Almost Hermitian Golden manifolds

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Abstract. In this paper, we discuss some geometric properties of almost complex Golden structure (i.e. a polynomial structure with the structure polynomial $Q(X) = X^2 - X + \frac{3}{2}I$) and we introduce such some new classes of almost Hermitian Golden structures. We give a concrete examples.

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1 Introduction

To equip a space with a structure leads to the production of a new mathematical object and consequently to contribute to the development of science. Manifolds equipped with certain differential-geometric structures are richer and more practical spaces, they have been studied widely in differential geometry. Indeed, D. Chinea and C. Gonzalez [1] obtained a classification of the (2n + 1)-dimensional almost contact metric manifold based on $U(n)$ representation theory, which is an analogy of the classification of the $2n$-dimensional almost Hermitian manifolds established by A. Gray and H. M. Hervella [4].
Being inspired by the Golden ratio, the notion of Golden manifold M was defined in [2] by a tensor field Φ on M satisfying $\Phi^2 = \Phi + I$. The authors studied some properties of this manifold and they showed that Φ is an automorphism of the tangent bundle TM and its eigenvalues are $\phi = \frac{1+\sqrt{5}}{2}$ and $1 - \phi$. There are also several recent works in this direction. And in the same article [2], they introduced the notion of complex Golden structure as a tensor Φ_c of type $(1, 1)$ satisfies $\Phi_c^2 = \Phi_c - \frac{3}{2}I$ and its eigenvalues are $\phi_c = \frac{1+i\sqrt{5}}{2}$ and $1 - \phi_c$.

In this work, rely on the relationship between the almost complex structure J and the almost complex Golden structure Φ_c given in [2], we extract the geometric tools for the almost Hermitian Golden structure and we use them to define certain new classes.

Aiming at our purpose, we organize this paper as follows:
Section 2 is devoted to the background of the almost complex Golden structure and

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we give some new and important properties such as Riemannian metric which is compatible with the structure, the fundamental 2-form and others.
In Section 3 we establish an important proposition that allows us to state our main theorem concerning the classes of almost Hermitian Golden structures. The last section is devoted to building a concrete example.

2 Almost complex Golden manifold

The complex Golden ratio section ϕ_c is the root of the polynomial equation $x^2 - x + \frac{3}{2} = 0$, i.e, $\phi_c = \frac{1+i\sqrt{5}}{2}$ where $i^2 = -1$ and the secod root denoted by ϕ_c^*, satisfies $\phi_c^* = \frac{1-i\sqrt{5}}{2} = 1 - \phi_c$ is his conjugate.

Definition 2.1. [2, 3]. Let M be a C^∞ differentiable manifold of an even dimension and let I be the identity $(1,1)$ tensor field. A tensor field F of type $(1,1)$ on M is said to define a polynomial structure if F satisfies the algebraic equation

$$Q(X) = X^n + a_n X^{n-1} + ... + a_2 X + a_1 I = 0,$$

where $F^{n-1}(p), F^{n-2}(p), ..., F(p)$ and I are linearly independent for every $p \in M$. The polynomial $Q(X)$ is called the structure polynomial.

Definition 2.2. [2]. A non-null tensor field Φ_c of type $(1,1)$ and of class C^∞ satisfying the equation

$$\Phi_c^2 = \Phi_c - \frac{3}{2}I, \tag{2.1}$$

is called an almost complex Golden structure on M of even dimensional.

A straightforward computation yields:

Proposition 2.1. • *The eigenvalues of an almost complex Golden structure Φ_c are the complex Golden ratio ϕ_c and $\phi_c^* = 1 - \phi_c$.*
• *An almost complex Golden structure Φ_c is an isomorphism on the tangent space of the manifold, T_pM, for every $p \in M$.*
• *It follows that Φ_c is invertible and its inverse Φ_c^{-1} given by*

$$\Phi_c^{-1} = \frac{-2}{3}\left(\Phi_c - I\right).$$

Remark 2.3. If Φ_c is an almost complex Golden structure then $\tilde{\Phi}_c = I - \Phi_c$ is also an almost complex Golden structure, where I is the identity transformation.

For an almost complex Golden structure Φ_c, is said to be integrable if its Nijenhuis tensor N_{Φ_c} vanishes, ([2]). That is,

$$N_{\Phi_c}(X,Y) = \Phi_c^2[X,Y] + [\Phi_c X, \Phi_c Y] - \Phi_c[\Phi_c X, Y] - \Phi_c[X, \Phi_c Y] = 0, \tag{2.2}$$

where X, Y any two vectors fields on M.
For an integrable almost complex Golden structure we drop the adjective " almost " and then simply call it complex Golden structure.

Proposition 2.2. *If J is an almost complex structure on M, then*

$$\Phi_c = \frac{1}{2}\left(I+\sqrt{5}J\right), \tag{2.3}$$

is an almost complex Golden structure. Conversely, if Φ_c is an almost Golden structure on M then

$$J = \frac{1}{\sqrt{5}}\left(2\Phi_c - I\right), \tag{2.4}$$

is an almost complex structure on M.

Proof.

$$\begin{aligned} J^2X &= J\big(\frac{1}{\sqrt{5}}(2\Phi_cX - X)\big) \\ &= \frac{1}{\sqrt{5}}\big(2\Phi_c\big(\frac{1}{\sqrt{5}}(2\Phi_cX-X)\big) - \frac{1}{\sqrt{5}}(2\Phi_cX-X)\big) \\ &= \frac{1}{\sqrt{5}}\big(\frac{4}{\sqrt{5}}\Phi_c^2X - \frac{2}{\sqrt{5}}\Phi_cX - \frac{2}{\sqrt{5}}\Phi_cX + \frac{1}{\sqrt{5}}X\big) \\ &= \frac{1}{\sqrt{5}}\big(\frac{4}{\sqrt{5}}(\Phi_cX - \frac{3}{2}X) - \frac{4}{\sqrt{5}}\Phi_cX + \frac{1}{\sqrt{5}}X\big) \\ &= -X. \end{aligned}$$

Conversely, we have,

$$\begin{aligned} \Phi_c^2 &= \big(\frac{1}{2}(I+\sqrt{5}J)\big)^2 = \frac{1}{4}(I+5J^2+2\sqrt{5}J) \\ &= -I + \frac{\sqrt{5}}{2}\big(\frac{1}{\sqrt{5}}(2\Phi_c - I)\big) = \Phi_c - \frac{3}{2}I \end{aligned}$$

□

Proposition 2.3. *For a twin pair $\{\Phi_c, J\}$, on an even dimensional manifold M with any free linear connection $\tilde{\nabla}$, one has*

$$4N_{\Phi_c} = 5N_J \qquad \text{and} \qquad 2\tilde{\nabla}\Phi_c = \sqrt{5}\tilde{\nabla}J. \tag{2.5}$$

Proof. Just using (2.2) and (2.3). □

Note that,

- For every almost complex structure J on M, the corresponding Φ_c is an almost complex Golden structure on M.
- For every almost complex Golden structure Φ_c on M, the corresponding J is an almost complex structure on M.
- There is a one-to-one correspondence between the set of all almost complex structures and the set of all almost complex Golden structures on a manifold M.

Example 2.4. Let (x, y, z, t) be Cartesian coordinates in $\mathbb{R}^4$, and $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\}$ is a local basis. Then the structure Φ_c defined by

$$\begin{cases} \Phi_c \frac{\partial}{\partial x} = \dfrac{1}{2}\big(\dfrac{\partial}{\partial x} + \sqrt{5}(\dfrac{\partial}{\partial y} + 2x\dfrac{\partial}{\partial z})\big) \\ \Phi_c \frac{\partial}{\partial y} = \dfrac{1}{2}\big(\dfrac{\partial}{\partial y} - \sqrt{5}(\dfrac{\partial}{\partial x} + 2xe^{-4t}\dfrac{\partial}{\partial t})\big) \\ \Phi_c \frac{\partial}{\partial z} = \dfrac{1}{2}\big(\dfrac{\partial}{\partial z} + \sqrt{5}e^{-4t}\dfrac{\partial}{\partial t}\big) \\ \Phi_c \frac{\partial}{\partial t} = \dfrac{1}{2}\big(\dfrac{\partial}{\partial t} - \sqrt{5}e^{4t}\dfrac{\partial}{\partial z}\big) \end{cases}$$

is an complex Golden structure on $\mathbb{R}^4$.

3 Almost Hermitian Golden manifold

Recall that an almost Hermitian structure is a pair (J, g) with g a fixed Riemannian metric on M and J an almost complex structure related by

$$g(JX, JY) = g(X, Y), \tag{3.1}$$

or equivalently, J is a g-anti-symmetric endomorphism

$$g(JX, Y) + g(X, JY) = 0, \tag{3.2}$$

Definition 3.1. An almost Hermitian Golden structure is a pair (Φ_c, g) where Φ_c is an almost complex Golden structure and g is a Riemannian metric,
with

$$g(\Phi_c X, \Phi_c Y) = \frac{3}{2} g(X, Y), \tag{3.3}$$

or equivalently,

$$g(\Phi_c X, Y) + g(X, \Phi_c Y) = g(X, Y). \tag{3.4}$$

The Riemannian metric (3.3) is called Φ_c-compatible and the triple (M, Φ_c, g) is an almost Hermitian Golden manifold.

Proposition 3.1. *The operator J is a g-anti-symmetric endomorphism but the associated almost complex Golden structure (2.3) is not.*

Proof. Just using (3.2) and (3.4). □

Definition 3.2. Let (M, Φ_c, g) be an almost Hermitian Golden manifold. Set

$$\Omega(X, Y) = \frac{1}{\sqrt{5}}\big(2g(X, \Phi_c Y) - g(X, Y)\big),$$

for all X, Y vectors fields on M. Ω is a 2-form on M and it is called "fundamental 2-form".

Remark 3.3. If (M, Φ_c, g) be an almost Hermitian Golden manifold and Ω is a fundamental 2-form, we have

1. $\Omega(X, Y) = -\Omega(Y, X)$
2. $\Omega(\Phi_c X, \Phi_c Y) = \dfrac{3}{2}\Omega(X, Y)$

for all $X, Y \in \Gamma(TM)$.

Lemma 3.2. *For an almost Hermitian Golden structure (Φ_c, g), we have:*

1. $g\big((\nabla_X \Phi_c)Y, Z\big) = -g\big(Y, (\nabla_X \Phi_c)Z\big)$,
2. $\big(\nabla_X \Phi_c\big)\Phi_c Y = (I - \Phi_c)\big(\nabla_X \Phi_c\big)Y$,
3. $g\big((\nabla_X \Phi_c)\Phi_c Y, Z\big) = g\big((\nabla_X \Phi_c)Y, \Phi_c Z\big)$,

for all vectors fields X, Y, Z on M where ∇ denotes the Levi-Civita connection.

Proof. 1. For all X, Y, Z vectors fields on M, using formula (3.4) we have:

$$\begin{aligned} g\big((\nabla_X \Phi_c)Y, Z\big) &= g\big(\nabla_X \Phi_c Y, Z\big) - g\big(\Phi_c \nabla_X Y, Z\big) \\ &= Xg(\Phi_c Y, Z) - g(\Phi_c Y, \nabla_X Z) - g(\nabla_X Y, Z) + g(\nabla_X Y, \Phi_c Z) \\ &= -g\big(Y, (\nabla_X \Phi_c)Z\big). \end{aligned}$$

2. Using formula (2.1) we get:

$$\begin{aligned} \big(\nabla_X \Phi_c\big)\Phi_c Y &= \nabla_X \Phi_c^2 Y - \Phi_c \nabla_X \Phi_c Y \\ &= \nabla_X \Phi_c Y - \frac{3}{2}\nabla_X Y - \Phi_c(\nabla_X \Phi_c)Y - \Phi_c^2 \nabla_X Y \\ &= (I - \Phi_c)\big(\nabla_X \Phi_c\big)Y. \end{aligned}$$

3. Using the equation 2 of this lemma and formula (3.4) we obtain:

$$\begin{aligned} g\big((\nabla_X \Phi_c)\Phi_c Y, Z\big) &= g\big((I - \Phi_c)(\nabla_X \Phi_c)Y, Z\big) \\ &= g\big((\nabla_X \Phi_c)Y, \Phi_c Z\big). \end{aligned}$$

□

Proposition 3.3. *For any almost Hermitian Golden structure (Φ_c, g), we have:*

$$\begin{aligned} 2g\left((\nabla_X \Phi_c)Y, (3I - \Phi_c)Z\right) &= 3\sqrt{5}\left(\mathrm{d}\Omega(X, \Phi_c Y, \Phi_c Z) - \frac{3}{2}\mathrm{d}\Omega(X, Y, Z)\right) \\ &+ g\left(\Phi_c X, N_{\Phi_c}(Y, Z)\right), \end{aligned}$$

for all vectors fields X, Y, Z on M where ∇ denotes the Levi-Civita connection, d *the exterior derivative.*

Proof. Ω is a two differential form on M, then

$$\begin{aligned} 3\mathrm{d}\Omega(X,Y,Z) &= X\big(\Omega(Y,Z)\big)+Y\big(\Omega(Z,X)\big)+Z\big(\Omega(X,Y)\big)\\ &- \Omega\big([X,Y],Z\big)-\Omega\big([Y,Z],X\big)-\Omega\big([Z,X],Y\big), \end{aligned}$$

knowing that

$$\begin{aligned} X\big(\Omega(Y,Z)\big) &= \frac{1}{\sqrt{5}}X\big(2g(Y,\Phi_c Z)-g(Y,Z)\big)\\ &= \frac{2}{\sqrt{5}}g\big(Y,(\nabla_X\Phi_c)Z\big)+\Omega(\nabla_X Y,Z)+\Omega(Y,\nabla_X Z), \end{aligned}$$

and

$$\frac{1}{2}N_{\Phi_c}(Y,Z)=(\nabla_{\Phi_c Y}\Phi_c)Z-(\nabla_{\Phi_c Z}\Phi_c)Y+\Phi_c\big((\nabla_Z\Phi_c)Y-(\nabla_Y\Phi_c)Z\big).$$

Then,

$$\frac{3\sqrt{5}}{2}\mathrm{d}\Omega(X,Y,Z)=g\big(Y,(\nabla_X\Phi_c)Z\big)+g\big(Z,(\nabla_Y\Phi_c)X\big)+g\big(X,(\nabla_Z\Phi_c)Y\big). \tag{3.5}$$

On the other hand, using lemma (3.2) we can get

$$\begin{aligned} \frac{3\sqrt{5}}{2}\mathrm{d}\Omega(X,\Phi_c Y,\Phi_c Z) &= g\big(\Phi_c Y,(\nabla_X\Phi_c)\Phi_c Z\big)+g\big(\Phi_c Z,(\nabla_{\Phi_c Y}\Phi_c)X\big)\\ &+ g\big(X,(\nabla_{\Phi_c Z}\Phi_c)\Phi_c Y\big)\\ &= -g\big((\nabla_X\Phi_c)Y,\Phi_c^2 Z\big)-g\big(\Phi_c X,(\nabla_{\Phi_c Y}\Phi_c)Z-(\nabla_{\Phi_c Z}\Phi_c)Y\big)\\ &= -g\big((\nabla_X\Phi_c)Y,\Phi_c^2 Z\big)-\frac{1}{2}g\big(\Phi_c X,N_{\Phi_c}(Y,Z)\big)\\ &+ g\Big(\Phi_c X,\Phi_c\big((\nabla_Z\Phi_c)Y-(\nabla_Y\Phi_c)Z\big)\Big), \end{aligned}$$

now, using formulas (3.3), (3.5) and lemma (3.2) we obtain

$$\begin{aligned} 2g\left((\nabla_X\Phi_c)Y,(3I-\Phi_c)Z\right) &= 3\sqrt{5}\left(\mathrm{d}\Omega(X,\Phi_c Y,\Phi_c Z)-\frac{3}{2}\mathrm{d}\Omega(X,Y,Z)\right)\\ &+ g\left(\Phi_c X,N_{\Phi_c}(Y,Z)\right). \end{aligned}$$

□

Theorem 3.4. *Let (M,Φ_c,g) be an almost Hermitian Golden manifold and ∇ denotes the Riemannian connection of g. The following conditions are equivalent:*

(a) $\nabla\Phi_c=0$

(b) $\nabla\Omega=0$

(c) $N_{\Phi_c}\equiv 0$ *and* $d\Omega=0$

Proof. For all vectors fields X, Y, Z on $\Gamma(M)$, we have

$$\begin{aligned}(\nabla_X \Omega)(Y,Z) &= X\Omega(Y,Z) - \Omega(\nabla_X Y, Z) - \Omega(Y, \nabla_X Z)\\ &= \frac{2}{\sqrt{5}}\Big(Xg(Y,\Phi_c Z) - g(\nabla_X Y, \Phi_c Z) - g(Y, \Phi_c \nabla_X Z)\Big)\\ &= \frac{2}{\sqrt{5}} g(Y, (\nabla_X \Phi_c)Z).\end{aligned}$$

Thus $\nabla\Phi_c = 0$ if and only if $\nabla\Omega = 0$. Hence (**a**) is equivalent to (**b**).
We suppose (**b**). Then $\mathrm{d}\Omega = 0$ obviously. Moreover, by proposition (3.3) we have $N_{\Phi_c} \equiv 0$.
Conversely, we suppose (**c**). Then proposition (3.3) implies $\nabla\Phi_c = 0$ and hence $\nabla\Omega = 0$. Hence (**b**) is equivalent to (**c**). □

Definition 3.4. Let (M, Φ_c, g) be an almost Hermitian Golden manifold. (M, Φ_c, g) is said to be:

1. Hermitian Golden (HG) manifold if and only if $N_{\Phi_c} = 0$
2. locally conformal Golden (l.c.G) manifold if there exists a closed one-form η such that:
$$\mathrm{d}\Omega = \eta \wedge \Omega.$$
3. Kähler-Golden (KG) manifold if and only if $N_{\Phi_c} = 0$ and $\mathrm{d}\Omega = 0$ or equivalently,
$$\nabla\Phi_c = 0.$$
4. Nearly Golden (NG) manifold if and only if $(\nabla_X \Phi_c)X = 0$.
5. Quasi Golden (QG) manifold if and only if
$$(\nabla_X \Phi_c)Y + (\nabla_{\Phi_c X} \Phi_c)\Phi_c Y = 0.$$

4 Construction of examples

Let (x, y, z, t) denote the Cartesian coordinates in $\mathbb{R}^4$. Let (θ^i) be the frame of differential 1-forms on $\mathbb{R}^4$ given by

$$\theta^1 = f dx, \quad \theta^2 = f dy, \quad \theta^3 = \frac{1}{f}\left(dz - 2x dy\right), \quad \theta^4 = f^3 dt,$$

where f is a non-zero function on $\mathbb{R}^4$, and let (e_i) be the dual frame of vector fields,

$$e_1 = \frac{1}{f}\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{f}\left(\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z}\right), \quad e_3 = f\frac{\partial}{\partial z}, \quad e_4 = \frac{1}{f^3}\frac{\partial}{\partial t}.$$

On $\mathbb{R}^4$, define an almost complex Golden structure Φ_c and a Riemannian metric g by

$$2\Phi_c = \left(e_1 + \sqrt{5}e_2\right)\otimes\theta^1 + \left(e_2 - \sqrt{5}e_1\right)\otimes\theta^2 + \left(e_3 + \sqrt{5}e_4\right)\otimes\theta^3 + \left(e_4 - \sqrt{5}e_3\right)\otimes\theta^4,$$

$$g=\sum_i \theta^i\otimes\theta^i.$$

The frame (e_i) is orthonormal with respect g,

$$\Phi_c e_1=\frac{1}{2}\left(e_1+\sqrt{5}e_2\right),\qquad \Phi_c e_2=\frac{1}{2}\left(e_2-\sqrt{5}e_1\right),$$

$$\Phi_c e_3=\frac{1}{2}\left(e_3+\sqrt{5}e_4\right),\qquad \Phi_c e_4=\frac{1}{2}\left(e_4-\sqrt{5}e_3\right),$$

and g is compatible with Φ_c. Let N_{Φ_c} be the Nijenhuis torsion tensor of Φ_c. $[,]$ being the Lie bracket of vector fields. By direct calculations, one checks that

$$N_{\Phi_c}(e_1,e_2)=N_{\Phi_c}(e_3,e_4)=0,$$

$$N_{\Phi_c}(e_1,e_3)=-\frac{5}{f^2}\big(f_1e_3+(f_2+2xf_3)e_4\big)=-N_{\Phi_c}(e_2,e_4),$$

$$N_{\Phi_c}(e_1,e_4)=-\frac{5}{f^2}\big((f_2+2xf_3)e_3-f_1e_4\big)=N_{\Phi_c}(e_2,e_3),$$

where $f_i=\frac{\partial f}{\partial x_i}$, which implies that, $(\mathbb{R}^4,\Phi_c,g)$ is a Hermitian Golden manifold if and only if

$$f_1=f_2=f_3=0.$$

Moreover, in our example, the fundamental 2-form Ω has the shape

$$\Omega=-2(\theta^1\wedge\theta^2+\theta^3\wedge\theta^4)=-2f^2\big(dx\wedge dy-2x\,dy\wedge dt+dz\wedge dt\big),$$

so, we obtain

$$\begin{aligned}\mathrm{d}\Omega &= -4f\Big(f_3\,dx\wedge dy\wedge dz-(2xf_1-f_4+f)dx\wedge dy\wedge dt\\ &+ f_1\,dx\wedge dz\wedge dt+(f_2+2xf_3)dy\wedge dz\wedge dt\Big)\\ &= 2\big(\mathrm{d}\ln f-dt\big)\wedge\Omega,\end{aligned}$$

for $\eta=2\big(\mathrm{d}\ln f-dt\big)$, $(\mathbb{R}^4,\Phi_c,g)$ is a locally conformal Golden manifold. Consequently, $(\mathbb{R}^4,\Phi_c,g)$ is a Kähler-Golden manifold if and only if $f=ce^t$ where $c\in\mathbb{R}$. For the last two classes, we calculate the components of the tensor $\nabla\Phi_c$. Using the Koszul formula for the Levi-Civita connection of a Riemannian metric

$$2g(\nabla_{e_i}e_j,e_k)=-g(e_i,[e_j,e_k])+g(e_j,[e_k,e_i])+g(e_k,[e_i,e_j]),$$

we get

$$\nabla_{e_1}e_1=\frac{-1}{f^2}(f_2+2xf_3)e_2-f_3e_3-\frac{f_4}{f^4}e_4,\quad \nabla_{e_1}e_2=\frac{1}{f^3}e_3+\frac{1}{f^2}(f_2+2xf_3)e_1,$$

$$\nabla_{e_1}e_3=f_3e_1-\frac{1}{f^3}e_2,\quad \nabla_{e_1}e_4=\frac{f_4}{f^4}e_1,\nabla_{e_2}e_1=\frac{f_1}{f^2}e_2-\frac{1}{f^3}e_3,$$

$$\nabla_{e_2}e_2=-\frac{f_1}{f^2}e_1-f_3e_3-\frac{f_4}{f^4}e_4,\quad \nabla_{e_2}e_3=\frac{1}{f^3}e_1+f_3e_2,\quad \nabla_{e_2}e_4=\frac{f_4}{f^4}e_2,\quad \nabla_{e_3}e_1=\frac{-1}{f^3}e_2-\frac{f_1}{f^2}e_3,$$

$$\nabla_{e_3}e_2 = \frac{1}{f^3}e_1 - \frac{1}{f^2}(f_2 + 2xf_3)e_3, \quad \nabla_{e_3}e_3 = \frac{1}{f^2}(f_2 + 2xf_3)e_2 + \frac{f_4}{f^4}e_4 + \frac{f_1}{f^2}e_1,$$

$$\nabla_{e_3}e_4 = -\frac{f_4}{f^4}e_3, \quad \nabla_{e_4}e_1 = \frac{3f_1}{f^2}e_4, \quad \nabla_{e_4}e_2 = \frac{3}{f^2}(f_2 + 2xf_3)e_4,$$

$$\nabla_{e_4}e_3 = 3f_3e_4, \quad \nabla_{e_4}e_4 = \frac{-3}{f^2}(f_2 + 2xf_3)e_2 - 3f_3e_3 - \frac{3f_1}{f^2}e_1.$$

Knowing that $(\nabla_{e_i}\Phi_c)e_j = \nabla_{e_i}\Phi_c e_j - \Phi_c\nabla_{e_i}e_j$, then we obtain

$$\frac{2}{\sqrt{5}}(\nabla_{e_1}\Phi_c)e_1 = \frac{-1}{f^2}(f_2 + 2xf_3)e_1 + \frac{1}{f^4}(f - f_4)e_3 + f_3e_4,$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_1}\Phi_c)e_2 = \frac{1}{f^2}(f_2 + 2xf_3)e_2 + f_3e_3 - \frac{1}{f^4}(f - f_4)e_4,$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_1}\Phi_c)e_3 = \frac{2}{\sqrt{5}}(\nabla_{e_2}\Phi_c)e_4 = -\frac{1}{f^4}(f - f_4)e_1 - f_3e_2,$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_1}\Phi_c)e_4 = \frac{2}{\sqrt{5}}(\nabla_{e_2}\Phi_c)e_3 = -f_3e_1 + \frac{1}{f^4}(f - f_4)e_2,$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_2}\Phi_c)e_1 = -f_3e_3 + \frac{1}{f^4}(f - f_4)e_4,$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_2}\Phi_c)e_2 = \frac{1}{f^4}(f - f_4)e_3 + f_3e_4,$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_3}\Phi_c)e_1 = \frac{-1}{f^2}(f_2 + 2xf_3)e_3$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_3}\Phi_c)e_2 = \frac{1}{f^2}(f_2 + 2xf_3)e_4$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_3}\Phi_c)e_3 = \frac{1}{f^2}(f_2 + 2xf_3)e_1$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_3}\Phi_c)e_4 = \frac{-1}{f^2}(f_2 + 2xf_3)e_2$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_4}\Phi_c)e_1 = \frac{3f_1}{f^2}e_3 + \frac{3}{f^2}(f_2 + 2xf_3)e_4,$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_4}\Phi_c)e_2 = \frac{3}{f^2}(f_2 + 2xf_3)e_3 + \frac{3f_1}{f^2}e_4,$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_4}\Phi_c)e_3 = \frac{-3}{f^2}(f_2 + 2xf_3)e_2,$$

$$\frac{2}{\sqrt{5}}(\nabla_{e_4}\Phi_c)e_4 = \frac{-3}{f^2}(f_2 + 2xf_3)e_1,$$

Now we will make $\nabla\Phi_c = 0$ (i.e., Kähler-Golden case) we get the following equations:

$$f_2 + 2xf_3 = 0, \quad f - f_4 = 0, \quad f_1 = f_3 = 0,$$

and moreover that these equations are equivalent to the following OED

$$f - f_4 = 0, \quad with \quad f = f(t).$$

Solving the differential equation we obtain $f = ce^t$ with $c \in \mathbb{R}$. Which confirms the previous result.

For the Nearly Golden case $\big($i.e. $(\nabla_X \Phi_c)X = 0$ $\big)$, we get the following equations:

$$f_2 + 2xf_3 = 0, \quad f - f_4 = 0, \quad f_3 = 0,$$

which give

$$f = A(x)e^t.$$

Unfortunately, in this family of almost Hermitian Golden manifolds, there are no manifolds properly Quasi Golden.

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