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Bernstein and Markov-type inequalities for polynomials on $L_p(\mu)$ spaces

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Abstract

In this work we discuss generalizations of the classical Bernstein and Markov type inequalities for polynomials and we present some new inequalities for the kth Fréchet derivative of homogeneous polynomials on real and complex $L_p(\mu)$ spaces. We also give applications to homogeneous polynomials and symmetric multilinear mappings in $L_p(\mu)$ spaces. Finally, Bernstein's inequality for homogeneous polynomials on both real and complex Hilbert spaces has been discussed.

1 Introduction and notation

We recall the basic definitions needed to discuss polynomials from X into Y, where X and Y are real or complex Banach spaces. We denote by B_X and S_X the closed unit ball and the unit sphere of X respectively. A map $P : X \to Y$ is a (continuous) *m-homogeneous polynomial* if there is a (continuous) symmetric m-linear mapping $L : X^m \to Y$ for which $P(x) = L(x, \dots, x)$ for all $x \in X$. In this case it is convenient to write $P = \widehat{L}$. We let $\mathcal{P}(^mX;Y)$, $\mathcal{L}(^mX;Y)$ and $\mathcal{L}^s(^mX;Y)$ denote respectively the spaces of continuous m-homogeneous polynomials from X into Y, the continuous m-linear mappings from X into Y and the continuous symmetric m-linear mappings from X into Y. If $\mathbb{K}$ is the real or complex field we use the notations $\mathcal{P}(^mX)$, $\mathcal{L}(^mX)$ and $\mathcal{L}^s(^mX)$ in place of $\mathcal{P}(^mX;\mathbb{K})$, $\mathcal{L}(^mX;\mathbb{K})$ and $\mathcal{L}^s(^mX;\mathbb{K})$ respectively. More generally, a map $P : X \to Y$ is a *continuous polynomial of degree* $\leq m$ if

$$P = P_0 + P_1 + \cdots + P_m\,,$$

where $P_k \in \mathcal{P}(^kX;Y)$, $1 \leq k \leq m$, and $P_0 : X \to Y$ is a constant function. The space of continuous polynomials from X to Y of degree at most m is denoted by $\mathcal{P}_m(X;Y)$. If $Y = \mathbb{K}$, then we use the notation $\mathcal{P}_m(X)$ instead of $\mathcal{P}_m(X;\mathbb{K})$. We define the norm of a continuous (homogeneous) polynomial $P : X \to Y$ by

$$\|P\|_{B_X} = \sup\{\|P(x)\|_Y :\ x \in B_X\}\,.$$

Similarly, if $L : X^m \to Y$ is a continuous m-linear mapping we define its norm by

$$\|L\|_{B_X^m} = \sup\{\|L(x_1, \dots, x_m)\|_Y :\ x_1, \dots, x_m \in B_X\}\,.$$

When convenient we shall denote $\|L\|_{B_X^m}$ by $\|L\|$ and $\|P\|_{B_X}$ by $\|P\|$. Note that $\mathcal{P}(^mX;Y)$ and $\mathcal{L}(^mX;Y)$ are Banach spaces.

The classical Bishop-Phelps theorem [13] asserts that the collection of norm attaining continuous linear functionals on a Banach space X is norm dense in $X^* := \mathcal{L}(^1X)$, the space of all continuous linear functionals on X. However, in contrast to the linear case, the set of norm attaining continuous symmetric m-linear forms ($m \geq 2$) on a Banach space X is not generally norm dense in the Banach space of all continuous symmetric m-linear forms on X, and the set of norm-attaining continuous m-homogeneous polynomials on X is not generally norm dense in the Banach space of all continuous m-homogeneous polynomials on X [1]. In fact, an example of a Banach space X was given in [1] such that the set of norm-attaining bilinear forms on $X \times X$ is not dense in the space of all continuous bilinear forms. We refer to [45] for the relationship between the norm-attaining condition for a continuous homogeneous polynomial on a Banach space and the norm-attaining condition for its associated continuous symmetric multilinear form.

If $P \in \mathcal{P}_m(X;Y)$ and $x \in X$, then $D^kP(x)$, $2 \leq k \leq m$, denotes the kth Fréchet derivative of P at x. Recall that $D^kP(x)$ would be, in fact, a symmetric k-linear mapping on X^k, whose associated k-homogeneous polynomial will be represented by $\widehat{D}^kP(x)$. So, $\widehat{D}^kP(x) := \widehat{D^kP(x)}$. We just write $DP(x)$ for the first Fréchet derivative of P at x. If $\widehat{L} \in \mathcal{P}(^mX;Y)$, for any vectors $x, y_1, \dots, y_k$ in X and any $k \leq m$ the following identity (see for instance [19, 7.7 Theorem]) holds

$$\frac{1}{k!}D^k\widehat{L}(x)(y_1, \dots, y_k) = \binom{m}{k}L(x^{m-k}, y_1, \dots, y_k)\,. \tag{1}$$

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In particular, for $x, y \in X$

$$\frac{1}{k!}\widehat{D}^k\widehat{L}(x)y = \binom{m}{k} L(x^{m-k}y^k) \tag{2}$$

and for $k = 1$

$$D\widehat{L}(x)y = \widehat{D}\widehat{L}(x)y = mL(x^{m-1}y). \tag{3}$$

Here, $L(x^{m-k}y^k)$ denotes $L(\underbrace{x,\dots,x}_{(m-k)},\underbrace{y,\dots,y}_{k})$. For general background on polynomials, we refer to [19] and [24].

Finally, observe that by composing $P \in \mathcal{P}_m(X;Y)$ with a given linear functional and applying the Hahn-Banach theorem, the upper bounds for $\|D^kP(x)\|$ and $\|\widehat{D}^kP(x)\|$ to be determined are unchanged when Y is replaced by $\mathbb{R}$ or $\mathbb{C}$. Therefore, in proving Bernstein and Markov-type inequalities on real or complex Banach spaces, without loss of generality we can restrict ourselves to scalar-valued polynomials.

Let $r_n(t) = \operatorname{sign}(\sin 2^n\pi t)$ be the nth Rademacher function on $[0,1]$. The Rademacher functions (r_n) form an orthonormal set in $L_2([0,1],dt)$ where dt denotes Lebesgue measure on $[0,1]$. The next formula expresses a well known *polarization formula* in a very convenient form (see [51, Lemma 2]):

$$L(x_1,\dots,x_m) = \frac{1}{m!}\int_0^1 r_1(t)\cdots r_m(t)\widehat{L}\Big(\sum_{n=1}^m r_n(t)x_n\Big)\,dt. \tag{4}$$

Therefore, each $\widehat{L} \in \mathcal{P}(^mX)$ is associated with a unique $L \in \mathcal{L}^s(^mX)$ with the property that $\widehat{L}(x) = L(x,\dots,x)$. In many circumstances [22, 23, 46, 58] it is of interest to compare the norm of $L \in \mathcal{L}^s(^mX)$ with the norm of $\widehat{L} \in \mathcal{P}(^mX)$. It follows from (4)(see [24]) that

$$\|\widehat{L}\| \le \|L\| \le \frac{m^m}{m!}\|\widehat{L}\|,$$

for every $L \in \mathcal{L}^s(^mX)$. However, the right hand inequality can be tightened for many Banach spaces, see for instance [24, 30, 51], and we call

$$\mathbb{K}(m,X) = \inf\big\{M > 0 : \|L\| \le M\|\widehat{L}\|,\ \forall L \in \mathcal{L}^s(^mX;\mathbb{K})\big\}$$

the *mth polarization constant* of the Banach space X. We shall write $\mathbb{R}(m,X)$, $\mathbb{C}(m,X)$ instead of $\mathbb{K}(m,X)$, if the space X is real, complex respectively.

For $L_p(\mu)$ spaces we also set

$$\mathbb{K}(m,p) = \sup\{\mathbb{K}(m,L_p(\mu)) : \mu \text{ is a measure}\}.$$

It is an interesting fact that $\mathbb{K}(m,p) = \mathbb{K}(m,L_p(\mu))$, for any μ with $L_p(\mu)$ infinite-dimensional (we refer to [51]).

2 Bernstein-Markov inequalities for polynomials on Banach spaces

2.1 Bernstein-Markov inequalities for polynomials: classical results

Let $\mathcal{P}_n(\mathbb{R})$ be the set of all algebraic polynomials of degree at most n with real coefficients. According to a well-known result of Bernstein [10], if $p \in \mathcal{P}_n(\mathbb{R})$ and $\|p\|_{[-1,1]} := \max_{-1\le t\le 1}|p(t)| \le 1$ then

$$|p'(t)| \le \frac{n}{\sqrt{1-t^2}}, \quad \forall t \in (-1,1). \tag{5}$$

It was proved by A. A. Markov that if $p \in \mathcal{P}_n(\mathbb{R})$ and $\|p\|_{[-1,1]} \le 1$, then

$$\|p'\|_{[-1,1]} \le n^2. \tag{6}$$

A. A. Markov's original paper [41] dates back to 1889 and it is not readily accessible. For a modern exposition on this and other related topics we refer to [48]. Note that the upper bounds in (5) and (6) are sharp since they are attained for the nth Chebyshev polynomial $T_n(t)$ (for certain values of t in the case of (5)), where $T_n(t)$ is the polynomial agreeing $\cos(n\arccos t)$ in the range $-1 < t < 1$. Inequality (5) yields a better estimate for $|p'(t)|$ when t is not near ± 1.

In the previous two inequalities we have estimates on the magnitude of the derivative of a polynomial, as compared to the polynomial itself. A related result is the following inequality known as Schur's inequality [16, p. 233]:

For every $p \in \mathcal{P}_{n-1}(\mathbb{R})$,

$$\|p\|_{[-1,1]} \le n\|p(t)\sqrt{1-t^2}\|_{[-1,1]}. \tag{7}$$

Observe that Markov's inequality follows immediately from inequalities (5) and (7).

V. A. Markov (brother of A. A. Markov) considered the problem of determining exact bounds for the kth derivative of an algebraic polynomial. For $1 \le k \le n$, if $p \in \mathcal{P}_n(\mathbb{R})$ and $\|p\|_{[-1,1]} \le 1$, V. A. Markov [42] has shown that

$$\|p^{(k)}\|_{[-1,1]} \le T_n^{(k)}(1) = \frac{n^2(n^2-1^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)}. \tag{8}$$

S. N. Bernstein presented a shorter variational proof of (8) in 1938 (see [11]). In 1938 Schaeffer and Duffin [25] have given a rather simple proof of V. A. Markov's inequality. The key in their proof is the following generalization of Bernstein's inequality.

Theorem A (A. C. Schaeffer & R. J. Duffin [25]). *If $p \in \mathcal{P}_n(\mathbb{R})$ with $\|p\|_{[-1,1]} \le 1$ and $1 \le k \le n$, then*

$$|p^{(k)}(t)|^2 \le \left(T_n^{(k)}(t)\right)^2 + \left(S_n^{(k)}(t)\right)^2, \quad \forall t \in (-1,1), \tag{9}$$

where $S_n(t)$ is the is the polynomial agreeing $\sin(n \arccos t)$ *in the range $-1 < t < 1$.*

In fact, if we define

$$\mathcal{M}_k(t) := \left(T_n^{(k)}(t)\right)^2 + \left(S_n^{(k)}(t)\right)^2, \quad \forall t \in (-1,1), \tag{10}$$

a close look at the proofs of Lemma 3 and Markoff's Theorem in [25] reveals that the following result holds true.

Theorem B (A. C. Schaeffer & R. J. Duffin [25]). *If $p \in \mathcal{P}_{n-k}(\mathbb{R})$, $1 \le k \le n$, is such that $|p(t)|^2 \le \mathcal{M}_k(t)$ $\forall t \in (-1,1)$, then*

$$\|p\|_{[-1,1]} \le T_n^{(k)}(1) = \frac{n^2(n^2-1^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)}.$$

Notice that V. A. Markov's inequality (8) and Theorem B together imply Theorem A. Observe also that inequality (7) (Schur's inequality) is a special case of Theorem B for $k=1$.

In studying extremal problems usually we normalize the set of polynomials, that is if $p \in \mathcal{P}_n(\mathbb{R})$ we take $|p(t)| \le 1$ for $-1 \le t \le 1$. In other words we require that the graph of p is contained in the square $[-1,1]\times[-1,1]$.

In the last twenty years extensions of the classical Bernstein and Markov-type inequalities to the multivariate case have been widely investigated. In [31] Harris considers the growth of the Fréchet derivatives of a polynomial on a normed space when the polynomial has restricted growth on the space. His main concern is with *real* normed spaces. Using the technique of potential theory with external fields, improved estimates on Markov constants of *homogeneous* polynomials over real normed spaces have been given in [49]. For the Markov inequality for multivariate polynomials we also refer to [38] and [47]. In 2012 Révész [50] has given a survey on conjectures and results on the multivariate Bernstein inequality on convex bodies. For more polynomial inequalities in Banach spaces we refer to [8].

Finally, it is of importance how the pluripotential theory approach of Baran [8] and the inscribed ellipse approach of Sarantopoulos [52] relate. This is far from obvious and it was in fact unknown for long. However, in 2010 it was fully clarified in [18]. In fact, Burns, Levenberg, Ma'u and Révész have shown in [18] that the "inscribed ellipse method" of Sarantopoulos in [52] to prove Bernstein-Markov inequalities and the "pluripotential" proof of Bernstein-Markov inequalities due to Baran [8] are equivalent.

2.2 Bernstein-Markov inequalities for polynomials on *real* Hilbert spaces

Let P be a polynomial of degree at most n with real coefficients on ℓ_2^m, the m-dimensional Euclidean space $(\mathbb{R}^m, \langle\cdot,\cdot\rangle)$. If $\|P\| \le 1$ and $\|x\|_2 < 1$, the first sharp Bernstein and Markov-type inequalities were obtained in 1928 by Kellogg [36]:

$$\|\nabla P(x)\|_2 \le \min\left\{\frac{n}{\sqrt{1-\|x\|_2^2}}, n^2\right\} \tag{11}$$

In other words, if $D_y P(x) = DP(x)y = \langle \nabla P(x), y\rangle$ is the directional derivative of P at x, in the direction of the unit vector y, then the maximum of the absolute value of $D_y P(x)$ in any direction y is just the maximum of the magnitude of the gradient of the polynomial and is dominated by the smaller of the two numbers $n/\sqrt{1-\|x\|_2^2}$ and n^2. In fact, Kellogg has derived (11) by showing (see Theorem V in [36]) that the tangential derivatives of P on the unit sphere $S_{\mathbb{R}^m}$ cannot exceed n in absolute value.

If K is a smooth compact algebraic curve in $\mathbb{R}^2$ and P is a polynomial of degree $\le n$ in two variables, Bos *et al.* [17] have shown that

$$\|D_T P\|_K \le Mn\|P\|_K,$$

where $D_T P$ denotes tangential derivative of P along K, $\|P\|_K := \sup |P|(K)$ and $M > 0$ is a constant depending only on K. If K is the unit circle, the previous inequality with $M = 1$ is just Kellogg's result. For a discussion on this last inequality and for some other related results see [7] and [29].

Harris [30] has extended Kellogg's argument and in the case of a *real* Hilbert space $(H, \langle\cdot,\cdot\rangle)$, if $P \in \mathcal{P}_n(H)$ and $\|P\| \le 1$, he has obtained the following generalization of (11):

$$|DP(x)y| \le \min\left\{ n\left[\frac{1-\|x\|^2+\langle x,y\rangle^2}{1-\|x\|^2}(1-P(x)^2)\right]^{1/2}, n^2\right\},$$

for all $\|x\| < 1$ and $y \in S_H$.

The generalization of Markov's inequality for any derivative of a polynomial on a *real* Hilbert space was given in [44]. The proof relies on the following extension of Theorem A for polynomials on a *real* Hilbert space and the generalization of Theorem B for polynomials on any *real* Banach space. Recall that $\mathcal{M}_k(t)$ is given by (10).

Theorem 2.1. *[44, Theorem 4] If $(H, \langle\cdot,\cdot\rangle)$ is a real Hilbert space, $P \in \mathcal{P}_n(H)$ with $\|P\| \le 1$ and $1 \le k \le n$, then*

$$\|D^k P(x)\|^2 = \|\widehat{D}^k P(x)\|^2 \le \mathcal{M}_k(\|x\|),$$

for every $x \in H$, $\|x\| < 1$.

Theorem 2.2. *[44, Lemma 1] If X is a real Banach space and $P \in \mathcal{P}_{n-k}(X)$, $n \geq k$, is such that $|P(x)|^2 \leq \mathcal{M}_k(\|x\|)$, $\forall \|x\| < 1$, then $|P(x)| \leq T_n^{(k)}(1)$, $\forall \|x\| \leq 1$.*

Now, the generalization of Markov's inequality (8) on a *real* Hilbert space follows immediately from the previous two theorems.

Theorem 2.3. **(V. A. Markov's theorem)** *[44, Theorem 5] If $(H, \langle\cdot,\cdot\rangle)$ is a real Hilbert space, $P \in \mathcal{P}_n(H)$ with $\|P\| \leq 1$ and $1 \leq k \leq n$, then*

$$\|D^k P(x)\| = \|\widehat{D}^k P(x)\| \leq T_n^{(k)}(1) = \frac{n^2(n^2-1^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)},$$

for every $x \in B_H$.

2.3 Bernstein-Markov inequalities for polynomials on *real* Banach spaces

Let $K \subset \mathbb{R}^m$ be a convex body, i.e. a convex compact set with non-empty interior. If u is a unit vector in $\mathbb{R}^m$ then there are precisely two support hyperplanes to K having u for a normal vector. The distance $w(u)$ between these parallel support hyperplanes is the width of K in the direction of u. The *minimal width* of K is

$$w(K) := \min_{\|u\|_2=1} w(u).$$

For general background on convexity, we refer to [26]. If $P \in \mathcal{P}_n(\mathbb{R}^m)$ with $\|P\|_K := \max_{x\in K} |P(x)|$, Wilhelmsen [59] has shown that

$$\|\nabla P\|_K = \max_{x\in K} \|\nabla P(x)\|_2 \leq \frac{4n^2}{w(K)} \|P\|_K. \tag{12}$$

Since $w(B_{\ell_2^m}) = 2$, the constant in (12) is two times the constant in (11).

Consider now the case where $K \subset \mathbb{R}^m$ is a centrally symmetric convex body with center at the origin, in other words K is invariant under $x \mapsto -x$. We call K a ball. A ball K is the unit ball of a unique Banach norm $\|\cdot\|_K$ defined by

$$\|x\|_K = \inf\{t > 0 : \ x/t \in K\}, \quad x \in \mathbb{R}^m.$$

If $P \in \mathcal{P}_n(\mathbb{R}^m)$, $x \in \operatorname{Int} K$ and $y \in S_{\mathbb{R}^m}$, the next sharp Bernstein and Markov-type inequalities follow from the work of Sarantopoulos[52]:

$$|DP(x)y| \leq \frac{2n}{w(K)\sqrt{1-\|x\|_K^2}} \|P\|_K, \tag{13}$$

$$\|\nabla P\|_K \leq \frac{2n^2}{w(K)} \|P\|_K. \tag{14}$$

In fact, if X is a *real* Banach space and $P \in \mathcal{P}_n(X)$, $\|P\| \leq 1$, for the first Fréchet derivative of P it has been proved in [52] that

$$\|DP(x)\| \leq \min\left\{ n\frac{\sqrt{1-P(x)^2}}{\sqrt{1-\|x\|^2}}, n^2 \right\}, \quad \text{for every } \|x\| < 1. \tag{15}$$

Using methods of several complex variables, inequalities (13) and (14) were proved independently by Baran [6]. For non-symmetric convex bodies, Kroó and Révész [37] have derived a Bernstein-type inequality and they have shown (12) with constant $(4n^2 - 2n)/w(K)$. They have also achieved a further improvement on the Markov constant in case K is a triangle in $\mathbb{R}^2$. But, as it has been shown in [12], in the non-symmetric case the Markov constant in (12) has to be larger than 2.

Finally, a proof of Markov's inequality for any derivative of a polynomial on a *real* Banach space was given by Skalyga [53] in 2005 and additional discussion is given in [54]. In 2010 Harris [33] has given another proof which depends on a Lagrange interpolation formula for the Chebyshev nodes and a Christoffel-Darboux identity for the corresponding bivariate Lagrange polynomials [32].

Theorem 2.4. **(V. A. Markov's theorem)** *[53, 54, 32] If X is a real Banach space, $P \in \mathcal{P}_n(X)$ with $\|P\| \leq 1$ and $1 \leq k \leq n$, then*

$$\|\widehat{D}^k P(x)\| \leq T_n^{(k)}(1),$$

for all $x \in X$, $\|x\| \leq 1$.

Kroó [39] has derived certain Bernstein-Markov inequalities for multivariate polynomials on convex and star-like domains in finite dimensional *real* $L_p(\mu)$ spaces, $1 \leq p < \infty$. In [27, Theorem 6] Eskenazis and Ivanisvili have obtained dimension independent Bernstein-Markov inequality in Gauss space. That is, for each $1 \leq p < \infty$ there is a constant $C_p > 0$ such that for any $k \geq 1$ and all polynomials P on $\mathbb{R}^k$

$$\|\nabla P\|_{L_p(d\gamma_k)} \leq C_p (\deg P)^\alpha \|P\|_{L_p(d\gamma_k)},$$

where $d\gamma_k(x) = \frac{e^{-\|x\|_2^2/2}}{\sqrt{(2\pi)^k}} dx$ is the standard Gaussian measure on $\mathbb{R}^k$, $\alpha = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{|p-2|}{2\sqrt{p-1}}\right)$ and

$$\|\nabla P\|_{L_p(d\gamma_k)} := \left(\int_{\mathbb{R}^k} \left(\sum_{j=1}^k (\partial_j P)^2(x) \right)^{p/2} d\gamma_k(x) \right)^{1/p}.$$

We also refer to [28] for polynomial inequalities on the Hamming cube.

3 Bernstein-Markov inequalities for homogeneous polynomials on $L_p(\mu)$ spaces

3.1 Bernstein and Markov-type estimates for homogeneous polynomials on $L_p(\mu)$ spaces

In the case of a continuous homogeneous polynomial $P \in \mathcal{P}(^mX;Y)$, where X and Y are *real* Banach spaces, the constant $c_{m,k}$ in V. A. Markov's inequality

$$\|\widehat{D}^k P\| \le c_{m,k}\|P\|,$$

can be improved and is considerably better than $T_m^{(k)}(1)$.

For continuous homogeneous polynomials on *real* Banach spaces we have the following Bernstein and Markov-type inequalities.

Theorem 3.1. *[52, Theorem 3] If X is a real Banach space and $\widehat{L}: X \to \mathbb{R}$ is a continuous m-homogeneous polynomial, then we have the following Bernstein-type inequalities*

$$(a) \qquad \|\widehat{D}^k\widehat{L}(x)\| \le \binom{m}{k}\frac{k!}{\left(\sqrt{1-\|x\|^2}\right)^k}\|\widehat{L}\| \tag{16}$$

and

$$(b) \qquad \|D^k\widehat{L}(x)\| \le \binom{m}{k}\frac{k^k}{\left(\sqrt{1-\|x\|^2}\right)^k}\|\widehat{L}\|, \tag{17}$$

for any $\|x\| < 1$ and $k \le m$.

Corollary 3.2. *[52, Corollary] If X is a real Banach space and $\widehat{L}: X \to \mathbb{R}$ is a continuous m-homogeneous polynomial, then we have the following Markov-type inequalities*

$$(a) \qquad \|\widehat{D}^k\widehat{L}(x)\| \le \binom{m}{k}\frac{k!m^{m/2}}{(m-k)^{(m-k)/2}k^{k/2}}\|\widehat{L}\| \tag{18}$$

and

$$(b) \qquad \|D^k\widehat{L}(x)\| \le \binom{m}{k}\frac{m^{m/2}k^{k/2}}{(m-k)^{(m-k)/2}}\|\widehat{L}\|, \tag{19}$$

for any $\|x\| \le 1$ and $k \le m$.

Now we prove Bernstein and Markov-type inequalities for homogeneous polynomials on any *complex* $L_p(\mu)$ space, $1 \le p \le \infty$. For the proof we need the *generalized Clarkson inequality*

$$\left(\|x_1+x_2\|_p^{\lambda'} + \|x_1-x_2\|_p^{\lambda'}\right)^{1/\lambda'} \le 2^{1/\lambda'}\left(\|x_1\|_p^{\lambda} + \|x_2\|_p^{\lambda}\right)^{1/\lambda}, \tag{20}$$

where $x_1, x_2 \in L_p(\mu)$ and $1 \le \lambda \le \min\{p, p'\}$. Here, as usual, $\lambda' = \lambda/(\lambda-1)$ and $p' = p/(p-1)$ are the conjugate exponents of λ and p respectively. Inequality (20) is a special case for $m = 2$ of the following L_p-inequality

$$\left(\int_0^1 \Big\|\sum_{i=1}^m r_i(t)x_i\Big\|_p^{\lambda'}\,dt\right)^{1/\lambda'} \le \Big(\sum_{i=1}^m \|x_i\|_p^{\lambda}\Big)^{1/\lambda}, \tag{21}$$

for $x_i \in L_p(\mu)$, $1 \le i \le m$ and $1 \le \lambda \le \min\{p, p'\}$. We refer to [60] for this and other similar L_p inequalities. Setting $\lambda = p$ or $\lambda = p'$, inequality (20) gives the classical Clarkson inequalities:

$$\left(\|x_1+x_2\|_p^{p'} + \|x_1-x_2\|_p^{p'}\right)^{1/p'} \le 2^{1/p'}\left(\|x_1\|_p^{p} + \|x_2\|_p^{p}\right)^{1/p}, \quad 1 \le p \le 2,$$
$$\left(\|x_1+x_2\|_p^{p} + \|x_1-x_2\|_p^{p}\right)^{1/p} \le 2^{1/p}\left(\|x_1\|_p^{p'} + \|x_2\|_p^{p'}\right)^{1/p'}, \quad 2 \le p \le \infty.$$

Theorem 3.3. *Let $\widehat{L}: L_p(\mu) \to \mathbb{C}$ be a continuous m-homogeneous polynomial, $m \ge 2$, on the complex $L_p(\mu)$ space. If m' and p' are the conjugate exponents of m and p respectively, for $k \le m$ and every $x \in L_p(\mu)$, $\|x\|_p < 1$, we have the following Bernstein-type inequalities*

$$\|\widehat{D}^k\widehat{L}(x)\| \le \begin{cases} \frac{k!}{(1-\|x\|_p^p)^{k/p}}\|\widehat{L}\| & 1 \le p \le m', \\[2ex] \frac{k!}{(1-\|x\|_p^{m'})^{k/m'}}\|\widehat{L}\| & m' \le p \le m, \\[2ex] \frac{k!}{(1-\|x\|_p^{p'})^{k/p'}}\|\widehat{L}\| & m \le p \le \infty. \end{cases} \tag{22}$$

Proof. <u>1st case</u>: Let $1 \le p \le m' \Leftrightarrow m \le p' \le \infty$ or $m \le p \le \infty \Leftrightarrow 1 \le p' \le m'$. If $\lambda = \min\{p, p'\}$, then $\lambda = p$, for $1 \le p \le m'$ and $\lambda = p'$, for $m \le p \le \infty$. For every $x, y \in L_p(\mu)$, $\|x\|_p < 1$, $\|y\|_p = 1$ and every $z \in \mathbb{C}$ put

$$q(z) := \widehat{L}\big(x + (1-\|x\|_p^{\lambda})^{1/\lambda}yz\big) + (-1)^k\widehat{L}\big(x - (1-\|x\|_p^{\lambda})^{1/\lambda}yz\big).$$

Then q is a polynomial of degree $\le m$ on $\mathbb{C}$ with

$$|q(z)| \le \|\widehat{L}\| \left\{ \|x + (1-\|x\|_p^\lambda)^{1/\lambda} yz)\|_p^m + \|x - (1-\|x\|_p^\lambda)^{1/\lambda} yz)\|_p^m \right\}.$$

Applying Hölder's inequality first and then Clarkson's inequality (20), for $|z| \le 1$ we have

$$\begin{aligned} &\|x + (1-\|x\|_p^\lambda)^{1/\lambda} yz)\|_p^m + \|x - (1-\|x\|_p^\lambda)^{1/\lambda} yz)\|_p^m \\ &\le 2^{1-m/\lambda'} \left\{ \|x + (1-\|x\|_p^\lambda)^{1/\lambda} yz)\|_p^{\lambda'} + \|x - (1-\|x\|_p^\lambda)^{1/\lambda} yz)\|_p^{\lambda'} \right\}^{m/\lambda'} \\ &\le 2^{1-m/\lambda'} \cdot 2^{m/\lambda'} \left\{ \|x\|_p^\lambda + \|(1-\|x\|_p^\lambda)^{1/\lambda} yz)\|_p^\lambda \right\}^{m/\lambda} \\ &\le 2 \left\{ \|x\|_p^\lambda + (1-\|x\|_p^\lambda) \right\}^{m/\lambda} = 2. \end{aligned}$$

Hence, $|q(z)| \le 2\|\widehat{L}\|$, for every $|z| \le 1$ and by the Cauchy estimates $|q^{(k)}(0)| \le k! \cdot 2\|\widehat{L}\|$. Since $q^{(k)}(0) = 2(1-\|x\|_p^\lambda)^{k/\lambda} \widehat{D}^k \widehat{L}(x) y$, we have

$$\|\widehat{D}^k \widehat{L}(x)\| \le \frac{k!}{(1-\|x\|_p^\lambda)^{k/\lambda}} \|\widehat{L}\|$$

and this proves the first and the third estimate in (22).

<u>2nd case</u>: Let $m' \le p \le m \Leftrightarrow m' \le p' \le m$. For every $x, y \in L_p(\mu)$, $\|x\|_p < 1$, $\|y\|_p = 1$ and every $z \in \mathbb{C}$ put

$$q(z) := \widehat{L}\big(x + (1-\|x\|_p^{m'})^{1/m'} yz\big) + (-1)^k \widehat{L}\big(x - (1-\|x\|_p^{m'})^{1/m'} yz\big).$$

Then q is a polynomial of degree $\le m$ on $\mathbb{C}$ with

$$|q(z)| \le \|\widehat{L}\| \left\{ \|x + (1-\|x\|_p^{m'})^{1/m'} yz)\|_p^m + \|x - (1-\|x\|_p^{m'})^{1/m'} yz)\|_p^m \right\}.$$

For every $|z| \le 1$, Clarkson's inequality (20) for $\lambda = m' \le \min\{p, p'\}$ implies

$$\begin{aligned} &\|x + (1-\|x\|_p^{m'})^{1/m'} yz)\|_p^m + \|x - (1-\|x\|_p^{m'})^{1/m'} yz)\|_p^m \\ &\le 2 \left\{ \|x\|_p^{m'} + \|(1-\|x\|_p^{m'})^{1/m'} yz)\|_p^{m'} \right\}^{m/m'} \\ &\le 2 \left\{ \|x\|_p^{m'} + (1-\|x\|_p^{m'}) \right\}^{m/m'} = 2. \end{aligned}$$

Hence, $|q(z)| \le 2\|\widehat{L}\|$, for every $|z| \le 1$ and by the Cauchy estimates $|q^{(k)}(0)| \le k! \cdot 2\|\widehat{L}\|$. Since $q^{(k)}(0) = 2(1-\|x\|_p^{m'})^{k/m'} \widehat{D}^k \widehat{L}(x) y$, we have

$$\|\widehat{D}^k \widehat{L}(x)\| \le \frac{k!}{(1-\|x\|_p^{m'})^{k/m'}} \|\widehat{L}\|$$

which is the second estimate in (22). □

Remark 1. Harris [30, Theorem 10] has proved a Bernstein-type inequality for a holomorphic function h satisfying certain conditions on a *complex* $L_p(\mu)$ space, $1 \le p \le \infty$. In particular, if h is a homogeneous polynomial $\widehat{L}$ of degree $m = 2k$ on some $L_p(\mu)$ space, he gives an upper bound for the norm $\|\widehat{D}^k \widehat{L}(x)\|$, for all $x \in L_p(\mu)$, $\|x\|_p \le 1/2$.

Proposition 3.4. *Let $\widehat{L} : L_p(\mu) \to \mathbb{C}$ be a continuous m-homogeneous polynomial, $m \ge 2$, on the complex $L_p(\mu)$ space. If m' and p' are the conjugate exponents of m and p respectively, for $k \le m$ we have the following Markov-type inequality*

$$\|\widehat{D}^k \widehat{L}\| \le C_{k,m} \|\widehat{L}\|, \tag{23}$$

where

$$C_{k,m} = \begin{cases} \frac{k! m^{m/p}}{(m-k)^{(m-k)/p} k^{k/p}} & 1 \le p \le m', \\ \frac{k! m^{m/m'}}{(m-k)^{(m-k)/m'} k^{k/m'}} & m' \le p \le m, \\ \frac{k! m^{m/p'}}{(m-k)^{(m-k)/p'} k^{k/p'}} & m \le p \le \infty. \end{cases} \tag{24}$$

In the case $1 \le p \le m'$ the estimate is best possible.

Proof. Consider the case $1 \le p \le m'$ or $m \le p \le \infty$. If $\lambda = \min\{p, p'\}$ and $x \in L_p(\mu)$, $\|x\|_p < 1$, from the previous theorem

$$\|\widehat{D}^k \widehat{L}(x)\| \le \frac{k!}{(1-\|x\|_p^\lambda)^{k/\lambda}} \|\widehat{L}\|$$

and so for $\|x\|_p \le 1$ and $0 < r < 1$ we have

$$r^{m-k} \|\widehat{D}^k \widehat{L}(x)\| = \|\widehat{D}^k \widehat{L}(rx)\| \le \frac{k!}{(1-r^\lambda)^{k/\lambda}} \|\widehat{L}\|.$$

Therefore,

$$\|\widehat{D}^k\widehat{L}(x)\| \le \frac{k!}{r^{m-k}(1-r^\lambda)^{k/\lambda}}\|\widehat{L}\|, \quad \text{for } \|x\|_p \le 1 \text{ and } 0<r<1\,.$$

Observe that

$$\min_{0<r<1}\frac{1}{r^{m-k}(1-r^\lambda)^{k/\lambda}} = \frac{m^{m/\lambda}}{(m-k)^{(m-k)/\lambda}k^{k/\lambda}}$$

and the minimum is attained for $r=\left(\frac{m-k}{m}\right)^{1/\lambda}$. Hence,

$$\|\widehat{D}^k\widehat{L}\| = \sup_{\|x\|_p\le 1}\|\widehat{D}^k\widehat{L}(x)\| \le \frac{k!\,m^{m/\lambda}}{(m-k)^{(m-k)/\lambda}k^{k/\lambda}}\|\widehat{L}\|\,.$$

Similar is the proof of the middle estimate in (24). Sharpness in the case $1\le p\le m'$ will follow from the next Example 3.1. □

Observe that in the case $1\le p\le m'$ the first inequality in (24) also follows from a special case of [51, Theorem 1].

Example 3.1. Consider the symmetric m-linear form L on the space of p-summable sequences ℓ_p given by

$$L(x_1,\dots,x_m) = \frac{1}{m!}\sum_{\sigma\in S_m} x_{1\sigma(1)}\cdots x_{m\sigma(m)}\,,$$

where $x_i=(x_{in})_{n=1}^\infty$, $i=1,\dots,m$, and S_m is the set of permutations of the first m natural numbers. Then, $\widehat{L}(u)=u_1\cdots u_m$, $u=(u_i)$, is the m-homogeneous polynomial associated to L. If (e_i) is the standard unit vector basis of ℓ_p, for the unit vectors

$$x=\frac{1}{(m-k)^{1/p}}(e_1+\cdots+e_{m-k}) \quad\text{and}\quad y=\frac{1}{k^{1/p}}(e_{m-k+1}+\cdots+e_m)$$

in ℓ_p we can easily verify (see [51, *Example 1*]) that

$$|L(x^{m-k}y^k)| = \frac{(m-k)!k!}{(m-k)^{(m-k)/p}k^{k/p}}\cdot\frac{m^{m/p}}{m!}\|\widehat{L}\|\,.$$

Observe that

$$|\widehat{L}(u)| = \{|u_1|^p\cdots|u_m|^p\}^{1/p} \le \left\{\frac{|u_1|^p+\cdots+|u_m|^p}{m}\right\}^{m/p}$$

by the arithmetic-geometric mean inequality and so $\|\widehat{L}\|\le 1/m^{m/p}$. In fact $\|\widehat{L}\|=1/m^{m/p}$ since for the unit vector $v=(v_i)$ in ℓ_p, with $v_i=m^{-1/p}$ for $1\le i\le m$ and $v_i=0$ for $i>m$, $|\widehat{L}(v)|=1/m^{m/p}$. Therefore, identity (2) implies

$$|\widehat{D}^k\widehat{L}(x)y| = k!\binom{m}{k}|L(x^{m-k}y^k)| = \frac{k!\,m^{m/p}}{(m-k)^{(m-k)/p}k^{k/p}}\|\widehat{L}\|\,.$$

Now we give Markov-type estimates in the case of *real* $L_p(\mu)$ spaces. For this we need some results related to complexification of *real* Banach spaces, polynomials and multilinear maps, see [43].

A *complex* vector space $\widetilde{X}$ is a *complexification* of a *real* vector space X if the following two conditions hold:

(i) there is a one-to-one real-linear map $j: X\to\widetilde{X}$ and

(ii) complex-span$\big(j(X)\big)=\widetilde{X}$.

If X is a real vector space, we can make $X\times X$ into a complex vector space by defining

$$(x,y)+(u,v) := (x+u,y+v) \quad \forall x,y,u,v\in X\,,$$
$$(\alpha+i\beta)(x,y) := (\alpha x-\beta y,\beta x+\alpha y) \quad \forall x,y\in X, \quad \forall \alpha,\beta\in\mathbb{R}.$$

The map $j: X\to X\times X;\ x\mapsto (x,0)$ clearly satisfies conditions (i) and (ii) above, and so this complex vector space is a complexification of X. It is convenient to denote it by

$$\widetilde{X} = X\oplus iX\,.$$

If X is a real-valued $L_p(\mu)$-space, the complexification procedure yields the corresponding complex-valued space. Since $X=L_p(\mu)$ is actually a Banach lattice, the norm on $\widetilde{X}$ can be specified by

$$\|(x,y)\| = \|(|x|^2+|y|^2)^{1/2}\|, \qquad \forall x,y\in X.$$

Bochnak and Siciak (see [15, Theorem 3]) observed that when X is a real Banach space, each $L\in\mathcal{L}(^mX;\mathbb{R})$ has a unique complex extension $\widetilde{L}\in\mathcal{L}(^m\widetilde{X};\mathbb{C})$, defined by the formula

$$\widetilde{L}(x_1^0+ix_1^1,\dots,x_m^0+ix_m^1) = \sum i^{\sum_{j=1}^m\epsilon_j}L(x_1^{\epsilon_1},\dots,x_m^{\epsilon_m}),$$

where x_k^0, x_k^1 are vectors in X, and the summation is extended over the 2^m independent choices of $\epsilon_k=0,1$ $(1\le k\le m)$. The norm of $\widetilde{L}$ depends on the norm used on $\widetilde{X}$, but continuity is always assured.

In the context of polynomials (see also [55, p.313]), any $P \in \mathcal{P}(^m X;\mathbb{R})$ has a unique complex extension $\widetilde{P} \in \mathcal{P}(^m \widetilde{X};\mathbb{C})$, given by the formula

$$\widetilde{P}(x+iy) = \sum_{k=0}^{[\frac{m}{2}]} (-1)^k \binom{m}{2k} L(x^{m-2k}y^{2k}) + i \sum_{k=0}^{[\frac{m-1}{2}]} (-1)^k \binom{m}{2k+1} L(x^{m-(2k+1)}y^{2k+1})$$

for x, y in X, where $P := \widehat{L}$ for some $L \in \mathcal{L}^s(^m X;\mathbb{R})$. Here also $\widetilde{P} = \widehat{\widetilde{L}}$.

If $\widetilde{X}$ is the complexification of a real Banach space X, each $L \in \mathcal{L}^s(^m X;\mathbb{R})$ has a unique complex extension $\widetilde{L} \in \mathcal{L}^s(^m \widetilde{X};\mathbb{C})$ with $\|L\| \le \|\widetilde{L}\|$ and $\|P\| \le \|\widetilde{P}\|$, where $P = \widehat{L}$. We also have [43, Proposition 18]

$$\|\widetilde{P}\| \le 2^{m-1}\|P\| \quad \text{and} \quad \|\widetilde{L}\| \le 2^{m-1}\|L\|. \tag{25}$$

Proposition 3.5. *Let $\widehat{L} : L_p(\mu) \to \mathbb{R}$ be a continuous m-homogeneous polynomial, $m \ge 2$, on the* real *$L_p(\mu)$ space. Then, for $k \le m$ we have the following Markov-type inequality*

$$\|\widehat{D}^k \widehat{L}\| \le 2^{m-1} C_{k,m} \|\widehat{L}\|, \tag{26}$$

where $C_{k,m}$ are the estimates in (24).

Proof. Let $P = \widehat{L} \in \mathcal{P}(^m L_p(\mu);\mathbb{R})$. If $\widetilde{P} \in \mathcal{P}(^m L_p(\mu);\mathbb{C})$ is the unique extension of P on the complex $L_p(\mu)$-space, it follows from (23) that

$$\|\widehat{D}^k \widetilde{P}\| \le C_{k,m}\|\widetilde{P}\|.$$

If we use the first inequality in (25), we have

$$\|\widehat{D}^k \widehat{L}\| = \|\widehat{D}^k P\| \le \|\widehat{D}^k \widetilde{P}\| \le 2^{m-1} C_{k,m} \|\widehat{L}\|.$$

□

The estimate in (26) is far from optimal.

3.2 An application: Polarization constants of $L_p(\mu)$ spaces

Let $L \in \mathcal{L}^s(^m L_p(\mu))$. Consider first the case $1 \le p \le m'$. Using formula (3), from inequality (23) and the estimate in (24) with $k = 1$ of Proposition 3.4 we have

$$|L(x^{m-1}y)| = \frac{1}{m}|D\widehat{L}(x)y| \le \frac{m^{m/p-1}}{(m-1)^{(m-1)/p}}\|\widehat{L}\|.$$

Now an induction on m implies that

$$\|L\| \le \frac{m^{m/p}}{m!}\|\widehat{L}\|, \quad \text{for every } L \in \mathcal{L}^s(^m L_p(\mu)),\ 1 \le p \le m'$$

and so $\mathbb{C}(m,p) \le m^{m/p}/m!$. If $m \le p \le \infty$, using the estimate in (24) a similar argument shows that $\mathbb{C}(m,p) \le m^{m/p'}/m!$. Finally, we consider the case $L \in \mathcal{L}^s(^m L_p(\mu))$, $m' \le p \le m$. Using the estimate in (24), an induction on m gives

$$\|L\| \le \frac{m^{m/m'}}{m!}\|\widehat{L}\| = \frac{m^{(m-1)}}{m!}\|\widehat{L}\|.$$

Notice that for the induction argument in the case $m' \le p \le 2$ we need to consider $m' \le p \le (m-1)'$ and $(m-1)' \le p \le 2$, while in the case $2 \le p \le m$ we need to consider $2 \le p \le m-1$ and $m-1 \le p \le m$, $m \ge 3$. We have proved the following result.

Proposition 3.6. *For the mth polarization constant $\mathbb{C}(m,p)$, $m \ge 2$, we have the estimates*

$$\mathbb{C}(m,p) \le \begin{cases} \frac{m^{m/p}}{m!} & 1 \le p \le m', \\ \frac{m^{m/m'}}{m!} & m' \le p \le m, \\ \frac{m^{m/p'}}{m!} & m \le p \le \infty. \end{cases} \tag{27}$$

Using the polarization formula (4) and inequality (21) we can show that the estimates in (27) hold for *complex* as well as for *real* $L_p(\mu)$ spaces.

Proposition 3.7. *For the mth polarization constant $\mathbb{K}(m,p)$, $m \ge 2$, we have the estimates*

$$\mathbb{K}(m,p) \le \begin{cases} \frac{m^{m/p}}{m!} & 1 \le p \le m', \\ \frac{m^{m/m'}}{m!} & m' \le p \le m, \\ \frac{m^{m/p'}}{m!} & m \le p \le \infty. \end{cases} \tag{28}$$

In the case $1 \le p \le m'$ the estimate is best possible.

Proof. Let $x_i \in L_p(\mu)$, $1 \le i \le m$, be unit vectors. From the polarization formula (4) we have

$$|L(x_1,\dots,x_m)| \le \frac{\|\widehat{L}\|}{m!}\int_0^1 \Big\|\sum_{i=1}^m r_i(t)x_i\Big\|_p^m \, dt\,.$$

Since $1 \le p \le m' \Leftrightarrow m \le p' \le \infty$, using Hölder's inequality first and then inequality (21), the previous inequality gives the first estimate in (28) (we also refer to the proof of Theorem 2 in [51]). The proof of the third estimate in (28) is similar. In particular, for $p = m$ we have

$$\mathbb{K}(m,m) \le \mathbb{K}(m,m') = \frac{m^{m/m'}}{m!} = \frac{m^{m-1}}{m!}\,.$$

Finally, consider the case $m' \le p \le m$. Since $\mathbb{K}(m,p)$, as a function of p, is decreasing on the interval $[1,2]$ and increasing for $p \ge 2$ (see [20]), for every $p \in [m',m]$ we have $\mathbb{K}(m,p) \le m^{m/m'}/m!$.

To see that the estimate $m^{m/p}/m!$ is best possible in the case $1 \le p \le m'$, we consider $L \in \mathcal{L}^s({}^m\ell_p;\mathbb{K})$ defined in Example 3.1. We have $\|L\| \ge L(e_1,\dots,e_m) = 1/m!$ and $\|\widehat{L}\| = 1/m^{m/p}$. Since by (28) $\mathbb{K}(m,p) \le m^{m/p}/m!$, we conclude that $\|L\| = L(e_1,\dots,e_m) = 1/m!$ and

$$\|L\| = \frac{m^{m/p}}{m!}\|\widehat{L}\|\,.$$

Thus, $\mathbb{K}(m,p) = m^{m/p}/m!$. □

(i) Special case $m = 2$.
From (28) and for $1 \le p \le \infty$ we have the estimate

$$\mathbb{K}(2,p) \le 2^{|p-2|/p}\,.$$

For $1 \le p \le 2$ we have $\mathbb{K}(2,p) = 2^{(2-p)/p}$. In fact, in the case $1 \le p \le m'$ it follows from Proposition 3.7 that $\mathbb{K}(m,p) = m^{m/p}/m!$, for every $m \ge 2$. Therefore, for $m = 2$ and for $1 \le p \le 2$ we have $\mathbb{K}(2,p) = 2^{2/p}/2! = 2^{(2-p)/p}$. To prove equality, as in Example 3.1 we consider the 2-homogeneous polynomial $\widehat{L}(x) = x_1x_2$ with $L(x,y) = \frac{1}{2}(x_1y_2 + x_2y_1)$, $x = (x_i)$, $y = (y_i)$, the corresponding symmetric bilinear form on the real or complex ℓ_p space, $1 \le p \le 2$. Since $\|L\| \ge |L(e_1,e_2)| = 1/2$ and for $x = (2^{-1/p}, 2^{-1/p}, 0, \dots)$ with $\|x\|_p = 1$, $\|\widehat{L}\| = |\widehat{L}(x)| = 2^{-2/p}$, we have $\|L\| \ge 2^{(2-p)/p}\|\widehat{L}\|$. But $\mathbb{K}(2,p) \le 2^{(2-p)/p}$ and so $\|L\| = 2^{(2-p)/p}\|\widehat{L}\|$.

For $2 \le p \le \infty$ we have $\mathbb{R}(2,p) = 2^{(p-2)/p}$ and in particular $\mathbb{R}(2,\infty) = 2$. To see this consider the 2-homogeneous polynomial $\widehat{L}(x) = x_1^2 - x_2^2$ with $L(x,y) = x_1y_1 - x_2y_2$, $x = (x_i)$, $y = (y_i)$, the corresponding symmetric bilinear form on the *real* ℓ_p space. Obviously $\|\widehat{L}\| = |\widehat{L}(e_1)| = 1$. On the other hand, for $x = (2^{-1/p}, 2^{-1/p}, 0, \dots)$ and $y = (2^{-1/p}, -2^{-1/p}, 0, \dots)$ we have $\|x\|_p = \|y\|_p = 1$ and $L(x,y) = 2^{1-2/p}$. Hence, $\|L\| = 2^{(p-2)/p}\|\widehat{L}\|$.

(ii) Case $p \ge m \ge 3$.
In this case the constant $\frac{m^{m/p'}}{m!}$ in (28) can be improved. It has been shown in [20] that for $p \ge m \ge 3$,

$$\mathbb{K}(m,p) \le \frac{(2m)^{m/2}}{m!}\left(\frac{\Gamma\left(\frac{1}{2}(p+1)\right)}{\sqrt{\pi}}\right)^{m/p}, \tag{29}$$

where Γ is the *gamma function*. For example, in the special case $p = m = 4$ inequality (29) gives

$$\mathbb{K}(4,4) \le \frac{8^2}{4!}\cdot\frac{\Gamma(5/2)}{\sqrt{\pi}} = 2\,.$$

Since $p' = 4/3$ is the conjugate exponent of $p = 4$, from (28) we have the estimate $\mathbb{K}(4,4) \le \frac{4^3}{4!} = \frac{8}{3}$ which is bigger than 2.

For the *complex* ℓ_∞ Harris [30, (16)], see also [24, Proposition 1.43], has shown that

$$\mathbb{C}(m,\infty) = \mathbb{C}(m,\ell_\infty) \le \frac{m^{m/2}(m+1)^{(m+1)/2}}{2^m m!}$$

and this upper estimate is smaller than $m^m/m!$. Tonge has also proved the same result by using a method very similar to the method which was used to prove that the *complex* Grothendieck constant $G(2)$ is bounded above by $\frac{3}{4}\sqrt{3}$, see [56].

Remark 2. Harris [30, Theorem 6] showed that if $1 \le p \le \infty$ and m is a power of 2, then

$$\mathbb{C}(m,p) \le (m^m/m!)^{|p-2|/p}\,. \tag{30}$$

He has also conjectured that (30) holds for all positive integers m and that the constant given is best possible. But, as we have stated in Proposition 3.7, in the case $1 \le p \le m'$, $m \ge 3$, the best constant is $\mathbb{C}(m,p) = m^{m/p}/m!$ and this is strictly less than $(m^m/m!)^{(2-p)/p}$. Observe that for $m = 2$

$$\mathbb{C}(2,p) = \frac{2^{2/p}}{2!} = 2^{(2-p)/p}, \quad 1 \le p \le 2,$$

and this is the constant given in (30).

On the other hand, for $m' \le p \le 2$, where $m = 2^n$, $n \ge 2$, the constant given in (30) has been improved in [51, Theorem 3′]. But, in case p is close to 2, and for m a power of 2, Harris' bound is better than that of Proposition 3.6.

4 Bernstein's inequality for homogeneous polynomials on Hilbert spaces

A famous result, investigated by Banach [5] and many other authors, for example [15, 21, 30, 34, 36, 45], asserts that if H is a Hilbert space, then $\mathbb{K}(n,H)=1$. In other words, $\|L\|=\|\widehat{L}\|$ for every $L\in\mathcal{L}^s(^nH)$. Recall that L is a continuous symmetric n-linear form on a Hilbert space H and $\widehat{L}$ is the associated continuous n-homogeneous polynomial. Since the Fréchet derivative of $\widehat{L}$ at $x\in H$ is given by $D\widehat{L}(x)(y)=nL(x^{n-1}y)$, $y\in H$, where $L(x^{n-1}y):=L(\underbrace{x,\dots,x}_{n-1},y)$, to prove $\|L\|=\|\widehat{L}\|$ by an inductive argument, it suffices to show that $|L(x^{n-1}y)|\le\|\widehat{L}\|$ for any unit vectors x and y in H. In other words, $\|L\|=\|\widehat{L}\|$ for any $\widehat{L}\in\mathcal{P}(^nH)$ if and only if

$$\|D\widehat{L}\|\le n\|\widehat{L}\|\,,\qquad \forall\,\widehat{L}\in\mathcal{P}(^nH)\,. \tag{31}$$

Banach proved this result for continuous symmetric n-linear forms and continuous n-homogeneous polynomials on *finite dimensional real* Hilbert spaces. The proof works equally well for real and complex Hilbert spaces, and the condition of finite dimensionality is only needed to ensure that the n-linear form attains its norm. The result that $\|L\|=\|\widehat{L}\|$ is true for all Hilbert spaces, and, as pointed out by Banach, can be obtained through a simple limit argument based on the finite dimensional case.

Clearly, if $\widehat{L}$ attains its norm at $x_0\in B_H$, the closed unit ball of the Hilbert space H, then L also attains its norm at $(x_0,\dots,x_0)\in B_H^n$. When H is finite dimensional, L will always attain its norm, since the closed unit ball of H is compact. However, when H is infinite dimensional, L need not attain its norm: if $H=\ell_2$, the space of square summable sequences, and $L(x,y)=\sum_{n=1}^{\infty}\frac{n}{n+1}x_ny_n$, it is easy to see that $\|L\|=1$, but that $|L(x,y)|<1$ for all unit vectors $x=(x_n)$ and $y=(y_n)$ in H.

It is true, but not obvious, that if L attains its norm at $(x_1,\dots,x_n)\in B_H^n$, then $\widehat{L}$ also attains its norm at some $x_0\in B_H$. When L does attain its norm, an explicit construction has been given in [45, section 2] to provide a unit vector x_0 with $\|\widehat{L}\|=|\widehat{L}(x_0)|$.

Theorem 4.1. *[45, Theorem 2.1] If L is a norm attaining continuous symmetric n-linear form, $n\ge 2$, on a Hilbert space, then the associated continuous symmetric n-homogeneous polynomial $\widehat{L}$ also attains its norm. Moreover, $\|L\|=\|\widehat{L}\|$.*

For *real* Hilbert spaces it is an interesting fact, see [30, Theorem 4], that the Bernstein-type inequality (31) is equivalent to Szegö's inequality for *real* trigonometric polynomials (see [21]). That is, if $T(t)=\sum_{k=-n}^{n}c_ke^{ikt}$, $c_{-k}=\bar{c}_k$, is a real trigonometric polynomial of degree n which satisfies $|T(t)|\le 1$ for all real t, then

$$n^2T(t)^2+T'(t)^2\le n^2\,,\quad \forall\, t\in\mathbb{R}\,. \tag{32}$$

But Szegö's inequality (32) is a special case of a more general inequality for entire functions of exponential type. Recall that an entire function $f:\mathbb{C}\to\mathbb{C}$ is of *exponential type* (EFET) if for some $A>0$ the inequality

$$M_f(r):=\max_{|z|=r}|f(z)|<e^{Ar}$$

holds for sufficiently large values of r. The greatest lower bound for those values of A for which the latter asymptotic inequality is fulfilled is called the *type* $\sigma=\sigma_f$ of the function f. It follows from the definition of the type that

$$\sigma_f=\limsup_{r\to\infty}\frac{\log M_f(r)}{r}\,.$$

For example, if $T(t)=\sum_{k=-n}^{n}c_ke^{ikt}$ is a trigonometric polynomial of degree $\le n$, then $T(z)=\sum_{k=-n}^{n}c_ke^{ikz}$ is an EFET of type $\le n$. A classical theorem due to Bernstein [11] states that if f is an EFET of type $\le\sigma$, then f satisfies the inequality

$$\sup_{t\in\mathbb{R}}|f'(t)|\le\sigma\sup_{t\in\mathbb{R}}|f(t)|\,.$$

The following theorem, see [2] or inequality (11.4.5) in [14], contains Bernstein's inequality as a special case.

Theorem 4.2. *Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function of exponential type $\le\sigma$ and let $\sup_{t\in\mathbb{R}}|f(t)|<\infty$. Then for all $\omega\in\mathbb{R}$*

$$\sup_{t\in\mathbb{R}}|f'(t)\cos\omega+\sigma f(t)\sin\omega|\le\sigma\sup_{t\in\mathbb{R}}|f(t)|\,. \tag{33}$$

Equality holds in (33) if and only if $f(z)=ae^{i\sigma z}+be^{-i\sigma z}$, where $a,b\in\mathbb{C}$.

In particular, if T is a *real* trigonometric polynomial of degree n with $|T(t)|\le 1$ for all real t, inequality (33) implies Szegö's inequality (32).

We prove now that the Bernstein-type inequality (31) on real or complex Hilbert spaces can be easily derived from inequality (33)(cf. [4, Theorem 2.2]).

Theorem 4.3. *Let $(H,\langle\cdot,\cdot\rangle)$ be a real or complex Hilbert space. If $P:H\to\mathbb{K}$ is a continuous polynomial of degree n and x is a unit vector in H, then*

$$\{n^2|P(x)|^2-|DP(x)x|^2+\|DP(x)\|^2\}^{1/2}\le n\|P\|\,. \tag{34}$$

In particular, if $P=\widehat{L}$ is a continuous n-homogeneous polynomial, then

$$\|D\widehat{L}\|\le n\|\widehat{L}\|\,.$$

In other words, $\|L\|=\|\widehat{L}\|$ for any $L\in\mathcal{L}^s(^nH)$.

Proof. Let x, y be orthogonal unit vectors in H and let $c \in \mathbb{K}$ satisfy $|c| = 1$. Then $T(t) := P(x \cos t + cy \sin t)$ is a trigonometric polynomial of degree $\leq n$. But $\|x \cos t + cy \sin t\| = 1$ and therefore $|T(t)| \leq \|P\|$, for any $t \in \mathbb{R}$. Since $T'(t) = DP(x \cos t + cy \sin t)(-x \sin t + cy \cos t)$, Bernstein's inequality (33), for $t = 0$, implies

$$|cDP(x)y \cos \omega + nP(x) \sin \omega| \leq n\|P\|, \quad \forall\, \omega \in \mathbb{R}.$$

By appropriate choice of c, $|c| = 1$ and $\omega \in \mathbb{R}$ we get

$$\{|DP(x)y|^2 + n^2|P(x)|^2\}^{1/2} \leq n\|P\|. \tag{35}$$

Now, let x be a fixed unit vector in H. Then, given a unit vector u in H it is possible to find a unit vector $y \in H$ orthogonal to x so that $u = \alpha x + \beta y$, where $|\alpha|^2 + |\beta|^2 = 1$. Since

$$|DP(x)u)|^2 = |\alpha DP(x)x + \beta DP(x)y|^2 \leq |DP(x)x|^2 + |DP(x)y|^2,$$

using (35) we have

$$\{|DP(x)u|^2 - |DP(x)x|^2 + n^2|P(x)|^2\}^{1/2} \leq n\|P\|, \quad \forall u \in S_H.$$

But $\|DP(x)\| = \sup_{\|u\|=1} |DP(x)u|$ and the proof of (34) follows.

If $P = \widehat{L}$ is a continuous n-homogeneous polynomial, then as a particular case of (3) $D\widehat{L}(x)x = n\widehat{L}(x)$ and (34) is equivalent to $\|D\widehat{L}(x)\| \leq n\|\widehat{L}\|$, for every $x \in S_H$. □

In 1990 Lomonosov [40] conjectured that Bernstein's inequality (31) for continuous 2-homogeneous polynomials characterizes *real* Hilbert spaces. Benítez and Sarantopoulos [9] proved this conjecture in 1993. In other words, it was shown that if X is a *real* Banach space, then $\|D\widehat{L}\| \leq 2\|\widehat{L}\|$ (or $\|L\| = \|\widehat{L}\|$) for any $\widehat{L} \in \mathcal{P}(^2X)$ if and only if X is a *real* Hilbert space.

However, Bernstein's inequality (31) for continuous homogeneous polynomials doesn't characterize *complex* Hilbert spaces. As it has been proved in [30], Bernstein's inequality for continuous homogeneous polynomials holds on the *complex* ℓ^2_∞, the 2-dimensional *complex* $C(K)$ space. This result cannot be extended to all $C(K)$ spaces. For instance, in [57] an example of a 2-homogeneous polynomial was given on the *complex* ℓ^3_∞ for which Bernstein's inequality fails. Recall that a $C(K)$ space is the Banach space of continuous functions on the compact Hausdorff space K, under the usual uniform norm. It is known that for any σ-finite measure μ the space $L_\infty(\mu)$ is isometric to a $C(K)$ space, see [3, Proposition 4.3.8(ii) and Theorem 4.3.7(Kelley [35])]. The simplest examples of $C(K)$ spaces are ℓ_∞ and $L_\infty[0,1]$.

Now we give another example of a *complex* Banach space for which Bernstein's inequality for continuous homogeneous polynomials does hold. For this we need the following result of Harris.

Proposition 4.4. *[30, Corollary 3] Let $(H, \langle\cdot,\cdot\rangle)$ be* complex *Hilbert space and let $P : H \to \mathbb{C}$ be a continuous polynomial of degree n. Then,*

$$|nP(x) - DP(x)x| + \|DP(x)\| \leq n\|P\|, \quad \forall x \in B_H. \tag{36}$$

Observe that $S(x) := nP(x) - DP(x)x$ is the sum of the first $n-1$ partial sums of the polynomial P.

Proposition 4.5. *If H is a complex Hilbert space, consider the complex Banach space $H \times \mathbb{C}$, with the supremum norm, which is a non-Hilbert space. Then,*

$$\|D\widehat{L}\| \leq n\|\widehat{L}\|, \qquad \forall\, \widehat{L} \in \mathcal{P}(^nH \times \mathbb{C}).$$

In other words, $\|L\| = \|\widehat{L}\|$ for any $L \in \mathcal{L}^s(^nH \times \mathbb{C})$.

Proof. Suppose $\dim(H) < \infty$. Any continuous n-homogeneous polynomial $\widehat{L}$ on $H \times \mathbb{C}$ can be written in the form

$$\widehat{L}(<x,z>) = z^n P\left(\frac{x}{z}\right), \quad \forall x \in H, z \in \mathbb{C},$$

where P is a polynomial of degree n on H. By the maximum modulus principle

$$\|\widehat{L}\| = \sup_{\|<x,z>\|=1} \left|z^n P\left(\frac{x}{z}\right)\right| = \sup_{\|x\|\leq 1} |P(x)| = \|P\|.$$

To prove

$$|D\widehat{L}(<x,z>)<y,w>| \leq n\|\widehat{L}\|, \quad \forall <x,z>, <y,w> \in B_{H\times\mathbb{C}},$$

by the maximum modulus principle is enough to show that

$$|D\widehat{L}(<x,1>)<y,1>| \leq n\|\widehat{L}\|, \quad \forall x,\, y \in B_H.$$

For this we need the following identity, which can be easily checked

$$D\widehat{L}(<x,1>)<y,1> = DP(x)y + nP(x) - DP(x)x.$$

Then, from inequality (36) it follows that

$$|D\widehat{L}(<x,1>)<y,1>| \leq n\|P\| = n\|\widehat{L}\|, \quad \forall x,\, y \in B_H.$$

Based on the finite dimensional case, a simple argument gives the proof in the case H is an arbitrary complex Hilbert space. □

Observe that in the special case $H = \mathbb{C}$, the space $H \times \mathbb{C}$ with the supremum norm is just the complex space ℓ^2_∞.

Problem. Characterize the *complex* Banach spaces X for which Bernstein's inequality holds for any continuous homogeneous polynomial on X. That is, the *complex* Banach spaces X which share the property

$$\|D\widehat{L}\| \leq n\|\widehat{L}\| \Leftrightarrow \|L\| = \|\widehat{L}\|, \quad \forall\, \widehat{L} \in \mathcal{P}(^nX).$$

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