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Some Open Problems Concerning Orthogonal Polynomials on Fractals and Related Questions

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Abstract

We discuss several open problems related to analysis on fractals: estimates of the Green functions, the growth rates of the Markov factors with respect to the extension property of compact sets, asymptotics of orthogonal polynomials and the Parreau-Widom condition, Hausdorff measures and the Hausdorff dimension of the equilibrium measure on generalized Julia sets.

1 Background and notation

1.1 Chebyshev and orthogonal polynomials

Let $K \subset \mathbb{C}$ be a compact set containing infinitely many points. We use $\|\cdot\|_{L^\infty(K)}$ to denote the sup-norm on K, $\mathcal{M}_n$ is the set of all monic polynomials of degree n. The polynomial $T_{n,K}$ that minimizes $\|Q_n\|_{L^\infty(K)}$ for $Q_n \in \mathcal{M}_n$ is called the n-th *Chebyshev polynomial* on K.

Assume that the logarithmic capacity $\mathrm{Cap}(K)$ is positive. We define the n-th *Widom factor* for K by

$$W_n(K) := \|T_{n,K}\|_{L^\infty(K)}/\mathrm{Cap}(K)^n.$$

In what follows we consider probability Borel measures μ with non-polar compact support $\mathrm{supp}(\mu)$ in $\mathbb{C}$. The n-th monic orthogonal polynomial $P_n(z;\mu) = z^n + \dots$ associated with μ has the property

$$\|P_n(\cdot;\mu)\|^2_{L^2(\mu)} = \inf_{Q_n \in \mathcal{M}_n} \int |Q_n(z)|^2 \, d\mu(z),$$

where $\|\cdot\|_{L^2(\mu)}$ is the norm in $L^2(\mu)$. Then the n-th *Widom-Hilbert factor* for μ is

$$W_n^2(\mu) := \|P_n(\cdot;\mu)\|_{L^2(\mu)}/(\mathrm{Cap}(\mathrm{supp}(\mu)))^n.$$

If $\mathrm{supp}(\mu) \subset \mathbb{R}$ then a three-term recurrence relation

$$xP_n(x;\mu) = P_{n+1}(x;\mu) + b_{n+1}P_n(x;\mu) + a_n^2 P_{n-1}(x;\mu)$$

is valid for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The initial conditions $P_{-1}(x;\mu) \equiv 0$ and $P_0(x;\mu) \equiv 1$ generate two bounded sequences $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty$ of *recurrence coefficients* associated with μ. Here, $a_n > 0$, $b_n \in \mathbb{R}$ for $n \in \mathbb{N}$ and

$$\|P_n(\cdot;\mu)\|_{L^2(\mu)} = a_1 \cdots a_n.$$

A bounded two sided $\mathbb{C}$-valued sequence $(d_n)_{n=-\infty}^\infty$ is called *almost periodic* if the set $\{(d_{n+k})_{n=-\infty}^\infty : k \in \mathbb{Z}\}$ is precompact in $l^\infty(\mathbb{Z})$. A one sided sequence $(c_n)_{n=1}^\infty$ is called almost periodic if it is the restriction of a two sided almost periodic sequence to $\mathbb{N}$. A sequence $(e_n)_{n=1}^\infty$ is called *asymptotically almost periodic* if there is an almost periodic sequence $(e_n')_{n=1}^\infty$ such that $|e_n - e_n'| \to 0$ as $n \to 0$.

The class of Parreau-Widom sets plays a special role in the recent theory of orthogonal and Chebyshev polynomials. Let K be a non-polar compact set and $g_{\mathbb{C}\setminus K}$ denote the Green function for $\overline{\mathbb{C}} \setminus K$ with a pole at infinity. Suppose K is regular with respect to the Dirichlet problem, so the set $\mathcal{C}$ of critical points of $g_{\mathbb{C}\setminus K}$ is at most countable (see e.g. Chapter 2 in [9]). Then K is said to be a *Parreau-Widom* set if $\sum_{c\in\mathcal{C}} g_{\mathbb{C}\setminus K}(c) < \infty$. Parreau-Widom sets on $\mathbb{R}$ have positive Lebesgue measure. For different aspects of such sets, see [8, 15, 23].

The class of regular measures in the sense of Stahl-Totik can be defined by the following condition

$$\lim_{n\to\infty} W_n(\mu)^{1/n} = 1.$$

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For a measure μ supported on $\mathbb{R}$ we use the Lebesgue decomposition of μ with respect to the Lebesgue measure:

$$d\mu(x) = f(x)dx + d\mu_s(x).$$

Following [9], we define the Szegő class $\mathrm{Sz}(K)$ of measures on a given Parreau-Widom set $K \subset \mathbb{R}$. Let μ_K be the equilibrium measure on K. By $\operatorname{ess\,supp}(\cdot)$ we denote the essential support of the measure, that is the set of accumulation points of the support. We have $\mathrm{Cap}(\mathrm{supp}(\mu)) = \mathrm{Cap}(\operatorname{ess\,supp}(\mu))$, see Section 1 of [21]. A measure μ is in the *Szegő class* of K if
(i) $\operatorname{ess\,supp}(\mu) = K$.
(ii) $\int_K \log f(x)\, d\mu_K(x) > -\infty$. (Szegő condition)
(iii) the isolated points $\{x_n\}$ of $\mathrm{supp}(\mu)$ satisfy $\sum_n g_{\mathbb{C}\setminus K}(x_n) < \infty$.

By Theorem 2 in [9] and its proof, (ii) can be replaced by one of the following conditions:
(ii') $\limsup_{n\to\infty} W_n^2(\mu) > 0$. (Widom condition)
(ii'') $\liminf_{n\to\infty} W_n^2(\mu) > 0$. (Widom condition 2)
One can show that any $\mu \in \mathrm{Sz}(K)$ is regular in the sense of Stahl-Totik.

1.2 Generalized Julia sets and $K(\gamma)$

Let $(f_n)_{n=1}^\infty$ be a sequence of rational functions with $\deg f_n \geq 2$ in $\overline{\mathbb{C}}$ and $F_n := f_n \circ f_{n-1} \circ \ldots \circ f_1$. The domain of normality for $(F_n)_{n=1}^\infty$ in the sense of Montel is called the *Fatou set* for (f_n). The complement of the Fatou set in $\overline{\mathbb{C}}$ is called the *Julia set* for (f_n). We denote them by $F_{(f_n)}$ and $J_{(f_n)}$ respectively. These sets were considered first in [11]. In particular, if $f_n = f$ for some fixed rational function f for all n then $F_{(f)}$ and $J_{(f)}$ are used instead. To distinguish the last case, the word *autonomous* is used in the literature.

Suppose $f_n(z) = \sum_{j=0}^{d_n} a_{n,j} \cdot z^j$ where $d_n \geq 2$ and $a_{n,d_n} \neq 0$ for all $n \in \mathbb{N}$. Following [**?**], we say that (f_n) is a *regular polynomial sequence* (write $(f_n) \in \mathcal{R}$) if positive constants A_1, A_2, A_3 exist such that for all $n \in \mathbb{N}$ we have the following three conditions:
$|a_{n,d_n}| \geq A_1$
$|a_{n,j}| \leq A_2 |a_{n,d_n}|$ for $j = 0, 1, \ldots, d_n - 1$
$\log |a_{n,d_n}| \leq A_3 \cdot d_n$
For such polynomial sequences, by [**?**], $J_{(f_n)}$ is a regular compact set in $\mathbb{C}$, so $\mathrm{Cap}(J_{(f_n)})$ is positive. In addition, $J_{(f_n)}$ is the boundary of

$$\mathcal{A}_{(f_n)}(\infty) := \{z \in \overline{\mathbb{C}} : F_n(z) \text{ goes locally uniformly to } \infty\}.$$

The following construction is from [12]. Let $\gamma := (\gamma_k)_{k=1}^\infty$ be a sequence provided that $0 < \gamma_k < 1/4$ holds for all $k \in \mathbb{N}$ and $\gamma_0 := 1$. Let $f_1(z) = 2z(z-1)/\gamma_1 + 1$ and $f_n(z) = \frac{1}{2\gamma_n}(z^2 - 1) + 1$ for $n > 1$. Then $K(\gamma) := \cap_{s=1}^\infty F_s^{-1}([-1,1])$ is a Cantor set on $\mathbb{R}$. Furthermore, $F_s^{-1}([-1,1]) \subset F_t^{-1}([-1,1]) \subset [0,1]$ whenever $s > t$.

Also we use an expanded version of this set. For a sequence γ as above, let $f_n(z) = \frac{1}{2\gamma_n}(z^2 - 1) + 1$ for $n \in \mathbb{N}$. Then $K_1(\gamma) := \cap_{s=1}^\infty F_s^{-1}([-1,1]) \subset [-1,1]$ and $F_s^{-1}([-1,1]) \subset F_t^{-1}([-1,1]) \subset [-1,1]$ provided that $s > t$. If there is a c with $0 < c < \gamma_k$ for all k then $(f_n) \in \mathcal{R}$ and $J_{(f_n)} = K_1(\gamma)$, see [5]. If $\gamma_1 = \gamma_k$ for all $k \in \mathbb{N}$ then $K_1(\gamma)$ is an autonomous polynomial Julia set.

1.3 Hausdorff measure

A function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *dimension function* if it is increasing, continuous and $h(0) = 0$. Given a set $E \subset \mathbb{C}$, its *h-Hausdorff measure* is defined as

$$\Lambda_h(E) = \lim_{\delta\to 0} \inf \left\{ \sum h(r_j) : E \subset \bigcup B(z_j, r_j) \text{ with } r_j \leq \delta \right\},$$

where $B(z, r)$ is the open ball of radius r centered at z. For a dimension function h, a set $K \subset \mathbb{C}$ is an *h-set* if $0 < \Lambda_h(K) < \infty$. To denote the Hausdorff measure for $h(t) = t^\alpha$, Λ_α is used. *Hausdorff dimension* of K is defined as $\mathrm{HD}(K) = \inf\{\alpha \geq 0 : \Lambda_\alpha(K) = 0\}$.

2 Smoothness of Green functions and Markov Factors

The next set of problems is concerned with the smoothness properties of the Green function $g_{\mathbb{C}\setminus K}$ near compact set K and related questions. We suppose that K is regular with respect to the Dirichlet problem, so the function $g_{\mathbb{C}\setminus K}$ is continuous throughout $\mathbb{C}$. The next problem was posed in [12].

Problem 1. Given modulus of continuity ω, find a compact set K such that the modulus of continuity $\omega(g_{\mathbb{C}\setminus K}, \cdot)$ is similar to ω.

Here, one can consider similarity either as coincidence of the values of moduli of continuity on some null sequence or in the sense of weak equivalence: $\exists C_1, C_2$ such that

$$C_1\, \omega(\delta) \leq \omega(g_{\mathbb{C}\setminus K}, \delta) \leq C_2\, \omega(\delta)$$

for sufficiently small positive δ.

We guess that a set $K(\gamma)$ from [12] is a candidate for the desired K provided a suitable choice of the parameters. We recall that, for many moduli of continuity, the corresponding Green functions were given in [12], whereas the characterization of optimal smoothness for $g_{\mathbb{C}\setminus K(\gamma)}$ is presented in [[5], Th.6.3].

A stronger version of the above problem concerns with the pointwise estimation of the Green function:

Problem 2. Given modulus of continuity ω, find a compact set K such that

$$C_1\,\omega(\delta) \le g_{\mathbb{C}\setminus K}(z) \le C_2\,\omega(\delta)$$

for $\delta = dist(z,K) \le \delta_0$, where C_1, C_1 and δ_0 do not depend on z.

In the most important case we get a problem of "two-sided Hölder" Green function, which was posed by A. Volberg on his seminar (quoted with permission):

Problem 3. Find a compact set K on the line such that for some $\alpha > 0$ and constants C_1, C_2, if $\delta = dist(z,K)$ is small enough then

$$C_1\,\delta^\alpha \le g_{\mathbb{C}\setminus K}(z) \le C_2\,\delta^\alpha. \tag{1}$$

Clearly, a closed analytic curve gives a solution for sets on the plane.

If $K \subset \mathbb{R}$ satisfies (1), then K is of Cantor-type. Indeed, if interior of K (with respect to $\mathbb{R}$) is not empty, let $(a,b) \subset K$, then $g_{\mathbb{C}\setminus K}$ has $Lip\,1$ behavior near the point $(a+b)/2$. On the other hand, near endpoints of K the function $g_{\mathbb{C}\setminus K}$ cannot be better than $Lip\,1/2$.

By the Bernstein-Walsh inequality, smoothness properties of the Green functions are closely related with a character of maximal growth of polynomials outside the corresponding compact sets, which, in turn, allows to evaluate the Markov factors for the sets. Recall that, for a fixed $n \in \mathbb{N}$ and (infinite) compact set K, the n−th *Markov factor* $M_n(K)$ is the norm of operator of differentiation in the space of holomorphic polynomials $\mathcal{P}_n$ with the uniform norm on K. In particular, the Hölder smoothness (the right inequality in (1)) implies the Markov property of the set K (a polynomial growth rate of $M_n(K)$). The problem of inverse implication (see e.g [20]) has attracted attention of many researches.

By W. Pleśniak [20], any Markov set $K \subset \mathbb{R}^d$ has the *extension property EP*, which means that there exists a continuous linear extension operator from the space of Whitney functions $\mathcal{E}(K)$ to the space of infinitely differentiable functions on $\mathbb{R}^d$. We guess that there is some extremal growth rate of M_n which implies the lack of EP. Recently it was shown in [14] that there is no complete characterization of EP in terms of growth rate of the Markov factors. Namely, two sets were presented, K_1 with EP and K_2 without it, such that $M_n(K_1)$ grows essentially faster than $M_n(K_2)$ as $n \to \infty$. Thus there exists non-empty zone of uncertainty where the growth rate of $M_n(K)$ is not related with EP of the set K.

Problem 4. Characterize the growth rates of the Markov factors that define the boundaries of the zone of uncertainty for the extension property.

3 Orthogonal polynomials

One of the most interesting problems concerning orthogonal polynomials on Cantor sets on $\mathbb{R}$ is the character of periodicity of recurrence coefficients. It was conjectured in p.123 of [7] that if f is a non-linear polynomial such that $J(f)$ is a totally disconnected subset of $\mathbb{R}$ then the recurrence coefficients for $\mu_{J(f)}$ are almost periodic. This is still an open problem. In [6], the authors conjectured that the recurrence coefficients for $\mu_{K(\gamma)}$ are asymptotically almost periodic for any γ. It may be hoped that a more general and slightly weaker version of Bellissard's conjecture can be valid.

Problem 5. Let (f_n) be a regular polynomial sequence such that $J_{(f_n)}$ is a Cantor-type subset of the real line. Prove that the recurrence coefficients for $\mu_{J(f_n)}$ are asymptotically almost periodic.

For a measure μ which is supported on $\mathbb{R}$, let $Z_n(\mu) := \{x : P_n(x;\mu) = 0\}$. We define $U_n(\mu)$ by

$$U_n(\mu) := \inf_{\substack{x,x'\in Z_n(\mu)\\ x\neq x'}} |x - x'|.$$

In [17] Krüger and Simon gave a lower bound for $U_n(\mu)$ depending on n where μ is the Cantor-Lebesgue measure of the (translated and scaled) Cantor ternary set. In [16], it was shown that Markov's inequality and spacing of the zeros of orthogonal polynomials are somewhat related.

Let $\gamma = (\gamma_k)_{k=1}^\infty$ and $n \in \mathbb{N}$ with $n > 1$ be given and define $\delta_k = \gamma_0 \cdots \gamma_k$ for all $k \in \mathbb{N}_0$. Let s be the integer satisfying $2^{s-1} \le n < 2^s$. By [2],

$$\delta_{s+2} \le U_n(\mu_{K(\gamma)}) \le \frac{\pi^2}{4}\cdot\delta_{s-2}$$

holds. In particular, if there is a number c such that $0 < c < \gamma_k < 1/4$ holds for all $k \in \mathbb{N}$ then, by [2], we have

$$c^2\cdot\delta_s \le U_n(\mu_{K(\gamma)}) \le \frac{\pi^2}{4c^2}\cdot\delta_s. \tag{2}$$

By [13], at least for small sets $K(\gamma)$, we have $M_{2^s}(K(\gamma)) \sim 2/\delta_s$, where the symbol $\sim$ means the strong equivalence.

Problem 6. Let K be a non-polar compact subset of $\mathbb{R}$. Is there a general relation between the zero spacing of orthogonal polynomials for μ_K and smoothness of $g_{\mathbb{C}\setminus K}$? Is there a relation between the zero spacing of μ_K and the Markov factors?

As mentioned in section 1, the Szegő condition and the Widom condition are equivalent for Parreau-Widom sets. Let K be a Parreau-Widom set. Let μ be a measure such that $\operatorname{ess\,supp}(\mu) = K$ and the isolated points $\{x_n\}$ of $\operatorname{supp}(\mu)$ satisfy $\sum_n g_{\mathbb{C}\setminus K}(x_n) < \infty$. Then, as it is discussed in Section 6 of [4], the Szegő condition is equivalent to the condition

$$\int_K \log(d\mu/d\mu_K)\, d\mu_K(x) > -\infty. \tag{3}$$

This condition is also equivalent to the Widom condition under these assumptions.

It was shown in [1] that $\inf_{n\in\mathbb{N}} W_n(\mu_K) \geq 1$ for non-polar compact $K \subset \mathbb{R}$. Thus the Szegő condition in the above form (3) and the Widom condition are related on arbitrary non-polar sets.

Problem 7. Let K be a non-polar compact subset of $\mathbb{R}$ which is regular with respect to the Dirichlet problem. Let μ be a measure such that $\operatorname{ess\,supp}(\mu) = K$. Assume that the isolated points $\{x_n\}$ of $\operatorname{supp}(\mu)$ satisfy $\sum_n g_{\mathbb{C}\setminus K}(x_n) < \infty$. If the condition (3) is valid for μ, is it necessarily true that the Widom condition or the Widom condition 2 holds? Conversely, does the Widom condition imply (3)?

It was proved in [10] that if K is a Parreau-Widom set which is a subset of $\mathbb{R}$ then $(W_n(K))_{n=1}^\infty$ is bounded above. On the other hand, $(W_n(K))_{n=1}^\infty$ is unbounded for some Cantor-type sets, see e.g. [13].

Problem 8. Is it possible to find a regular non-polar compact subset K of $\mathbb{R}$ which is not Parreau-Widom but $(W_n(K))_{n=1}^\infty$ is bounded? If K has zero Lebesgue measure then is it true that $(W_n(K))_{n=1}^\infty$ is unbounded? We can ask the same problems if we replace $(W_n(K))_{n=1}^\infty$ by $(W_n^2(\mu_K))_{n=1}^\infty$ above.

Let T_N be a real polynomial of degree N with $N \geq 2$ such that it has N real and simple zeros $x_1 < \cdots < x_n$ and $N-1$ critical points $y_1 < \cdots < y_{n-1}$ with $|T_N(y_i)| \geq 1$ for each $i \in \{1, \ldots, N-1\}$. We call such a polynomial *admissible*. If $K = T_N^{-1}([-1,1])$ for an admissible polynomial T_N then K is called a *T-set*. The following result is well known, see e.g. [22].

Theorem 3.1. *Let $K = \cup_{j=1}^n [\alpha_j, \beta_j]$ be a union of n disjoint intervals such that α_1 is the leftmost end point. Then K is a T-set if and only if $\mu_K([\alpha_1, c])$ is in $\mathbb{Q}$ for all $c \in \mathbb{R}\setminus K$.*

For $K(\gamma)$, it is known that $\mu_{K(\gamma)}([0,c]) \in \mathbb{Q}$ if $c \in \mathbb{R}\setminus K(\gamma)$, see Section 4 in [2].

Problem 9. Let K be a regular non-polar compact subset of $\mathbb{R}$ and α be the leftmost end point of K. Let $\mu_K([\alpha, c]) \in \mathbb{Q}$ for all $c \in \mathbb{R}\setminus K$. What can we say about K? Is it necessarily a polynomial generalized Julia set? Does this imply that there is a sequence of admissible polynomials $(f_n)_{n=1}^\infty$ such that $(F_n^{-1}[-1,1])_{n=1}^\infty$ is a decreasing sequence of sets such that $K = \cap_{n=1}^\infty F_n^{-1}[-1,1]$?

4 Hausdorff measures

It is valid for a wide class of Cantor sets that the equilibrium measure and the corresponding Hausdorff measure on this set are mutually singular, see e.g. [18].

Let $\gamma = (\gamma_k)_{k=1}^\infty$ with $0 < \gamma_k < 1/32$ satisfy $\sum_{k=1}^\infty \gamma_k < \infty$. This implies that $K(\gamma)$ has Hausdorff dimension 0. In [3], the authors constructed a dimension function h_γ that makes $K(\gamma)$ an h-set. Provided also that $K(\gamma)$ is not polar it was shown that there is a $C > 0$ such that for any Borel set B,

$$C^{-1} \cdot \mu_{K(\gamma)}(B) < \Lambda_{h_\gamma}(B) < C \cdot \mu_{K(\gamma)}(B)$$

and in particular the equilibrium measure and Λ_{h_γ} restricted to $K(\gamma)$ are mutually absolutely continuous. In [14], it was shown that indeed these two measures coincide. To the best of our knowledge, this is the first example of a subset of $\mathbb{R}$ such that the equilibrium measure is a Hausdorff measure restricted to the set.

Problem 10. Let K be a non-polar compact subset of $\mathbb{R}$ such that μ_K is equal to a Hausdorff measure restricted to K. Is it necessarily true that the Hausdorff dimension of K is 0?

Hausdorff dimension of a probability Borel measure μ supported on $\mathbb{C}$ is defined by $\dim(\mu) := \inf\{\mathrm{HD}(K) : \mu(K) = 1\}$ where $\mathrm{HD}(\cdot)$ denotes Hausdorff dimension of the given set. For polynomial Julia sets which are totally disconnected there is a formula for $\dim(\mu_{J(f)})$, see e.g.p. 23 in [18] and p.176-177 in [20].

Problem 11. Is it possible to find simple formulas for $\dim\left(\mu_{J_{(f_n)}}\right)$ where (f_n) is a regular polynomial sequence?

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