Almost Ricci-Bourguignon solitons and geometrical structure in a relativistic perfect fluid spacetime

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Abstract. The present study is based on the geometrical bearing of relativistic perfect fluid spacetime and GRW-spacetime in terms of almost Ricci-Bourguignon solitons with torse-forming vector field ξ. A condition for the almost Ricci-Bourguignon solitons to be steady, expanding or shrinking is also given. In particular, when the potential vector field ξ of the soliton is of gradient type, we derive a Poisson-Laplacian equation from the almost η-Ricci-Bourguignon soliton. Finally, we provide an example of 4-dimensional relativistic spacetime admitting the almost Ricci-Bourguignon and almost η-Ricci-Bourguignon solitons.

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1 Introduction

General Relativity is the theory of gravity (GR) put forward by Albert Einstein in 1915. In this theory the gravitational field is the spacetime curvature and its source is energy-momentum tensor. This is the origin of all field theories. In GR the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be effectively modeled by some Lorentzian metric defined on a suitable 4-dimensional manifold M.

The Einstein's equations are fundamental tools in the construction of cosmological models which imply that the matter determines the geometry of the spacetime and conversely the motion of matter is determined by the metric tensor of the space which is non-flat. Relativistic fluid models are of considerable interest in several areas of astrophysics, plasma physics and nuclear physics. Theories of relativistic stars (which would be models for supermassive stars) are also based on relativistic fluid models. The problem of accretion onto a neutron stars or a black hole is usually set in the framework of relativistic fluid models.

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Besides its essential role in the theoretical study, general relativity has also gained great success in engineering when applying to our daily life. After being proposed, seeking the various solution to Einstein's field equation become one of the most important problems. The most obvious solution is the *Minikowski' spacetime*, which is the four dimensional Euclidean space $\mathbb{R}^4$ equipped with Lorentzian metric.

A connected 4-dimensional time oriented Lorentzian manifold is a special subclass of pseudo-Riemannian manifolds with Lorentzian metric g with signature $(-,+,+,+)$ has great importance in general relativity. The geometry of 4-dimensional Lorentzian manifold begins with the study of nature of vectors on the manifold. Therefore, 4-dimensional Lorentzian manifold becomes most suitable choice for the study of general relativity.

A perfect fluid is to be one with no heat conduction and no viscosity or it can be defined as a fluid which looks isotropic or star in its rest frame. The most simple example of the perfect fluid is dust. Perfect fluids are often used in general relativity to model idealized distribution.

Definition 1.1. An n-dimensional Lorentzian manifolds is said to be a perfect fluid spacetime if its non-vanishing Ricci tensor S satisfies

$$S = ag + b\eta \otimes \eta, \tag{1.1}$$

where a, b are scalars fields (not simultaneously zero) and η is a 1-form, that is $g(X, \xi) = \eta(X)$ for all X and $g(\xi, \xi) = -1$.

Definition 1.2. [14] A Lorentzian manifold M with $dim(M) \geq 3$ is said to be a generalized Robertson-Walker spacetime (GRW) if and only if it admits a unit timelike torse-forming vector field $\nabla_E \zeta = \omega E + \gamma(E)\zeta$, that is also an eigenvector of the Ricci tensor.

The energy-momentum tensor plays the major role as a matter content of the spacetime, matter is assumed to be fluid having density, pressure and having dynamical and kinematic quantities like velocity, acceleration, vorticity, shear and expansion [19]. The matter content of the universe is assumed to perform like a perfect fluid in standard cosmological models. Therefore, a perfect fluid can be completely characterized by its rest mass density and isotropic pressure. It has neither shear, stresses, viscosity, nor heat condition and is characterized by an energy-momentum tensor of the form ([15], [16]):

$$T(X, Y) = pg(X, Y) + (\sigma + p)\eta(X)\eta(Y), \tag{1.2}$$

where σ, p are the energy density and isotropic pressure respectively, g is the metric tensor of *Minkowski spacetime*, $\eta(X) = g(X, \xi)$ is 1-form, equivalent to the velocity vector of the perfect fluid ξ and $g(\xi, \xi) = -1$.

Further, example of energy-momentum tensor are energy-momentum tensor of electromagnetism and scalar field theory.

The field equation governing the perfect fluid motion is Einstein's gravitational equation [16]

$$S(X, Y) + \left(\lambda - \frac{r}{2}\right) g(X, Y) = \kappa T(X, Y), \tag{1.3}$$

for any $X, Y \in \chi(M)$, where λ is the cosmological constant, κ is the gravitational constant (which can be taken $8\pi G$, with G the universal gravitational constant), S is the Ricci tensor and r is the scalar curvature of g. They are obtained from Einstein's equations by adding a cosmological constant in order to get a static universe, according to Einstein's idea. In modern cosmology, it is considered as a candidate for dark energy, the cause of the acceleration of the expansion of the universe.

From equations (1.2) and (1.3) we obtain the Einstein's equation for perfect fluid as

$$S(X,Y) = -\left(\lambda - \frac{r}{2} + \kappa p\right) g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y). \tag{1.4}$$

By the the property of the manifold that the Ricci tensor S is a functional combination of g and $\eta \otimes \eta$, for η a 1-form g dual to a unitary vector field, is called quasi-Einstein ([8], [9]). Perfect fluid spacetime are extensively studied in many manners of views, we may refer to (see [2], [7],[11],[13], [18] and references therein). In [1], [3], [4], [20], [21], [23], Ricci solitons are studied extensively within the background of pseudo-Riemannian geometry.

On the other hand, geometric flows are most significant tools to explain the geometric structures in relativistic perfect fluid spacetime (semi-Riemannian geometry). A special class of solutions on which the metric evolves by dilations and diffeomorphisms plays a vital part in the study of singularities of the flows as they appear as possible singularity models. They are often called soliton solutions.

In 1981, the notion of Ricci-Bourguignon flow as a generalization of Ricci flow [12] has been introduced by J. P. Bourguinon [5]. Ricci-Bourguignon flow is an intrinsic geometric flow on pseudo-Riemannian manifolds, whose fixed points are solitons. The Ricci-Bourguignon-soliton, generates self-similar solution to the Ricci-Bourguignon flow is described by [6]

$$\frac{\partial g}{\partial t} = -2(S - \rho R g), \quad g(0) = g_0, \tag{1.5}$$

where S is the Ricci curvature tensor, R is the scalar curvature with respect to the g and ρ is a real non-zero constant. It should be noticed that for special values of the constant ρ in equation (1.5), we obtain the following situations for the tensor $S - \rho R g$ appearing in equation. The PDE system (1.5) defines the evolution equation of special interest, in particular [6]

1. $\rho = \frac{1}{2}$, the Einstein tensor $S - \frac{R}{2}g$ (Einstein soliton),
2. $\rho = \frac{1}{n}$, the traceless Ricci tensor $S - \frac{R}{n}g$,
3. $\rho = \frac{1}{2(n-1)}$, the Schouten tensor $S - \frac{R}{2(n-1)}$ (Schouten soliton),
4. $\rho = 0$, the Ricci tensor S (Ricci soliton).

In dimension two, the first three tensors are zero, hence the flow is static and in higher dimension the value of ρ are strictly ordered as above in descending order.

Short time existence and uniqueness for the solution of this geometric flow has been proved in [6]. In fact, for sufficiently small t the equation has a unique solution for $\rho < \frac{1}{2(n-1)}$.

In the other hand, quasi Einstein metrics or Ricci solitons serve as a solution to Ricci flow equation. This motivates a more general type of Ricci soliton by considering the Ricci-Bourguignon flow. In fact, a pseudo-Riemannian manifold of dimension $n \geq 3$ is said to be Ricci-Bourguignon soliton if

$$\mathcal{L}_V g(X,Y) + 2S(X,Y) + 2(\mu + \rho R)g(X,Y) = 0, \tag{1.6}$$

where $\mathcal{L}_V$ denotes the Lie derivative operator along vector field V and μ is an arbitrary real constant. Similar to Ricci solitons, a Ricci-Bourguignon soliton is called expanding if $\mu > 0$, steady if $\mu = 0$ and shrinking if $\mu < 0$.

Perturbing the equation that defines (1.6) Ricci-Bourguignon soliton by multiple of a certain $(0,2)$-tensor field $\eta \otimes \eta$, we obtain slightly more general notion, namely η-Ricci-Bourguignon soliton, which we shall consider in a relativistic perfect fluid spacetime, that is, in a 4-dimensional pseudo-Riemannian manifold M with Lorentzian metric g whose content is perfect fluid.

According to Pigola et al. [17] if we assume that the constant μ in (1.6) as a smooth function $\mu \in C^\infty(M)$, called soliton function, then we say that (M,g) is almost Ricci-Bourguignon soliton and almost η-Ricci-Bourguignon soliton see. This concept drags the attention of many geometers. Therefore, in recent years much effort has been devoted to the classification of self-similar solutions of geometric flows.

As an application to relativity by investigating the kinematic and dynamic nature of relativistic spacetime, we present a physical models of three classes namely, shrinking, steady and expanding of perfect fluid solution of Ricci-Bourguignon soliton and spacetime.

Geometry of almost Ricci-Bourguignon solitons, can develop a bridge between a curvature inheritance symmetry of imperfect fluid spacetime (semi-Riemannian manifold) and class of Ricci-Yamabe solitons. In support of this affair we construct three mathematical models of semi-conformally flat almost Ricci-Bourguignon soliton manifolds. As an application to relativity by investigating the kinematic and dynamic nature of spacetime, we present a physical models of three classes namely, shrinking, steady and expanding of perfect fluid solution of almost Ricci-Bourguignon soliton spacetime.

To deal with three special classes of almost Ricci-Bourguignon solitons, namely, shrinking ($\lambda < 0$) which exists on a maximal time interval $-\infty < t < b$ where $b < \infty$, steady ($\lambda = 0$) that which exists for all time or expanding ($\lambda > 0$) which exists on maximal time interval $a < t < \infty$, where $a > -\infty$ [10]. These classes yields example of *ancient*, *eternal* and *immortal solution*, respectively. Also, solutions of Einstein gravity coupled to a free mass less scalar field with nonzero cosmological constant are associated with shrinking or expanding almost Ricci-Bourguignon solitons.

In this paper, we will study some geometrical aspects of almost Ricci-Bourguignon soliton and almost η-Ricci-Bourguignon soliton in relativistic perfect fluid spacetime with torse-forming vector field ξ.

2 Properties of relativistic perfect fluid spacetime with torse-forming vector field

Let (M^4, g) be a relativistic viscous fluid spacetime satisfying (1.5). Contracting (1.3) and assumed that $g(\xi, \xi) = -1$, we obtain

$$r = 4\lambda + \kappa[(\sigma - 3p)]. \tag{2.1}$$

Therefore,

$$S(X,Y) = \left(\lambda + \frac{\kappa(\sigma - p)}{2}\right) g(X,Y) + \kappa(\sigma + p)\eta(X)\eta(Y). \tag{2.2}$$

Also,

$$S(\xi, \xi) = -\lambda + \frac{\kappa}{2}[\sigma + 3p]. \tag{2.3}$$

Example 2.1. A radiation fluid ($\sigma = 3p$) has constant scalar curvature r equal to 4λ.

Now, we have the following useful definitions:

Definition 2.2. A vector field ξ is called *torse-forming* if it satisfies [24]

$$\nabla_X \xi = fX + \eta(X)\xi, \tag{2.4}$$

for a vector field X on M^4 and η is a 1-form and a smooth function $f \in C^\infty(M)$.

Definition 2.3. A vector field ξ is called *conformal* vector field if

$$\mathcal{L}_\xi g = \alpha g, \tag{2.5}$$

for some smooth function $\alpha : M \longrightarrow \mathbb{R}$, particularly ξ is called *conformal killing* if $\alpha = 0$.

Definition 2.4. A Riemannian manifold M is said to admit a *Ricci collineation* if there is a vector field ξ such that

$$\mathcal{L}_\xi S = 0, \tag{2.6}$$

where S is the Ricci curvature tensor. It is clear that every killing vector field is a curvature collineation.

Theorem 2.1. *On a relativistic perfect fluid spacetime with torse-forming vector field ξ, the following relations hold [24]:*

$$\eta(\nabla_\xi \xi) = 0, \qquad \nabla_\xi \xi = 0, \tag{2.7}$$

$$(\nabla_X \eta)(Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.8}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.9}$$

$$R(X,\xi)\xi = -X - \eta(X)\xi, \tag{2.10}$$

$$\eta(R(X,Y)Z) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z), \tag{2.11}$$

$$(\mathcal{L}_\xi g)(X,Y) = 2f[g(X,Y) + \eta(X)\eta(Y)]. \tag{2.12}$$

Proof. We calculate

$$\begin{aligned}(\nabla_X\eta)(Y) &= X(\eta(Y)) - \eta(\nabla_X Y) = X(g(Y,\xi)) - g(\nabla_X Y,\xi)\\ &= g(Y,\nabla_X\xi) = f[g(X,Y)+\eta(X)\eta(Y)].\end{aligned}$$

In particular $(\nabla_\xi\eta)(Y) = 0$. The relation (2.8) can be obtained by (2.4).

Now, using (2.4) in

$$R(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi,$$

and from direct calculation, we get the relation (2.9). Additionally (2.10) and (2.11) follow from (2.9). □

3 Almost Ricci-Bourguignon soliton in relativistic perfect fluid spacetime

In this section, we study almost Ricci-Bourguignon soliton structure in a relativistic perfect fluid spacetime whose timelike velocity vector field ξ is torse-forming [3].

Now replacing $V = \xi$, equation (1.6)

$$\mathcal{L}_\xi g(X,Y) + 2S(X,Y) + 2(\mu+\rho R)g(X,Y) = 0. \tag{3.1}$$

From (2.12), we have

$$S(X,Y) = -\left[\mu + f + \rho R\right] g(X,Y) - f\eta(X)\eta(Y). \tag{3.2}$$

Now, from (1.1) and (3.2) we can conclude that

Theorem 3.1. *A Lorentzian manifold of dimension $n \geq 4$ admits almost Ricci-Bourguignon soliton, whose soliton filed is a unit time like torse-forming filed, is a perfect fluid spacetime.*

The Definition 1.2 together with Theorem 3.1 state the following result:

Theorem 3.2. *A generalized Robertson-Walker (GRW)-spacetime admitting almost Ricci-Bourguignon soliton is a perfect fluid spacetime.*

Next, putting $X = Y = \xi$ in (3.2), we obtain

$$S(\xi,\xi) = (\mu + \rho R). \tag{3.3}$$

Now, using (2.3) in the equation (3.3), we obtain

$$\mu = \frac{\kappa}{2}[\sigma + 3p] - (\lambda + \rho R). \tag{3.4}$$

Thus, we have the following theorem:

Theorem 3.3. *If a relativistic perfect fluid spacetime with torse-forming vector field ξ admits an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ), then almost Ricci-Bourguignon soliton is expending, steady and shrinking according as*

1. $\frac{\kappa}{2}(\sigma + 3p) > \lambda + \rho R$,
2. $\frac{\kappa}{2}(\sigma + 3p) = \lambda + \rho R$, *and*
3. $\frac{\kappa}{2}(\sigma + 3p) < \lambda + \rho R$,

respectively.

Corollary 3.4. *If a GRW-spacetime with torse-forming vector field ξ admits an almost Ricci-Bourguignon soliton (g, ξ, μ, ρ), then almost Ricci-Bourguignon soliton is expending, steady and shrinking according as $\frac{\kappa}{2}(\sigma+3p) > \lambda+\rho R$, $\frac{\kappa}{2}(\sigma+3p) = \lambda+\rho R$, and $\frac{\kappa}{2}(\sigma + 3p) < \lambda + \rho R$,*

Corollary 3.5. *If a relativistic perfect fluid spacetime with torse-forming vector field ξ admits an almost Einstein soliton (g, ξ, μ), then almost Einstein soliton is expending, steady and shrinking according as $\frac{\kappa}{2}(\sigma + 3p) > \lambda + \frac{R}{2}$, $\frac{\kappa}{2}(\sigma + 3p) = \lambda + \frac{R}{2}$, and $\frac{\kappa}{2}(\sigma + 3p) < \lambda + \frac{R}{2}$, respectively.*

Corollary 3.6. *If a relativistic perfect fluid spacetime with torse-forming vector field ξ admits an almost Schouten soliton (g, ξ, μ), then almost Schouten soliton is expending, steady and shrinking according as $\frac{\kappa}{2}(\sigma + 3p) > \lambda + \frac{R}{2(n-1)}$, $\frac{\kappa}{2}(\sigma + 3p) = \lambda + \frac{R}{2(n-1)}$, and $\frac{\kappa}{2}(\sigma + 3p) < \lambda + \frac{R}{2(n-1)}$, respectively.*

Corollary 3.7. *If a relativistic perfect fluid spacetime with torse-forming vector field ξ admits an almost Ricci soliton (g, ξ, μ), then almost Ricci soliton is expending, steady and shrinking according as $\frac{\kappa}{2}(\sigma + 3p) > \lambda$, $\frac{\kappa}{2}(\sigma + 3p) = \lambda$, and $\frac{\kappa}{2}(\sigma + 3p) < \lambda$, respectively.*

Remark 3.1. According to the above corollaries (3.5), (3.6), and (3.7) we can easily obtain the similar results for *GRW*-spacetime with almost Einstein soliton, almost Schouten soliton, and almost Ricci soliton, respectively.

4 Almost η-Ricci-Bourguignon soliton in relativistic perfect fluid spacetime

Consider the equation

$$\mathcal{L}_\xi g + 2S + 2(\mu + \rho R)g + 2\omega\eta \otimes \eta = 0, \tag{4.1}$$

where g is a pseudo-Riemannian metric, S is the Ricci curvature, ξ is a vector field, η is a 1-form and μ and ω are real constant. The data (g, ξ, μ, ω) which satisfy the equation (4.1) is said to be an almost η-Ricci-Bourguignon soliton in M [23]. In particular if $\omega = 0$, (g, ξ, μ) is an almost Ricci-Bourguignon soliton ([4], [21]) and it is called *shrinking*, *steady* or *expanding* according as μ is negative, zero or positive, respectively [21].

Writing explicitly the Lie derivative $\mathcal{L}_\xi g$ we get

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \tag{4.2}$$

and form (4.1) we obtain

$$S(X,Y) = -(\mu+\rho R)g(X,Y) - \omega\eta(X)\eta(Y) - \frac{1}{2}[g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi)], \tag{4.3}$$

for any $X, Y \in \chi(M)$.

Contracting (4.3) we get

$$r = -(\mu - \rho R)dim(M) + \omega - div(\xi). \tag{4.4}$$

Let (M^4, g) be a general relativistic perfect fluid spacetime and (g, ξ, μ, ω) be an almost η-Ricci-Bourguignon soliton in M. From (1.3) and (4.3) we obtain

$$\begin{aligned}\left[\lambda + \frac{\kappa(\sigma - p + J)}{2} + \mu + \rho R\right] g(X,Y) + [\kappa(\sigma + p) + \omega]\eta(X)\eta(Y) \\ +\kappa P(X,Y) + \frac{1}{2}g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) = 0,\end{aligned} \tag{4.5}$$

for any $X, Y \in \chi(M)$.

Consider $\{e_i\}_{1\leq i\leq 4}$ an orthonormal frame field and $\xi = \sum_{i=1}^4 \xi^i e_i$. We have $\sum_{i=1}^4 \varepsilon_{ii}(\xi^i)^2 = -1$ and $\eta(e_i) = \varepsilon_{ii}\xi^i$. Multiplying (4.5) by ε_{ii} and summing over i for $X = Y = e_i$, we get

$$4\mu - \omega = -4\lambda - \kappa(\sigma - 3p) - \rho R - div(\xi). \tag{4.6}$$

Writing (4.5) for $X = Y = \xi$, we obtain

$$\mu - \omega = -\lambda + \frac{\kappa}{2}[\sigma - 3p] - \rho R. \tag{4.7}$$

Therefore, we have

$$\begin{cases} \mu = \lambda + \frac{\kappa}{2}(\frac{\sigma}{3} - 3p) - \rho R - \frac{div(\xi)}{3} \\ \omega = \kappa(\frac{2}{3}\sigma - 3p) - \frac{div(\xi)}{3} \end{cases} \tag{4.8}$$

Using (4.8), we can state the following result:

Theorem 4.1. *Let (M^4, g) be a 4-dimensional pseudo-Riemannaian manifold and η be the g-dual 1-form of the gradient vector field $\xi = grad(\psi)$ with $g(\xi, \xi) = -1$. If (4.1) defines an almost η-Ricci-Bourguignon soliton for relativistic perfect fluid spacetime in M^4, then the Poisson-Laplacian equation for relativistic perfect fluid spacetime, satisfied by ψ, becomes*

$$\nabla^2(\psi) = 3[\omega - \kappa(\frac{2}{3}\sigma - 3p)]. \tag{4.9}$$

From *Plebanski energy conditions* for relativistic perfect fluid we deduce that $\sigma \geq \max\left\{-\frac{\lambda}{\kappa}, \frac{\lambda}{2\kappa}\right\}$ for steady case, $\sigma > \frac{\lambda}{2\kappa}$ and $\sigma > -\frac{\lambda}{\kappa}$ for the expanding and shrinking case, respectively.

Example 4.1. An almost η-Ricci-Bourguignon soliton (g, ξ, μ, ω) in a relativistic radiation fluid is give by

$$\begin{cases} \mu = \lambda + \frac{\kappa}{2}\left(\frac{\sigma}{3} - 3p\right) - \rho R - \frac{div(\xi)}{3} \\ \omega = \kappa\left(\frac{2}{3}\sigma - 3p\right) - \frac{div(\xi)}{3} \end{cases}$$

Example 4.2. An almost η-Einstein soliton (g, ξ, μ, ω) in a relativistic radiation fluid is give by

$$\begin{cases} \mu = \lambda + \frac{\kappa}{2}\left(\frac{\sigma}{3} - 3p\right) - \frac{R}{2} - \frac{div(\xi)}{3} \\ \omega = \kappa\left(\frac{2}{3}\sigma - 3p\right) - \frac{div(\xi)}{3} \end{cases}$$

Example 4.3. An almost η-Schouten soliton soliton (g, ξ, μ, ω) in a relativistic radiation fluid is give by

$$\begin{cases} \mu = \lambda + \frac{\kappa}{2}\left(\frac{\sigma}{3} - 3p\right) - \frac{R}{2(n-1)} - \frac{div(\xi)}{3} \\ \omega = \kappa\left(\frac{2}{3}\sigma - 3p\right) - \frac{div(\xi)}{3} \end{cases}$$

From this example 4.3, we deduce that almost Ricci-Bourguignon soliton in radiation fluid is steady if $p = \frac{\lambda}{3\kappa}$, expanding if $p > \frac{\lambda}{3\kappa}$ and shrinking if $p < \frac{\lambda}{3\kappa}$.

5 Physical significance of Poisson-Laplace equation

Now, we consider the case if ψ be the gravitational field, ρ the mass density and G the gravitational constant. The Gauss's law of gravitational in differential form is

$$\nabla\psi = -4\pi G\rho. \tag{5.1}$$

In case of gravitational field ψ is conservative and can be expressed as the negative gradient of gravitational potential, that is, $\psi = -gradf$ then by the Gauss's law of gravitational, we have

$$\nabla^2 f = 4\pi G\rho. \tag{5.2}$$

This physical phenomena is directly identical with Theorem 4.1 and equation (4.9), which is Poisson-Laplacian equation with potential vector field of gradient type. Poisson-Laplace equation for gravitational fields if the right hand side is specified as given function h and for homogeneous version. The basis for Newtonian cosmology is Poisson-Laplace equation for the gravitational field $\nabla^2 f = 4\pi G\rho$ this equation for the universe pre suppose that matter is continuously distributed with mass density ρ, while G stands for Newton's gravitational constant and f is the gravitational potential. Therefore, Newtonian gravitational potential also satisfy the Poisson-Laplace equation with Newtonian cosmological constant Λ such that

$$\nabla^2 f = 4\pi G\rho - \Lambda. \tag{5.3}$$

Poisson-Laplace equation obey the principal of relativity, it describes gravitational field. The Azimuthally symmetric theory of gravitons (ASTG-model), Magneto-Hydro-Dynamic (MHD) modelling of molecular clouds are also based on the Poisson-Laplace equation.

Remark 5.1. If the vector field ξ is *conformally killing, that is, $\mathcal{L}_\xi g = \alpha g$ with α a nonzero real number, then the existence of almost Ricci-Bourguignon soliton given by (4.1) for $\omega = 0$, implies the vacuum case. Moreover, the almost Ricci-Bourguignon soliton is steady if $p = \frac{\lambda}{\kappa} + \frac{\alpha}{2\kappa} - \frac{\sigma}{3} + \frac{\rho R}{3}$, expanding if $p > \frac{\lambda}{\kappa} + \frac{\alpha}{2\kappa} - \frac{\sigma}{3} + \frac{\rho R}{3}$ and shrinking if $p < \frac{\lambda}{\kappa} + \frac{\alpha}{2\kappa} - \frac{\sigma}{3} + \frac{\rho R}{3}$.*

Let ξ is a killing vector field and from (1.6), we have

$$\frac{1}{2}\mathcal{L}_\xi \mathcal{L}_\xi g(X,Y) + \mathcal{L}_\xi S(X,Y) = (\mu + \rho R)\mathcal{L}_\xi g(X,Y). \tag{5.4}$$

A vector field ξ is killing if $\mathcal{L}_\xi \mathcal{L}_\xi g = 0$. Thus, the equation (5.4) reveals the following results:

Theorem 5.1. *Let the data (g, μ, ξ, ρ) be an almost Ricci-Bourguignon soliton where ξ is conformal killing vector field if and only if (M, g) is Einstein and Einstein factor is $(\mu + \rho R)$.*

Theorem 5.2. *Let the data (g, μ, ξ, ρ) be a almost Ricci-Bourguignon soliton where ξ is conformal killing vector field if and only if ξ is an almost Ricci-Bourguignon-collineation.*

6 Example of a 4-dimensional Lotrentzian manifold admitting an almost η-Ricci-Bourguignon soliton

Example 6.1. Let 4-dimensional manifold $M = \{(x, y, z, t) \in \mathbb{R}^4 : t \neq 0\}$ where (x, y, z, t) are the standard coordinates of $\mathbb{R}^4$.

Let (e_1, e_2, e_3, e_4) be the set of linearly independent vector fields of M, and is defined as

$$e_1 = t\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \quad e_2 = t\frac{\partial}{\partial y}, \quad e_3 = t\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right), \quad e_4 = (t)^3\frac{\partial}{\partial t}. \tag{6.1}$$

Let g be the Riemannian metric M defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_4, e_4) = -1, \quad g(e_i, e_j) = 0, \tag{6.2}$$

for $i \neq j$, $i, j = 1, 2, 3, 4$.

Let η be the 1-form defined by $\eta(Z) = g(Z, e_4)$, for any $Z \in \chi(M)$. Also, let φ be the $(1,1)$ tensor field, defined by

$$\varphi(e_1) = e_1, \quad \varphi(e_2) = e_2, \quad \varphi(e_3) = e_3, \quad \varphi(e_4) = 0, \quad \xi = (t)^3\frac{\partial}{\partial t}. \tag{6.3}$$

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then, by using the linearity of φ and g, we have

$$[e_1, e_2] = -(t)e_2, \quad [e_1, e_4] = -(t)^2 e_1, \quad [e_2, e_4] = -(t)^2 e_2, \quad [e_3, e_4] = -(t)^2 e_3. \tag{6.4}$$

Then for $e_4 = \xi$ and using Koszul's formula for the Lorentzian metric g, we have

$$\nabla_{e_1} e_1 = -(t)^2 e_4, \quad \nabla_{e_2} e_1 = t e_2, \quad \nabla_{e_1} e_4 = -(t)^2 e_1, \quad \nabla_{e_2} e_4 = -(t)^2 e_2$$
$$\nabla_{e_3} e_4 = -(t)^2 e_3, \quad \nabla_{e_3} e_3 = -(t)^2 e_4, \quad \nabla_{e_2} e_2 = -(t)^2 e_4 - t e_1.$$

We find that the structure (φ, ξ, η, g) is a Lorentzian structure on M. Consequently, $M^4(\varphi, \xi, \eta, g)$ is an Lorentzian manifold (4-dimensional relativistic spacetime model).

The non-vanishing components of Riemannian curvature and the Ricci tensors are given by

$$\begin{aligned}
&R(e_1, e_4)e_1 = (t)^4 e_4, \quad R(e_2, e_4)e_2 = (t)^4 e_4, \quad R(e_3, e_4)e_3 = (t)^4 e_4,\\
&R(e_1, e_3)e_3 = (t)^4 e_1, \quad R(e_1, e_3)e_1 = -(t)^4 e_3, \quad R(e_2, e_3)e_2 = -(t)^4 e_3,\\
&R(e_1, e_4)e_4 = (t)^4 e_1, \quad R(e_2, e_4)e_4 = (t)^4 e_2, \quad R(e_1, e_2)e_2 = [(t)^4 - (t)^2]e_1,\\
&R(e_2, e_3)e_3 = (t)^4 e_2, \quad R(e_3, e_4)e_4 = (t)^4 e_3, \quad R(e_1, e_2)e_1 = -[(t)^4 - (t)^2]e_2.
\end{aligned}$$

From the above expression of the curvature tensor we can easily calculate the non-vanishing components of the Ricci tensor S as

$$S(e_1, e_1) = 3(t)^4 - (t)^2, \quad S(e_2, e_2) = 3(t)^4 - (t)^2. \tag{6.5}$$

Similarly, we have

$$S(e_3, e_3) = 3(t)^4, \quad S(e_4, e_4) = 3(t)^4. \tag{6.6}$$

Therefore,

$$R = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) + S(e_4, e_4) = 2[6(t)^4 - (t)^2]. \tag{6.7}$$

Now, from equations (2.12) and (4.1), we obtain

$$2[g(e_i, e_i) + \eta(e_i)\eta(e_i)] + 2S(e_i, e_i) + 2(2\mu + \rho R)g(e_i, e_i) + 2\omega\eta(e_i)\eta(e_i) = 0, \tag{6.8}$$

for all $i \in \{1, 2, 3, 4\}$, and we have

$$2[(-1 + \delta_{i4}] + 2S(e_i, e_i) + 2(2\mu + \rho R)g(e_i, e_i) + 2\omega\delta_{i4} = 0, \tag{6.9}$$

for all $i \in \{1, 2, 3, 4\}$, we get

$$\mu = [(t)^2(1 + \rho) - 3(t)^4(1 + 4\rho) + 1], \quad \omega = [(t)^2(1 - \rho) - 3(t)^2 + 2]. \tag{6.10}$$

Thus the data (g, ξ, μ, ω) is an η-Ricci-Bourguignon on $(M^4, \phi, \xi, \eta, g)$, which is expanding if $(t)^2(1+\rho) > 3(t)^4(1+4\rho)+1$, shrinking if $(t)^2(1+\rho) < 3(t)^4(1+4\rho)+1$ or steady if $(t)^2(1 + \rho) = 3(t)^4(1 + 4\rho) + 1$.

Moreover,

1. for $\rho = \frac{1}{2}$, $(M^4, \phi, \xi, \eta, g)$ also admits η-*Einstein soliton*, which is expanding if $\frac{3}{2}(t)^2 + 1 > 9(t)^4$, shrinking if $\frac{3}{2}(t)^2 + 1 < 9(t)^4$ or steady if $\frac{3}{2}(t)^2 + 1 = 9(t)^4$;
2. for $\rho = \frac{1}{2(n-1)}$, $(M^4, \phi, \xi, \eta, g)$ admits η-*Schouten soliton*, which is expanding if $(t)^2 + \frac{t^2}{2(n-1)} > 3(t)^4 + \frac{6t^4}{(n-1)}$, shrinking if $(t)^2 + \frac{t^2}{2(n-1)} < 3(t)^4 + \frac{6t^4}{(n-1)}$ or steady if $(t)^2 + \frac{t^2}{2(n-1)} = 3(t)^4 + \frac{6t^4}{(n-1)}$;
3. for $\rho = 0$, $(M^4, \phi, \xi, \eta, g)$ admits η-*Ricci soliton*, which is expanding if $(t)^2+1 > 3(t)^4$, shrinking if $(t)^2 + 1 < 3(t)^4$ or steady if $(t)^2 + 1 = 3(t)^4$.

7 Conclusions

In general theory of relativity, the matter of content of the relativistic spacetime is described by choosing the suitable energy-momentum tensor T. Since the matter content of the universe is considered to working like a perfect fluid such as dust fluid and viscous fluid in the standard cosmological models as a connected 4-dimensional Lorentzian manifold. In this framework Einstein's equation play the fundamental role to construct the cosmological model.

The relativistic perfect fluid spacetime manifold modeled as 4-dimensional Lorentzian manifold admitting the almost Ricci-Bourguignon soliton and also almost η-Ricci-Bourguignon soliton. Solitons are the natural extension of the Einstein's metric. Therefore, Einstein manifolds arose during the study of exact solution of the Einstein's field equation. We have obtained the condition such as steady, expanding and shrinking for the almost Ricci-Bourguignon soliton. Further, we generalized the notion of almost Ricci-Bourguignon soliton called almost η-Ricci-Bourguignon soliton. Moreover, we have proved that relativistic perfect fluid spacetime admitting the almost η-Ricci-Bourguignon soliton and satisfies the Poission-Laplacian equation with potential vector field ψ of gradient type. Therefore, gradient almost Ricci-Bourguignon soliton are natural generalization of Einstein manifold.

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