle H_i, H_i\rangle, \quad \mathrm{Tr}_g\, \Psi_i^\perp = \langle h_i^\perp, h_i^\perp\rangle - \langle T_i^\perp, T_i^\perp\rangle,$$
$$\mathrm{Tr}\, \mathcal{A}_i = \langle h_i, h_i\rangle, \quad \mathrm{Tr}\, \mathcal{T}_i = -\langle T_i, T_i\rangle, \quad \mathrm{Tr}_g\,(\mathrm{Def}_{\mathcal{D}_i^\perp} H_i^\perp) = \mathrm{div}\, H_i^\perp + g(H_i^\perp, H_i^\perp),$$

we get (2.3) with $\mathcal{D} = \mathcal{D}_i$.

Remark 2.3. For an almost multi-product manifold $(M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ we have

$$\operatorname{div} \sum\nolimits_i (H_i + H_i^\perp) = 2\,\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \sum\nolimits_i Q(\mathcal{D}_i, g), \tag{2.7}$$

see [13]. To illustrate the proof of (2.7) for $k > 2$, consider the case of $k = 3$. Using (2.3) for two distributions, $\mathcal{D}_1$ and $\mathcal{D}_1^\perp = \mathcal{D}_2 \oplus \mathcal{D}_3$, according to (2.4) and (2.5) with $\mathcal{D} = \mathcal{D}_1$, we get

$$\operatorname{div}(H_1 + H_1^\perp) \ = \ 2\,\mathrm{S}_{\mathcal{D}_1,\mathcal{D}_1^\perp} - Q(\mathcal{D}_1, g),$$

and similarly for ($\mathcal{D}_2$, $\mathcal{D}_2^\perp$) and ($\mathcal{D}_3$, $\mathcal{D}_3^\perp$). Summing 3 copies of (2.8), we obtain (2.7) for $k = 3$. Applying Stokes' Theorem for (2.7) on a closed manifold M yields the integral formulas for all $k \in \{2, \ldots, n\}$, which for $k = 2$ directly follows from (2.3).

3 Adapted variations of metric

We consider smooth 1-parameter variations $\{g_t \in \mathrm{Riem}(M) : |t| < \varepsilon\}$ of the metric $g_0 = g$. Let the infinitesimal variations, represented by a symmetric $(0, 2)$-tensor

$$B(t) \equiv \partial g_t/\partial t,$$

be supported in a relatively compact domain Ω in M, i.e., $g_t = g$ and $B_t = 0$ outside Ω for $|t| < \varepsilon$. We adopt the notations $\partial_t \equiv \partial/\partial t,\ B \equiv \partial_t g_{t\,|\,t=0} = \dot g$, but we shall also write B instead of B_t to make formulas easier to read, wherever it does not lead to confusion. Since B is symmetric, then $\langle C,\, B\rangle = \langle \mathrm{Sym}(C),\, B\rangle$ for any $(0, 2)$-tensor C. Denote by $\otimes$ the product of tensors.

Definition 3.1. A family of adapted pseudo-Riemannian metrics

$$\{g(t) \in \mathrm{Riem}(M; \mathcal{D}_1, \ldots \mathcal{D}_k) : |t| < \varepsilon\}$$

will be called an *adapted variation.* In other words, $\mathcal{D}_i$ and $\mathcal{D}_j$ are g_t-orthogonal for all $i \neq j$ and t. An adapted variation g_t is called a $\mathcal{D}_j$*-variation* (for some $j \in [1, k]$) if

$$g_t(X, Y) = g_0(X, Y), \quad X, Y \in \mathcal{D}_j^\perp, \quad |t| < \varepsilon.$$

For an adapted variation we have $g_t = g_1(t) \oplus \ldots \oplus g_k(t)$, where $g_j(t) = g_t|_{\mathcal{D}_j}$. Thus, the tensor $B_t = \partial_t\, g_t$ of an adapted variation of metric on $(M; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is decomposed into the sum of derivatives of $\mathcal{D}_j$-variations; namely, $B_t = \sum_{j=1}^k B_j(t)$, where $B_j(t) = \partial_t\, g_j(t) = B_t|_{\mathcal{D}_j}$.

Lemma 3.1. *Let a local adapted frame $\{E_a\}$ evolve by $g_t \in \mathrm{Riem}(M; \mathcal{D}_1, \ldots \mathcal{D}_k)$ according to*

$$\partial_t E_a = -(1/2)\, B_t^\sharp(E_a).$$

Then, $\{E_a(t)\}$ is a g_t-orthonormal adapted frame for all t.

Proof. For $\{E_a(t)\}$ we have

$$\begin{aligned}\partial_t(g_t(E_a, E_b)) &= g_t(\partial_t E_a(t), E_b(t)) + g_t(E_a(t), \partial_t E_b(t)) + (\partial_t g_t)(E_a(t), E_b(t))\\ &= B_t(E_a(t), E_b(t)) - \frac12\, g_t(B_t^\sharp(E_a(t)), E_b(t)) - \frac12\, g_t(E_a(t), B_t^\sharp(E_b(t))) = 0.\end{aligned}$$

From this the claim follows. □

Lemma 3.2. *If g_t is a $\mathcal{D}_j$-variation of $g \in \mathrm{Riem}(M;\mathcal{D}_1,\ldots,\mathcal{D}_k)$, then*

$$\partial_t\langle h_j^\perp, h_j^\perp\rangle = -\langle (1/2)\Upsilon_{h_j^\perp,h_j^\perp},\ B_j\rangle,$$
$$\partial_t\langle h_j,\, h_j\rangle = \langle\, \mathrm{div}\, h_j + \mathcal{K}_j^\flat,\ B_j\rangle - \mathrm{div}\langle h_j, B_j\rangle,$$
$$\partial_t g(H_j^\perp, H_j^\perp) = -\langle\, (H_j^\perp)^\flat \otimes (H_j^\perp)^\flat,\ B_j\rangle,$$
$$\partial_t g(H_j, H_j) = \langle\, (\mathrm{div}\, H_j)\, g_j,\ B_j\rangle - \mathrm{div}((\mathrm{Tr}\, B_j^\sharp)H_j),$$
$$\partial_t\langle T_j^\perp, T_j^\perp\rangle = \langle\, (1/2)\Upsilon_{T_j^\perp,T_j^\perp},\ B_j\rangle,$$
$$\partial_t\langle T_j,\, T_j\rangle = \langle 2\,\mathcal{T}_j^\flat,\ B_j\rangle, \tag{3.1}$$

and for $i \neq j$ (when $k > 2$) we have dual equations

$$\partial_t\langle h_i, h_i\rangle = \langle -(1/2)\Upsilon_{h_i,h_i},\ B_j\rangle,$$
$$\partial_t\langle h_i^\perp,\, h_i^\perp\rangle = \langle\, \mathrm{div}\, h_i^\perp + (\mathcal{K}_i^\perp)^\flat,\ B_j\rangle - \mathrm{div}\langle h_i^\perp, B_j\rangle,$$
$$\partial_t g(H_i, H_i) = -\langle\, H_i^\flat \otimes H_i^\flat,\ B_j\rangle,$$
$$\partial_t g(H_i^\perp, H_i^\perp) = \langle\, (\mathrm{div}\, H_i^\perp)\, g_j,\ B_j\rangle - \mathrm{div}((\mathrm{Tr}\, B_j^\sharp)H_i^\perp),$$
$$\partial_t\langle T_i, T_i\rangle = \langle\, (1/2)\Upsilon_{T_i,T_i},\ B_j\rangle,$$
$$\partial_t\langle T_i^\perp,\, T_i^\perp\rangle = \langle 2\,(\mathcal{T}_i^\perp)^\flat,\ B_j\rangle, \tag{3.2}$$

Proof. The equations (3.1) coincide with equations from [15, Proposition 2] for a pair $(\mathcal{D}_j, \mathcal{D}_j^\perp)$, and equations (3.2) are dual to (3.1). □

For any variation g_t of metric g on M with $B = \partial_t g$ we have, e.g., [14],

$$\partial_t\big(\mathrm{d}\,\mathrm{vol}_g\big) = \frac{1}{2}\,(\mathrm{Tr}_g\, B)\,\mathrm{d}\,\mathrm{vol}_g = \frac{1}{2}\,\langle B,\ g\rangle\,\mathrm{d}\,\mathrm{vol}_g\,. \tag{3.3}$$

By (3.3), using the Divergence Theorem, for any variation g_t with $\mathrm{supp}\,(\partial_t g) \subset \Omega$, and t-dependent $X \in \mathfrak{X}_M$ with $\mathrm{supp}\,(\partial_t X) \subset \Omega$ we have

$$\frac{d}{dt}\int_M (\mathrm{div}\, X)\,\mathrm{d}\,\mathrm{vol}_g = \int_M \mathrm{div}\,\big(\partial_t X + \frac{1}{2}\,(\mathrm{Tr}_g\, B)X\big)\,\mathrm{d}\,\mathrm{vol}_g = 0. \tag{3.4}$$

From Lemmas 3.1 and 3.2 and the equality, see (2.4),

$$Q(\mathcal{D}_i, g) = \langle H_i^\perp, H_i^\perp\rangle + \langle H_i, H_i\rangle - \langle h_i, h_i\rangle - \langle h_i^\perp, h_i^\perp\rangle + \langle T_i, T_i\rangle + \langle T_i^\perp, T_i^\perp\rangle,$$

we obtain the following.

Proposition 3.3. *For a $\mathcal{D}_j$-variation of metric $g \in \mathrm{Riem}(M;\mathcal{D}_1,\ldots,\mathcal{D}_k)$ we have*

$$\partial_t \sum\nolimits_i Q(\mathcal{D}_i, g) = \langle \mathcal{Q}_j,\ B_j\rangle - \mathrm{div}\, X_j, \tag{3.5}$$

where $B_j = \partial_t g_{t\,|\,t=0}$, and (0,2)-tensors $\mathcal{Q}_j$ on $\mathcal{D}_j \times \mathcal{D}_j$ and vector fields X_j are given by

$$2X_j = \langle h_j, B_j\rangle - (\mathrm{Tr}\, B_j^\sharp)H_j + \sum\nolimits_{i\neq j}(\langle h_i^\perp, B_j\rangle - (\mathrm{Tr}\, B_j^\sharp)H_i^\perp),$$
$$\mathcal{Q}_j = -\,\mathrm{div}\, h_j - \mathcal{K}_j^\flat + \frac{1}{2}\Upsilon_{h_j^\perp,h_j^\perp} + \frac{1}{2}\Upsilon_{T_j^\perp,T_j^\perp} + 2\,\mathcal{T}_j^\flat - (H_j^\perp)^\flat \otimes (H_j^\perp)^\flat + (\mathrm{div}\, H_j)\, g_j$$
$$+ \sum\nolimits_{i\neq j}\big(-\,\mathrm{div}\, h_i^\perp|_{\mathcal{D}_j} - (P_j\mathcal{K}_i^\perp)^\flat + \frac{1}{2}\Upsilon_{P_j h_i, P_j h_i} + \frac{1}{2}\Upsilon_{P_j T_i, P_j T_i} + 2\,(P_j\mathcal{T}_i^\perp)^\flat$$
$$-\,(P_j H_i)^\flat \otimes (P_j H_i)^\flat + (\mathrm{div}\, H_i^\perp)\, g_j\big).$$

The summation part related to $\mathcal{D}_j^\perp$ (in X_j and $\mathcal{Q}_j$) is dual to the part related to $\mathcal{D}_j$.

The next theorem allows us to restore the partial Ricci curvature, see (1.4). It is based on calculating the variations with respect to g of components in (2.3) and using (3.4) for divergence terms. By this theorem and Definition 3.5 in what follows we conclude that an adapted metric g is critical for the action (1.2) with respect to adapted variations of metric preserving the volume of Ω, i.e., $\mathrm{Vol}(\Omega, g_t) = \mathrm{Vol}(\Omega, g)$ for all t, if and only if (1.4) holds.

Theorem 3.4 (see [15]). *An adapted metric $g \in \mathrm{Riem}(M; \mathcal{D}_1, \ldots, \mathcal{D}_k)$ is critical for the geometrical part of (1.2) (i.e., $\Lambda = 0 = \mathcal{L}$) with respect to adapted variations preserving the volume of Ω if and only if the following Euler-Lagrange equations hold:*

$$\begin{aligned} &\operatorname{div} h_j + \mathcal{K}_j^\flat - \frac{1}{2}\Upsilon_{h_j^\perp, h_j^\perp} - \frac{1}{2}\Upsilon_{T_j^\perp, T_j^\perp} - 2\,\mathcal{T}_j^\flat + (H_j^\perp)^\flat \otimes (H_j^\perp)^\flat \\ &+\sum\nolimits_{i\neq j}\big(\operatorname{div} h_i^\perp|_{\mathcal{D}_j} + (P_j\mathcal{K}_i^\perp)^\flat - \frac{1}{2}\Upsilon_{P_j h_i, P_j h_i} - \frac{1}{2}\Upsilon_{P_j T_i, P_j T_i} - 2\,(P_j\mathcal{T}_i^\perp)^\flat \\ &+ (P_j H_i)^\flat \otimes (P_j H_i)^\flat\big) = \big(\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \operatorname{div}(H_j + \sum\nolimits_{i\neq j} H_i^\perp) + \lambda\big)\, g_j \end{aligned} \tag{3.6}$$

for some $\lambda \in \mathbb{R}$ and $1 \le j \le k$, or, in a short form,

$$\mathcal{Q}_j = -\big(\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k} - \frac{1}{2}\operatorname{div}\sum\nolimits_i (H_i + H_i^\perp) + \lambda\big)\, g_j, \quad 1 \le j \le k. \tag{3.7}$$

Proof. Let g_t be a $\mathcal{D}_j$-variation of g compactly supported in $\Omega \subset M$. Using Divergence theorem to (3.5) and removing integrals of divergences of vector fields supported in $\Omega \subset M$, we get

$$\int_\Omega \sum\nolimits_i \partial_t Q(\mathcal{D}_i, g_t)_{\,|\,t=0}\,\mathrm{d}\,\mathrm{vol}_g = \int_\Omega \langle \mathcal{Q}_j,\ B_j\rangle\,\mathrm{d}\,\mathrm{vol}_g\,. \tag{3.8}$$

By (3.4) with $X = \sum_i (H_i + H_i^\perp)$ we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \operatorname{div}\sum\nolimits_i (H_i + H_i^\perp)\,\mathrm{d}\,\mathrm{vol}_g = 0.$$

Thus, for the action (1.3), using (2.7), (3.3), (3.5) and (3.8), we get

$$\begin{aligned} 2\,\frac{\mathrm{d}}{\mathrm{dt}}\,J_{\mathcal{D}}^g(g_t)_{\,|\,t=0} &= \frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \sum\nolimits_i Q(\mathcal{D}_i, g_t)\,\mathrm{d}\,\mathrm{vol}_{g_t\,|\,t=0} \\ &= \int_\Omega \sum\nolimits_i \partial_t Q(\mathcal{D}_i, g_t)_{|\,t=0}\,\mathrm{d}\,\mathrm{vol}_g + \int_\Omega \sum\nolimits_i Q(\mathcal{D}_i, g)\,\partial_t(\mathrm{d}\,\mathrm{vol}_{g_t})_{|\,t=0} \\ &= \int_\Omega \langle\, \mathcal{Q}_j + \frac{1}{2}\sum\nolimits_i Q(\mathcal{D}_i, g)\, g,\ B_j\,\rangle\,\mathrm{d}\,\mathrm{vol}_g\,. \end{aligned} \tag{3.9}$$

If g is critical for the action $J_{\mathcal{D}}^g$ with respect to $\mathcal{D}_j$-variations of g, then the integral in (3.9) is zero for any symmetric $(0,2)$-tensor B_j. This yields the $\mathcal{D}_j$-component of Euler-Lagrange equation

$$\mathcal{Q}_j + \frac{1}{2}\sum\nolimits_i Q(\mathcal{D}_i, g)\, g_j = 0, \quad 1 \le j \le k. \tag{3.10}$$

For adapted variations preserving the volume of Ω, using (3.3), we have

$$0=\partial_t\int_M \mathrm{d}\operatorname{vol}_g=\int_M \partial_t \,\mathrm{d}\operatorname{vol}_g=\int_M \frac12\,(\operatorname{Tr} B)\,\mathrm{d}\operatorname{vol}_g=\frac12\int_\Omega \langle g,\,B\rangle\,\mathrm{d}\operatorname{vol}_g\,.$$

Thus, the Euler-Lagrange equation of (1.3) with respect to $\mathcal{D}_j$-variations preserving the volume of Ω are

$$\mathcal{Q}_j+(\frac12\sum\nolimits_i Q(\mathcal{D}_i,g)+\lambda)\,g_j=0$$

instead of (3.10). Replacing here $\sum_i Q(\mathcal{D}_i,g)$ according to (2.7), we get (3.6). □

Remark 3.2. Using the partial Ricci tensor (2.2) and replacing $\operatorname{div} h_j$ and $\operatorname{div} h_i^\perp$ for $i\ne j$ in (3.6) according to (2.6), we can rewrite (3.6) as

$$\begin{aligned}
& r_{\mathcal{D}_j}-\langle h_j,\,H_j\rangle+\mathcal{A}_j^\flat-\mathcal{T}_j^\flat+\Psi_j^\perp-\operatorname{Def}_{\mathcal{D}_j^\perp}H_j^\perp+\mathcal{K}_j^\flat+(H_j^\perp)^\flat\otimes(H_j^\perp)^\flat\\
& -\frac12\,\Upsilon_{h_j^\perp,h_j^\perp}-\frac12\Upsilon_{T_j^\perp,T_j^\perp}+\sum\nolimits_{i\ne j}\big(r_{\mathcal{D}_i^\perp}|_{\mathcal{D}_j}-\langle h_i^\perp|_{\mathcal{D}_j},\,H_i^\perp\rangle+(P_j\mathcal{A}_i^\perp)^\flat-(P_j\mathcal{T}_i^\perp)^\flat\\
& +\Psi_i|_{\mathcal{D}_j}-\operatorname{Def}_{\mathcal{D}_j}H_i+(P_j\mathcal{K}_i^\perp)^\flat+(P_jH_i)^\flat\otimes(P_jH_i)^\flat-\frac12\Upsilon_{P_jh_i,P_jh_i}-\frac12\Upsilon_{P_jT_i,P_jT_i}\big)\\
& =\big(\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k}-\operatorname{div}(H_j+\sum\nolimits_{i\ne j}H_i^\perp)+\lambda\big)\,g_j,\quad j=1,\ldots,k.
\end{aligned}\tag{3.11}$$

Example 3.3. A pair $(\mathcal{D}_i,\mathcal{D}_j)$ with $i\ne j$ of distributions on a Riemannian almost multi-product manifold $(M,g;\mathcal{D}_1,\ldots,\mathcal{D}_k)$ is called *mixed integrable*, see [13], if

$$T_{i,j}(X,Y)=0\quad(X\in\mathcal{D}_i,\ Y\in\mathcal{D}_j).$$

Let $(M,g;\mathcal{D}_1,\ldots,\mathcal{D}_k)$ with $k>2$ has integrable distributions $\mathcal{D}_1,\ldots,\mathcal{D}_k$ and each pair $(\mathcal{D}_i,\mathcal{D}_j)$ is mixed integrable. Then $T_l^\perp(X,Y)=0$ for all $l\le k$ and $X\in\mathcal{D}_i$, $Y\in\mathcal{D}_j$ with $i\ne j$, see [13, Lemma 2]. In this case, (3.11) reads as

$$\begin{aligned}
& r_{\mathcal{D}_j}-\langle h_j,\,H_j\rangle+\mathcal{A}_j^\flat+\Psi_j^\perp-\operatorname{Def}_{\mathcal{D}_j^\perp}H_j^\perp-\frac12\,\Upsilon_{h_j^\perp,h_j^\perp}+(H_j^\perp)^\flat\otimes(H_j^\perp)^\flat\\
& +\sum\nolimits_{i\ne j}\big(r_{\mathcal{D}_i^\perp}|_{\mathcal{D}_j}-\langle h_i^\perp|_{\mathcal{D}_j},\,H_i^\perp\rangle+(P_j\mathcal{A}_i^\perp)^\flat+\Psi_i|_{\mathcal{D}_j}-\operatorname{Def}_{\mathcal{D}_j}H_i-\frac12\,\Upsilon_{P_jh_i,P_jh_i}\\
& +(P_jH_i)^\flat\otimes(P_jH_i)^\flat\big)=\big(\mathrm{S}_{\mathcal{D}_1,\ldots,\mathcal{D}_k}-\operatorname{div}(H_j+\sum\nolimits_{i\ne j}H_i^\perp)+\lambda\big)\,g_j,\quad j=1,\ldots,k.
\end{aligned}$$

Definition 3.4. The Ricci type symmetric $(0,2)$-tensor $\mathcal{R}ic_{\mathcal{D}}$ in (1.4) is defined by its restrictions $\mathcal{R}ic_{\mathcal{D}\,|\,\mathcal{D}_j\times\mathcal{D}_j}$ on k subbundles $\mathcal{D}_j$ of TM,

$$\mathcal{R}ic_{\mathcal{D}\,|\,\mathcal{D}_j\times\mathcal{D}_j}=-\mathcal{Q}_j+\mu_j\,g_j,\quad j=1,\ldots,k\tag{3.12}$$

(in a short form, using $\mathcal{Q}_j$ in the LHS of (3.7)), where (μ_j) are uniquely determined (see (3.15) and Theorem 3.5 below) so that critical metrics satisfy Einstein type equation (1.4). Using (2.6), this can be written in more detail as

$$\begin{aligned}
\mathcal{R}ic_{\mathcal{D}\,|\,\mathcal{D}_j\times\mathcal{D}_j} & =r_{\mathcal{D}_j}-\langle h_j,\,H_j\rangle+\mathcal{A}_j^\flat+\mathcal{T}_j^\flat+\Psi_j^\perp-\operatorname{Def}_{\mathcal{D}_j^\perp}H_j^\perp+\mathcal{K}_j^\flat+(H_j^\perp)^\flat\otimes(H_j^\perp)^\flat\\
& -\frac12\Upsilon_{h_j^\perp,h_j^\perp}-2\,\mathcal{T}_j^\flat-\frac12\Upsilon_{T_j^\perp,T_j^\perp}+\sum\nolimits_{i\ne j}\big(r_{\mathcal{D}_i^\perp}|_{\mathcal{D}_j}-\langle h_i^\perp|_{\mathcal{D}_j},\,H_i^\perp\rangle+(P_j\mathcal{A}_i^\perp)^\flat\\
& +(P_j\mathcal{T}_i^\perp)^\flat+\Psi_i|_{\mathcal{D}_j}-\operatorname{Def}_{\mathcal{D}_j}H_i+(P_j\mathcal{K}_i^\perp)^\flat+(P_jH_i)^\flat\otimes(P_jH_i)^\flat\\
& -\frac12\,\Upsilon_{P_jh_i,P_jh_i}-\frac12\,\Upsilon_{P_jT_i,P_jT_i}-2\,(P_j\mathcal{T}_i^\perp)^\flat\big)+\mu_j\,g_j.
\end{aligned}\tag{3.13}$$

Theorem 3.5. *A metric $g \in \mathrm{Riem}(M;\mathcal{D}_1,\dots\mathcal{D}_k)$ is critical for the geometrical part of* (1.2)*, i.e., $\Lambda = 0 = \mathcal{L}$, with respect to adapted variations if and only if g satisfies Einstein type equation* (1.4)*, where the tensor $\mathcal{R}ic_{\mathcal{D}}$ is defined in* (3.12).

Proof. The Euler-Lagrange equations (3.6) consist of $\mathcal{D}_j\times\mathcal{D}_j$-components. Thus, for (1.3) we obtain (3.13). If $n=2$ (and $k=2$), then we take $\mu_1=\mu_2=0$, see [11]. Assume that $n>2$. Substituting (3.13) with arbitrary (μ_j) into (1.4) along $\mathcal{D}_j$, we conclude that if the Euler Lagrange equations

$$\mathcal{Q}_j = -b_j\, g_j \quad (1\le j\le k)$$

hold, where $b_j\, g_j$ is the RHS of (3.6), then $\mathcal{R}ic_{\mathcal{D}} - (1/2)\,\mathcal{S}_{\mathcal{D}}\cdot g = 0$, see (1.4) with $\Lambda = 0 = \Xi$, if and only if (μ_j) satisfy the linear system

$$\sum\nolimits_i n_i\,\mu_i - 2\,\mu_j = a_j,\quad j=1,\dots,k, \tag{3.14}$$

with coefficients $a_j = \mathrm{Tr}_g(\sum_i \mathcal{Q}_i) - 2\,\mathcal{Q}_j$. The matrix of (3.14) is invertible. Its determinant $2^{k-1}(2-n)$ is negative when $n>2$. Hence, the system (3.14) has a unique solution $(\mu_1,\dots,\mu_k)$ given by

$$\mu_{\,i} = -\frac{1}{2n-4}\,\big(\sum\nolimits_j (a_{\,i}-a_j)\,n_j - 2\,a_{\,i}\big), \tag{3.15}$$

and $\mathcal{R}ic_{\mathcal{D}\,|\,\mathcal{D}_j\times\mathcal{D}_j}$ satisfies (3.13). □

Example 3.5 (see [11])**.** The symmetric Ricci type tensor $\mathcal{R}ic_{\mathcal{D}}$ in (1.4) with $k=2$, is defined by its restrictions on two complementary subbundles $\mathcal{D}$ and $\mathcal{D}^\perp$ of TM,

$$\begin{aligned}
\mathcal{R}ic_{\mathcal{D}\,|\,\mathcal{D}^\perp\times\mathcal{D}^\perp} &= r - \langle h^\perp,\, H^\perp\rangle + (\mathcal{A}^\perp)^\flat - (\mathcal{T}^\perp)^\flat + \Psi - \mathrm{Def}_{\mathcal{D}}\,H + (\mathcal{K}^\perp)^\flat\\
&\quad + H^\flat\otimes H^\flat - \frac12\,\Upsilon_{h,h} - \frac12\,\Upsilon_{T,T} + \mu_1\, g^\perp,\\
\mathcal{R}ic_{\mathcal{D}|\,\mathcal{D}\times\mathcal{D}} &= r^\perp - \langle h,\, H\rangle + \mathcal{A}^\flat - \mathcal{T}^\flat + \Psi^\perp - \mathrm{Def}_{\mathcal{D}^\perp}\,H^\perp + \mathcal{K}^\flat\\
&\quad + (H^\perp)^\flat\otimes(H^\perp)^\flat - \frac12\,\Upsilon_{h^\perp,h^\perp} - \frac12\,\Upsilon_{T^\perp,T^\perp} + \mu_2\, g^\top,
\end{aligned} \tag{3.16}$$

where $\mu_1 = -\frac{n_1-1}{n-2}\,\mathrm{div}(H^\perp - H)$ and $\mu_2 = \frac{n_2-1}{n-2}\,\mathrm{div}(H^\perp-H)$. Here $(3.16)_2$ is dual to $(3.16)_1$ with respect to interchanging distributions $\mathcal{D}$ and $\mathcal{D}^\perp$, and their last terms vanish if $n_1=n_2=1$. Also, we have

$$\mathcal{S}_{\mathcal{D}} := \mathrm{Tr}_g\,\mathcal{R}ic_{\mathcal{D}} = \mathrm{S}_{\,\mathcal{D},\mathcal{D}^\perp} + \frac{n_2-n_1}{n-2}\,\mathrm{div}(H^\perp - H).$$

Example 3.6. Totally umbilical and totally geodesic integrable distributions appear on multiply twisted products. A *multiply twisted product* $F_1\times_{u_2} F_2\times\ldots\times_{u_k} F_k$ of Riemannian manifolds (F_i, g_{F_i}), $1\le i\le k$, is the product $M=\prod_i F_i$ with the metric $g = g_{F_1}\oplus u_2^2\, g_{F_2}\oplus\ldots\oplus u_k^2\, g_{F_k}$, where $u_i: F_1\times F_i\to(0,\infty)$ for $i\ge 2$ are smooth functions, see [19]. Twisted products (i.e., $k=2$) and multiply warped products (i.e., $u_i: F_1\to(0,\infty)$, see [6]) are special cases of multiply twisted products. Let $\mathcal{D}_i$ be the distribution on M obtained from vectors tangent to horizontal lifts of F_i. The leaves tangent to $\mathcal{D}_i$ $(i\ge 2)$, are totally umbilical, with the mean curvature vector fields

$$H_i = -n_i P_1\nabla(\log u_i),$$

and the fibers (tangent to $\mathcal{D}_1$) are totally geodesic ($h_1 = 0$). For $k > 2$ each pair of distributions is mixed totally geodesic (since M is the product and the Lie bracket does not depend on metric). Using

$$\operatorname{div} H_i = -n_i\,(\Delta_1\, u_i)/u_i - (n_i^2 - n_i)\,\|P_1\nabla u_i\|^2/u_i^2,$$

where Δ_1 is the Laplacian on (F_1, g_{F_1}), we find

$$\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} = \sum\nolimits_{i\ge 2} n_i\,(\Delta_1\, u_i)/u_i\,. \tag{3.17}$$

Let a multiply twisted product $F_1 \times_{u_2} F_2 \times \ldots \times_{u_k} F_k$ with $k > 2$, see Example 3.6, be critical for (1.3) with respect to adapted variations of g. Then the system (3.6) takes the form

$$\begin{aligned}&\operatorname{div} h_j - \frac12 \Upsilon_{h_j^\perp, h_j^\perp} + (H_j^\perp)^\flat \otimes (H_j^\perp)^\flat + \sum\nolimits_{i\ne j} \big(\operatorname{div} h_i^\perp|_{\mathcal{D}_j} - \frac12 \Upsilon_{h_i,h_i} + H_i^\flat \otimes H_i^\flat\big)\\ &= \big(\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} - \operatorname{div}(H_j + \sum\nolimits_{i\ne j} H_i^\perp) + \lambda\big)\, g_j.\end{aligned} \tag{3.18}$$

(a) Let $\dim F_1 = n_1 > 2$ and $\dim F_i = n_i > 1$ for $i \ne 1$. In addition, assume that

$$\langle H_i,\, H_j\rangle = 0, \quad i \ne j.$$

From (3.18) with $j = 1$, using $H_1^\perp = \sum_{i\ne 1} H_i$ and equalities

$$\begin{aligned}&\frac12 \Upsilon_{h_1^\perp, h_1^\perp} = \sum\nolimits_{i\ne 1} \frac{1}{n_i}\, H_i^\flat \otimes H_i^\flat = \frac12 \sum\nolimits_{i\ne 1} \Upsilon_{h_i,h_i},\\ &\sum\nolimits_{i\ne 1} \operatorname{div} h_i^\perp|_{\mathcal{D}_1} = (k-2) \sum\nolimits_{i\ne 1} \frac{1}{n_i}\, \operatorname{div} H_i,\\ &\operatorname{div}(\sum\nolimits_{i\ne 1} H_i^\perp) = (k-2) \sum\nolimits_{i\ne 1} \operatorname{div} H_i,\\ &(H_1^\perp)^\flat \otimes (H_1^\perp)^\flat = \sum\nolimits_{i\ne 1} H_i^\flat \otimes H_i^\flat,\end{aligned}$$

we obtain

$$2 \sum\nolimits_{i\ne 1} \big(1 - \frac{1}{n_i}\big) H_i^\flat \otimes H_i^\flat = \big(\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} - (k-2) \sum\nolimits_{i\ne 1} \big(1 + \frac{1}{n_i}\big)\, \operatorname{div} H_i + \lambda\big)\, g_1. \tag{3.19}$$

Comparing ranks (2 and $n_1 > 2$) of matrices $H_i^\flat \otimes H_i^\flat$ and g_1 in (3.19), we get $H_i = 0$ ($i > 1$). Hence, each distribution $\mathcal{D}_i$ is totally geodesic, and our multiply twisted product is the product of $(F_1,\, g_{F_1})$ and $(F_i,\, u_i^2 g_{F_i})$ for $i > 1$.

(b) Let $\dim F_i = 1$ for $i \ne 1$. Then the system (3.6) takes the form

$$\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} - \operatorname{div}(2\, H_j + \sum\nolimits_{i\ne j} H_i^\perp) + \lambda = 0, \quad 1 \le j \le k. \tag{3.20}$$

Using (3.17) and equality $\operatorname{div} H_i = -(\Delta_1\, u_i)/u_i$, we get the linear system

$$(k-2) y_j + (k-1) \sum\nolimits_{i\ne j} y_i + \lambda = 0, \quad 1 \le j \le k, \tag{3.21}$$

where $y_i = (\Delta_1\, u_i)/u_i$. The unique solution of (3.21) is $y_i = \tilde\lambda$, where $\tilde\lambda = \lambda/(\frac{1}{k-1} - k)$. Thus, $\tilde\lambda$ is the eigenvalue of the laplacian Δ_1 on (F_1, g_{F_1}), and u_i are the eigenfunctions: $\Delta_1\, u_i = \tilde\lambda\, u_i$. The mixed scalar curvature in this case is constant:

$$\mathrm{S}_{\mathcal{D}_1,\dots,\mathcal{D}_k} = \sum\nolimits_{i\ne 1} (\Delta_1\, u_i)/u_i = (k-1)\tilde\lambda.$$

Similarly, we can find critical multiply twisted products for the action (1.2).

References

[1] M.A. Akivis, V.V. Goldberg, Differential geometry of webs. In *Handbook of differential geometry*, Vol. I, pp. 1–152, North-Holland, Amsterdam, 2000.

[2] E. Barletta, S. Dragomir, V. Rovenski and M. Soret, Mixed gravitational field equations on globally hyperbolic space-times, Classical and Quantum Gravity, 30 : 8, (2013) 085015, 26 pp.

[3] E. Barletta, S. Dragomir and V. Rovenski, The mixed Einstein-Hilbert action and extrinsic geometry of foliated manifolds, Balkan J. of Geometry and Its Appl. 22 : 1, (2017), 1–17.

[4] A. Bejancu and H. Farran, *Foliations and geometric structures*, Springer, 2006.

[5] A. L. Besse, *Einstein manifolds.* Springer, 1987.

[6] B.-Y. Chen, *Differential geometry of warped product manifolds and submanifolds.* World Scientific, 2017.

[7] F. Dobarro and B. Unal, Curvature of multiply warped products, J. Geom. Phys. 55(1) (2005) 75–106.

[8] A. Gray, Pseudo-Riemannian almost product manifolds and submersions, J. Math. Mech. 16:7 (1967), 715–737.

[9] F. Karaca (2020): Multiply warped products with compact Einstein manifolds, Quaestiones Mathematicae: 1–9, DOI: 10.2989/16073606.2020.1746933

[10] M. Neagu, A distinguished geometry perspective on multi-time affine quadratic Lagrangians. Preprint, arXiv:1611.01413v2, (2020).

[11] V. Rovenski, Einstein-Hilbert type action on space-times, Publications de l'Institut Mathématique, Issue: (N.S.) 103 (117) (2018), 199–210.

[12] V. Rovenski, Prescribing the mixed scalar curvature of a foliation. Balkan J. of Geometry and Its Applications, 24 : 1, (2019), 73–92.

[13] V. Rovenski, *Integral formulas for a Riemannian manifold with several orthogonal complementary distributions*, Global J. of Advanced Research on Classical and Modern Geometries, 10, 1 (2021), 32–42.

[14] V. Rovenski and T. Zawadzki, The Einstein-Hilbert type action on pseudo-Riemannian almost product manifolds, J. Math. Physics, Analysis and Geometry, 15, 1 (2019), 86–121.

[15] V. Rovenski and T. Zawadzki, Variations of the total mixed scalar curvature of a distribution, Ann. Glob. Anal. Geom. 54 (2018), 87–122.

[16] S.E. Stepanov and J. Mikeš, Liouvile-type theorems for some classes of Riemannian almost product manifolds and for special mappings of Riemannian manifolds, Differential Geom. and its Appl. 54, Part A (2017), 111–121.

[17] C. Udrişte and I. Tevy, Geometric Dynamics on Riemannian Manifolds. Mathematics 8, 79 (2020). https://doi.org/10.3390/math8010079.

[18] P. Walczak, An integral formula for a Riemannian manifold with two orthogonal complementary distributions. Colloq. Math., 58 (1990), 243–252.

[19] Y. Wang, Multiply warped products with a semi-symmetric metric connection, Abstr. Appl. Anal., 2014, 12, (2014)

Author's address:

Vladimir Rovenski
Department of Mathematics, University of Haifa,
Mount Carmel, 31905 Haifa, Israel.
E-mail: vrovenski@univ.haifa.ac.il