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Markov's inequality on some cuspidal domains in the L^p norm

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Abstract

In this note we give an example of a cuspidal set for which the exact value of Markov's exponent is calculated. More precisely, we show that if

$$\Omega = \{(x, y) \in \mathbb{R}^2 \colon |x| \le y + 1,\, x^2 \ge 4y\},$$

then the Markov exponent for Ω, with respect to L^p norm, is equal to 4.

1 Introduction

For a Lebesgue-measurable set $E \subset \mathbb{R}^N$, a bounded real-valued function w defined on E, $1 \le p < \infty$ and $h : E \to \mathbb{R}$ for which the p-th power of the absolute value is Lebesgue integrable, we set

$$\|h\|_{L^p(E,w)} = \left(\int_E |h(x)|^p |w(x)|\, dx \right)^{1/p}.$$

If $w \equiv 1$ on E. Instead of writing $\|h\|_{L^p(E,w)}$, let us write $\|h\|_{L^p(E)}$.

Let $\mathcal{P}(\mathbb{R}^N)$ denote the space of algebraic polynomials of N real variables. Moreover, $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Definition 1.1. We say that a compact set $\emptyset \ne E \subset \mathbb{R}^N$ satisfies Markov's inequality (or: is a Markov set) if there exist $M, r > 0$ such that, for each polynomial $P \in \mathcal{P}(\mathbb{R}^N)$ and each $\alpha \in \mathbb{N}_0^N$,

$$\|D^\alpha P\|_E \le (M(\deg P)^r)^{|\alpha|} \|P\|_E, \tag{1}$$

where $\|\cdot\|_E$ is the supremum norm on E, $D^\alpha P = \frac{\partial^{|\alpha|} P}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$.

Clearly, by iteration, it is enough to consider in the above definition multi-indices α with $|\alpha| = 1$. We consider the following generalization of Markov's inequality:

Definition 1.2. Let $1 \le p < \infty$. We say that a compact set $\emptyset \ne E \subset \mathbb{R}^N$ satisfies L^p Markov type inequality (or: is a L^p Markov set) if there exist $\kappa, C > 0$ such that, for each polynomial $P \in \mathcal{P}(\mathbb{R}^N)$ and each $\alpha \in \mathbb{N}_0^N$,

$$\|D^\alpha P\|_{L^p(E)} \le (C(\deg P)^\kappa)^{|\alpha|} \|P\|_{L^p(E)}. \tag{2}$$

The inequalities (1) and (2) are natural generalizations of the classical Markov inequality proved by A.A. Markov in 1889. Markov's inequality and its various generalizations (restricted not only to nonpluripolar subsets of $\mathbb{R}^N$ or $\mathbb{C}^N$ but also their versions for pieces of semialgebraic sets or other "small" subsets of $\mathbb{R}^N$ ($\mathbb{C}^N$)) found many applications in approximation theory, analysis, constructive function theory, but also in other branches of science (for example, in physics or chemistry). Markov's inequality is still an active and fruitful area of approximation theory (see, for instance, [5, 7, 16]). For a given $E \subset \mathbb{R}^N$, an important problem is to determine $\mu(E)$, where $\mu(E) = \inf\{r : E \text{ satisfies } (1)\}$ is the Markov exponent of E (see [4] for more details on this matter). This is related to the linear extension operator for C^∞ functions with restricted growth of derivatives (see [22, 23]). For any compact set E in $\mathbb{R}^N$ we have $\mu(E) \ge 2$. If E is a fat convex subset of $\mathbb{R}^N$, then $\mu(E) = 2$. If $E = \{(x, y) \in \mathbb{R}^2 \colon 0 \le x \le 1,\, 0 \le y \le x^l\}$, for $l \ge 1$, then by [11] $\mu(E) = 2l$. The interested reader may refer to [8, 11, 20, 24, 25] for more on this topic. Markov's inequality was also considered in the L^p norm (see [1, 6, 12, 13, 14, 15]). The problem of determining the L^p Markov exponent $\mu_p(E)$ for a L^p Markov set E seems to be more complicated. Here $\mu_p(E) := \inf\{\kappa : E \text{ satisfies } (2)\}$. In particular, to the best of our knowledge there is no example of a bounded set in $\mathbb{R}^N$ with cusps for which L^p Markov exponent (with respect to the Lebesgue measure) is known. Attempts to solve this problem led, among others, to the so-called Milówka-Ozorka identity (see [3, 21] for discussion). The purpose of this note is to give such an example. More precisely, we show that if $1 \le p < \infty$, then $\mu_p(\Omega) = 4$, where $\Omega = \{(x, y) \in \mathbb{R}^2 \colon |x| \le y + 1,\, x^2 \ge 4y\}$ which is depicted in Figure 1. Since T. Koornwinder has constructed orthogonal polynomials on Ω (see [17, 18]) we call this set the Koornwinder domain.

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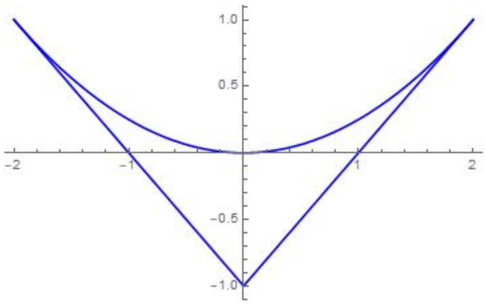


Figure 1: The Koornwinder domain.

2 Some weighted polynomial inequalities on simplex

The following lemma will be particularly useful in the proof of our main result.

Lemma 2.1. *Let* $S = \{(x_1, x_2) \in \mathbb{R}^2 \colon -1 \le x_1 \le x_2 \le 1\}$, $w(x_1, x_2) = x_2 - x_1$ *and* $1 \le p < \infty$. *Then there exist constants* $C, \tilde{C} > 0$ *such that*

$$\left\| w \frac{\partial P}{\partial x_i} \right\|_S \le C (\deg P)^2 \|wP\|_S \tag{3}$$

$$\left\| \frac{\partial P}{\partial x_i} \right\|_{L^p(S,w)} \le \tilde{C} (\deg P)^2 \|P\|_{L^p(S,w)} \tag{4}$$

for every $P \in \mathcal{P}(\mathbb{R}^2)$ *and* $i = 1, 2$.

Proof. First consider the inequality (3). By Wilhelmsen's result (see Theorem 3.1 in [27]), we have

$$\max\left\{ \left\| \frac{\partial P}{\partial x_1} \right\|_S, \left\| \frac{\partial P}{\partial x_2} \right\|_S \right\} \le \frac{4(\deg P)^2}{\delta_S} \|P\|_S, \tag{5}$$

where δ_S is the width of the convex body (the minimal distance between parallel supporting hyperplanes). Applying (5) to the polynomial wP yields

$$\max\left\{ \left\| w \frac{\partial P}{\partial x_1} - P \right\|_S, \left\| w \frac{\partial P}{\partial x_2} + P \right\|_S \right\} \le \frac{4(\deg P + 1)^2}{\delta_S} \|wP\|_S,$$

Then (see Lemma 3 of [13]) there is a constant $\kappa > 0$ such that

$$\left\| w \frac{\partial P}{\partial x_i} \right\|_S \le \frac{(\kappa \delta_S + 4)(\deg P + 1)^2}{\delta_S} \|wP\|_S \tag{6}$$

for all $P \in \mathcal{P}(\mathbb{R}^2)$ and $i = 1, 2$. Hence we may conclude that (3) holds.

For each $1 \le p < \infty$ it is clear that

$$\left\| \frac{\partial P}{\partial x_i} \right\|_{L^p(S,w)} \le \sum_{j=0}^{2} \left(\int_{D_j} \left| \frac{\partial P}{\partial x_i}(x_1, x_2) \right|^p (x_2 - x_1)\, dx_1 dx_2 \right)^{1/p},$$

where

$$D_0 = \{(x_1, x_2) \in \mathbb{R}^2 \colon -1 \le x_1 \le 0,\ x_1 + 1 \le x_2 \le 1\},$$
$$D_1 = \{(x_1, x_2) \in \mathbb{R}^2 \colon -1 \le x_1 \le 0,\ x_1 \le x_2 \le x_1 + 1\},$$
$$D_2 = \{(x_1, x_2) \in \mathbb{R}^2 \colon 0 \le x_2 \le 1,\ x_2 - 1 \le x_1 \le x_2\}.$$

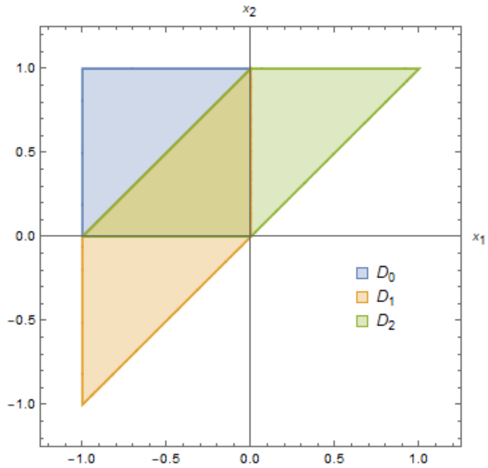
We shall show that there is a constant $\tilde{C} > 0$ such that

$$\left(\int_{D_j} \left| \frac{\partial P}{\partial x_i}(x_1, x_2) \right|^p (x_2 - x_1)\, dx_1 dx_2 \right)^{1/p} \le \tilde{C} (\deg P)^2 \|P\|_{L^p(S,w)} \tag{7}$$

for all $P \in \mathcal{P}(\mathbb{R}^2)$ and $j = 0, 1, 2$.

By the definition of D_0 it is clear that

$$\left(\int_{D_0} \left| \frac{\partial P}{\partial x_i}(x_1, x_2) \right|^p (x_2 - x_1)\, dx_1 dx_2 \right)^{1/p} \le 2 \left(\int_{D_0} \left| \frac{\partial P}{\partial x_i}(x_1, x_2) \right|^p dx_1 dx_2 \right)^{1/p} \tag{8}$$



Using the main result of [12] (see also the work in [9, 13, 19]) there is a constant C_0 so that

$$\left(\int_{D_0}\left|\frac{\partial P}{\partial x_i}(x_1,x_2)\right|^p dx_1dx_2\right)^{1/p} \le C_0(\deg P)^2\left(\int_{D_0}|P(x_1,x_2)|^p\, dx_1dx_2\right)^{1/p} \tag{9}$$

Since $w \ge 1$ on D_0, we obtain

$$\left(\int_{D_0}|P(x_1,x_2)|^p\, dx_1dx_2\right)^{1/p} \le \left(\int_{D_0}|P(x_1,x_2)|^p\,(x_2-x_1)\,dx_1dx_2\right)^{1/p} \tag{10}$$

By the inequalities (8), (9) and (10) it follows that

$$\left(\int_{D_0}\left|\frac{\partial P}{\partial x_i}(x_1,x_2)\right|^p (x_2-x_1)\,dx_1dx_2\right)^{1/p} \le 2C_0(\deg P)^2\|P\|_{L^p(D_0,w)} \le 2C_0(\deg P)^2\|P\|_{L^p(S,w)}.$$

Now consider the case $j = 1$. So we need to examine

$$\left(\int_{D_1}\left|\frac{\partial P}{\partial x_i}(x_1,x_2)\right|^p (x_2-x_1)\,dx_1dx_2\right)^{1/p}.$$

By making the change of variables $t = x_1$, $s = x_2 - x_1$, we obtain

$$\left(\int_{D_1}\left|\frac{\partial P}{\partial x_i}(x_1,x_2)\right|^p (x_2-x_1)\,dx_1dx_2\right)^{1/p} = \left(\int_{-1}^{0}\int_{0}^{1}\left|\frac{\partial P}{\partial x_i}(t,s+t)\right|^p s\,ds\,dt\right)^{1/p}.$$

If we define the polynomial Q by setting $Q(t,s) = P(t,s+t)$, then

$$\frac{\partial Q}{\partial t}(t,s) - \frac{\partial Q}{\partial s}(t,s) = \frac{\partial P}{\partial x_1}(t,s+t), \quad \frac{\partial Q}{\partial s}(t,s) = \frac{\partial P}{\partial x_2}(t,s+t). \tag{11}$$

Using the result of [10] (see Theorem 3), we can show that

$$\int_0^1\left|\frac{\partial P}{\partial x_2}(t,s+t)\right|^p s\,ds \le C_1^p(\deg Q)^{2p}\int_0^1|Q(t,s)|^p\,s\,ds. \tag{12}$$

Therefore

$$\begin{aligned}\left\|\frac{\partial P}{\partial x_2}\right\|_{L^p(D_1,w)} &\le \left(\int_{-1}^{0}\left[C_1^p(\deg P)^{2p}\int_0^1|Q(t,s)|^p\,s\,ds\right]dt\right)^{1/p}\\ &= C_1(\deg P)^2\left(\int_{-1}^{0}\int_0^1|P(t,s+t)|^p\,s\,ds\,dt\right)^{1/p} \le C_1(\deg P)^2\|P\|_{L^p(S,w)}.\end{aligned}$$

On the other hand,

$$\left\|\frac{\partial P}{\partial x_1}\right\|_{L^p(D_1,w)} \le \left(\int_{-1}^{0}\int_0^1\left|\frac{\partial Q}{\partial t}(t,s)\right|^p s\,ds\,dt\right)^{1/p} + \left(\int_{-1}^{0}\int_0^1\left|\frac{\partial Q}{\partial s}(t,s)\right|^p s\,ds\,dt\right)^{1/p}.$$

Using the result of [15] (see theorem in sec. 3), we can show that there exists constant $\hat{C}_1$ such that

$$\int_{-1}^{0}\left|\frac{\partial Q}{\partial t}(t,s)\right|^p dt \le \hat{C}_1^p(\deg Q)^{2p}\int_{-1}^{0}|Q(t,s)|^p\, dt \tag{13}$$

for every polynomial $Q \in \mathcal{P}(\mathbb{R}^2)$. By (11), (12) and (13), we see that

$$\left\|\frac{\partial P}{\partial x_1}\right\|_{L^p(D_1,w)} \le \left(\int_0^1\left[\hat{C}_1^p(\deg P)^{2p}s\int_{-1}^{0}|Q(t,s)|^p\, dt\right]ds\right)^{1/p}+\left(\int_{-1}^{0}\left[C_1^p(\deg P)^{2p}\int_0^1|Q(t,s)|^p s\, ds\right]dt\right)^{1/p}.$$

Thus we finally have

$$\left\|\frac{\partial P}{\partial x_1}\right\|_{L^p(D_1,w)} \le \hat{C}_1(\deg P)^2\|P\|_{L^p(D_1,w)}+C_1(\deg P)^2\|P\|_{L^p(D_1,w)} \le (\hat{C}_1+C_1)(\deg P)^2\|P\|_{L^p(S,w)}.$$

A similar result for D_2 can be obtained if one considers the change of variables $t = x_2$, $s = x_2 - x_1$ and the polynomial $\tilde{Q}(t,s) = P(t-s,t)$. Since the proof for D_2 is quite similar to the one that we carry out in detail for D_1, we omit the details.

Thus we have shown that, if $\tilde{C} = 2\max\{C_0, \hat{C}_1, C_1, \hat{C}_2, C_2\}$, then (7) holds. That completes the proof. □

Now we shall prove the following weighted Schur-type inequality.

Lemma 2.2. *Let $S = \{(x_1,x_2) \in \mathbb{R}^2 \colon -1 \le x_1 \le x_2 \le 1\}$, $w(x_1,x_2) = x_2 - x_1$, $1 \le p < \infty$, $d \in \mathbb{N}_0$ and $R \in \mathcal{P}(\mathbb{R}^2)$. Let A be a Lebesgue-measurable subset of S. Assume that there exists $\alpha \in \mathbb{N}_0^2$ such that $\alpha_1 + \alpha_2 \le d$ and*

$$\forall_{x\in A}\ |D^\alpha R(x)| \ge m > 0.$$

Then there exist constants $C_d, \tilde{C}_d$ such that

$$\|wP\|_A \le C_d m^{-1}\epsilon^{-d}(\deg P+\deg R)^{2d}\|wPR\|_S + \epsilon\|wP\|_S \tag{14}$$

$$\|P\|_{L^p(A,w)} \le \tilde{C}_d m^{-1}\epsilon^{-d}(\deg P+\deg R)^{2d}\|PR\|_{L^p(S,w)} + \epsilon\|P\|_{L^p(S,w)} \tag{15}$$

for any $0 < \epsilon < 1$ and every $P \in \mathcal{P}(\mathbb{R}^2)$.

Proof. At first, we prove the inequality (15). The idea of proof comes from [13]. Thus we proceed by induction on d, starting with $d = 0$. If $\alpha_1 = \alpha_2 = 0$, then

$$|P(x)| \le m^{-1}|P(x)R(x)| \quad \text{for} \quad x \in A.$$

Therefore

$$\|P\|_{L^p(A,w)} \le m^{-1}\|PR\|_{L^p(A,w)} \le m^{-1}\|PR\|_{L^p(S,w)} + \epsilon\|P\|_{L^p(S,w)}.$$

Now suppose that the theorem has been proved for $d = 0,1,2,\ldots,d_0-1$. We then prove it for $d = d_0$. Let

$$I = \left\{(\beta_1,\beta_2) \in \mathbb{N}^2 : 0 < |\beta|,\ 0 \le \beta_1 \le \alpha_1,\ 0 \le \beta_2 \le \alpha_2\right\}.$$

Here $|\alpha|$ denotes the length of α. Notice that the set I contains at most $\frac{(d_0+1)(d_0+2)}{2} - 1$ elements. By Leibniz's rule, if $x \in A$, then

$$|P(x)| \le m^{-1}\left[|D^\alpha(PR)(x)| + \sum_{\beta\in I}\binom{\alpha}{\beta}|D^{\alpha-\beta}R(x)|\,|D^\beta P(x)|\right].$$

Let $\eta = \frac{(d_0+1)(d_0+2)}{2}$ and $\deg P = n$. We set

$$B_0 = \{x \in A \colon |D^{\alpha-\beta}R(x)| \le \frac{m\epsilon}{\eta^2}\binom{\alpha}{\beta}^{-1}(\tilde{C}n^2)^{-|\beta|},\ \beta \in I\},$$

where $\tilde{C}$ is the constant from Lemma 2.1. Then

$$|P(x)| \le m^{-1}|D^\alpha(PR)(x)| + \frac{\epsilon}{\eta^2}\sum_{\beta\in I}(\tilde{C}n^2)^{-|\beta|}|D^\beta P(x)|.$$

for all $x \in B_0$. This yields

$$\begin{aligned}\|P\|_{L^p(B_0,w)} &\le m^{-1}\|D^\alpha(PR)\|_{L^p(B_0,w)} + \frac{\epsilon}{\eta^2}\sum_{\beta\in I}(\tilde{C}n^2)^{-|\beta|}\|D^\beta P\|_{L^p(B_0,w)}\\ &\le m^{-1}\|D^\alpha(PR)\|_{L^p(S,w)} + \frac{\epsilon}{\eta^2}\sum_{\beta\in I}(\tilde{C}n^2)^{-|\beta|}\|D^\beta P\|_{L^p(S,w)}.\end{aligned}$$

Therefore by the preceding lemma,

$$\|P\|_{L^p(B_0,w)} \le m^{-1}\tilde{C}^{|\alpha|}(n+k)^{2|\alpha|}\|PR\|_{L^p(S,w)} + \frac{\epsilon}{\eta}\|P\|_{L^p(S,w)}.$$

On the other hand, if $x \in A \setminus B_0$ then there exists $\beta \in I$ such that

$$|D^{\alpha-\beta}R(x)| > \binom{\alpha}{\beta}^{-1}\frac{m\epsilon(\tilde{C}n^2)^{-|\beta|}}{\eta^2}. \tag{16}$$

Hence, we can divide the set $A \setminus B_0$ into at most $\eta - 1$ disjoint subsets B_j such that, for each j, there exists an index β_j satisfying

$$\inf\{|D^{\alpha-\beta_j}R(x)| : x \in B_j\} \ge \binom{\alpha}{\beta_j}^{-1}\frac{m\epsilon(\tilde{C}n^2)^{-|\beta_j|}}{\eta^2}. \tag{17}$$

Since $|\beta_j| > 0$ for all j, the induction hypothesis implies that

$$\|P\|_{L^p(B_j,w)} \le (\tilde{C}n^2)^{|\beta_j|}\frac{\eta^2}{m\epsilon}\binom{\alpha}{\beta_j}C_{d_0-|\beta_j|}\left(\frac{\eta}{\epsilon}\right)^{d_0-1}(n+\deg R)^{2(d_0-|\beta_j|)}\|PR\|_{L^p(S,w)} + \frac{\epsilon}{\eta}\|P\|_{L^p(S,w)}.$$

Thus we see that

$$\|P\|_{L^p(A,w)} \le C_{d_0}m^{-1}\epsilon^{-d_0}(\deg P + \deg R)^{2d_0}\|PR\|_{L^p(S,w)} + \epsilon\|P\|_{L^p(S,w)},$$

where

$$C_{d_0} = \tilde{C}^{d_0} + \left(\frac{(d_0+1)(d_0+2)}{2}\right)^{d_0+1}\sum_{\beta\in I}\binom{\alpha}{\beta}C_{d_0-|\beta|}\tilde{C}^{|\beta|}.$$

This completes the induction and the proof of (15).

Since wP is a polynomial, the inequality (14) follows from Lemma 3 in [13]. □

3 Main result

The principal result of this paper is the following theorem:

Theorem 3.1. *Let $1 \le p < \infty$ and $\Omega = \{(x,y) \in \mathbb{R}^2 : |x| \le y+1,\ x^2 \ge 4y\}$. Then there exists constants $M, \tilde{M} > 0$ such that*

$$\max\left\{\left\|\frac{\partial P}{\partial x}\right\|_{\Omega}, \left\|\frac{\partial P}{\partial y}\right\|_{\Omega}\right\} \le M(\deg P)^4\,\|P\|_{\Omega} \tag{18}$$

$$\max\left\{\left\|\frac{\partial P}{\partial x}\right\|_{L^p(\Omega)}, \left\|\frac{\partial P}{\partial y}\right\|_{L^p(\Omega)}\right\} \le \tilde{M}(\deg P)^4\,\|P\|_{L^p(\Omega)} \tag{19}$$

for every polynomial $P \in \mathcal{P}(\mathbb{R}^2)$. Moreover, $\mu(\Omega) = \mu_p(\Omega) = 4$.

Proof. Let $P \in \mathcal{P}(\mathbb{R}^2)$ and $S = \{(u,v) \in \mathbb{R}^2 : -1 \le u \le v \le 1\}$. Observe first that the integrals

$$\int_\Omega \left|\frac{\partial P}{\partial y}(x,y)\right|^p dx\,dy, \quad \int_\Omega |P(x,y)|^p\,dx\,dy$$

become, under the change of variables $x = u+v$, $y = uv$,

$$\int_S \left|\frac{\partial P}{\partial y}(u+v,uv)\right|^p (v-u)\,du\,dv, \quad \int_S |P(u+v,uv)|^p\,(v-u)\,du\,dv.$$

Define Q by $Q(u,v) = P(u+v,uv)$. Then

$$(v-u)\frac{\partial P}{\partial y}(u+v,uv) = \frac{\partial Q}{\partial u}(u,v) - \frac{\partial Q}{\partial v}(u,v). \tag{20}$$

We now see, using Lemma 2.1, that

$$\left\|(v-u)\frac{\partial P}{\partial y}(u+v,uv)\right\|_{L^p(S,w)} = \left\|\frac{\partial Q}{\partial u} - \frac{\partial Q}{\partial v}\right\|_{L^p(S,w)} \le 2\tilde{C}(2\deg P)^2\,\|Q\|_{L^p(S,w)}.$$

Applying Lemma 2.2, with $R(u,v) = v-u$, $A = S$ and $\epsilon = 1/2$, to $\frac{\partial P}{\partial y}$ we may derive that

$$\left\|\frac{\partial P}{\partial y}(u+v,uv)\right\|_{L^p(S,w)} \le 4\tilde{C}_1(\deg P + 1)^2\left\|(v-u)\frac{\partial P}{\partial y}(u+v,uv)\right\|_{L^p(S,w)}.$$

Hence

$$\left\| \frac{\partial P}{\partial y}(u+v,uv) \right\|_{L^p(S,w)} \le 8\tilde{C}\tilde{C}_1(2\deg P+1)^4 \|Q\|_{L^p(S,w)} .$$

Thus

$$\left\| \frac{\partial P}{\partial y} \right\|_{L^p(\Omega)} \le 8\tilde{C}\tilde{C}_1(2\deg P+1)^4 \|P\|_{L^p(\Omega)} .$$

To prove the remainder, we need to consider the polynomials uQ and vQ. Then

$$(v-u)\frac{\partial P}{\partial x}(u+v,uv) = \frac{\partial vQ}{\partial v}(u,v) - \frac{\partial uQ}{\partial u}(u,v).$$

Hence

$$\left\| (v-u)\frac{\partial P}{\partial x}(u+v,uv) \right\|_{L^p(S,w)} \le \tilde{C}(2\deg P+1)^2 \left(\|vQ\|_{L^p(S,w)} + \|uQ\|_{L^p(S,w)} \right) \le 2\tilde{C}(2\deg P+1)^2 \|Q\|_{L^p(S,w)} .$$

Thus using an argument similar to the one that we carry out in detail for $\partial P/\partial y$, one can obtain the desired estimate.

To prove the inequality (18), let $Q(u,v) = P(u+v,uv)$, $G(u,v) = \frac{\partial P}{\partial y}(u+v,uv)$ and $w(u,v) = v-u$. Then, by (20) and Theorem 3.1 in [27], we see that

$$\|wG\|_S = \left\| \frac{\partial Q}{\partial u} - \frac{\partial Q}{\partial v} \right\|_S \le \frac{8}{\delta_S}(2\deg P)^2 \|Q\|_S . \tag{21}$$

Applying Lemma 3 from [13], with $R(u,v) = v-u$, $\Omega = S$, $A = S$ and $\epsilon = 1/2$, to G yields the following inequality

$$\|G\|_S \le 4\tilde{C}_1(2\deg P+1)^2 \|wG\|_S . \tag{22}$$

Since $T(x,y) = (x+y,xy)$ maps Ω to S,

$$\|G\|_S = \left\| \frac{\partial P}{\partial y} \right\|_\Omega \quad \text{and} \quad \|Q\|_S = \|P\|_\Omega . \tag{23}$$

Now (18) follows from (21), (22) and (23).

Since Ω is a compact subanalytic subset of $\mathbb{R}^2$, the Corollary 6.6 from [22] implies that Ω is UPC. Thus, by Corollary 26 in [2], $\mu_p(\Omega) \ge \mu(\Omega)$. From the inequality (19), we see now that $4 \ge \mu_p(\Omega)$. Therefore it remains to show that $\mu(\Omega) \ge 4$.

The discussion here is based on unpublished work of M. Baran. Let us consider the following sequence of polynomials

$$P_k(x,y) = \left[\frac{1}{k} T_k'\left(\frac{2-x}{4} \right) \right]^5 \left(\frac{1+x+y}{4} \right), \tag{24}$$

where T_k is the kth Chebyshev polynomial of the first kind. Note that the polynomial P_k has degree $5k-4$. It is known (see [25, Chap. 1.5]) that

$$\frac{1}{k+1} T_{k+1}'(x) = U_k(x), \tag{25}$$

where U_k is the kth Chebyshev polynomial of the second kind defined by

$$U_k(x) = \frac{\sin(k+1)\theta}{\sin\theta}, \quad \theta = \arccos x. \tag{26}$$

If $x \in [0,1]$, then $\sin(\arccos x) = \sqrt{1-x^2}$. Therefore by (25) and (26),

$$\frac{\sqrt{1-x}}{k} |T_k'(x)| \le \frac{\sqrt{1-x^2}}{k} |T_k'(x)| \le 1, \quad x \in [0,1]. \tag{27}$$

Hence

$$\left| \frac{\sqrt{x}}{k} T_k'(1-x) \right| \le 1 \quad \text{if} \quad x \in [0,1]. \tag{28}$$

If $(x,y) \in \Omega$, then $4y \le x^2$. Thus

$$\frac{1+x+y}{4} \le \left(\frac{1}{2} + \frac{x}{4} \right)^2 \quad \text{for } (x,y) \in \Omega. \tag{29}$$

Then, by (24), (28) and (29), we have

$$|P_k(x,y)| \le \left| \frac{1}{k} T_k'\left(\frac{2-x}{4} \right) \sqrt{\frac{1}{2} + \frac{x}{4}} \right|^4 \left| \frac{1}{k} T_k'\left(\frac{2-x}{4} \right) \right| \le \frac{1}{k} \|T_k'\|_{[-1,1]} = k \tag{30}$$

for any $(x,y)\in\Omega$. On the other hand,

$$\left|\frac{\partial P_k}{\partial y}(-2,1)\right| = \frac{1}{4}\left|\frac{1}{k}T_k'(1)\right|^5 = \frac{k^5}{4} \ge \frac{k^4}{4}\,\|P_k\|_\Omega. \tag{31}$$

Similarly, if $Q_k = \left[\frac{1}{k}T_k'\left(\frac{1+y}{2}\right)\right]^5\left(\frac{x^2}{4}-y\right)$, then

$$\|Q_k\|_\Omega \le k \quad and \quad \left|\frac{\partial Q_k}{\partial x}(2,1)\right| = k^5. \tag{32}$$

By the inequalities (30), (31) and (32), we have $\mu(\Omega)\ge 4$. Thus we finally have $\mu_p(\Omega)=\mu(\Omega)=4$. □

Remark 1. In the same fashion, we may prove that there exists a positive constant C_l such that

$$\max\left\{\left\|\frac{\partial P}{\partial x}\right\|_{L^p(\Delta_l)}, \left\|\frac{\partial P}{\partial y}\right\|_{L^p(\Delta_l)}\right\} \le C_l(\deg P)^{2l}\,\|P\|_{L^p(\Delta_l)} \tag{33}$$

for every $P\in\mathcal{P}(\mathbb{R}^2)$. Here $\Delta_l = \left\{(x,y)\in\mathbb{R}^2 : |x|^{1/l}+|y|^{1/l}\le 1\right\}$ and l is a positive odd number.

4 Sharpness of the exponents

In this section we shall analyze the inequality (33). Let $P_n^{(\alpha,\beta)}$ denote the nth Jacobi polynomial. Define $W_n(x,y)=yP_n^{(\alpha,\alpha)}(x)$. Then

$$\int_{\Delta_l}\left|\frac{\partial W_n}{\partial y}(x,y)\right|^p dxdy = 2\int_{-1}^{1}\left|P_n^{(\alpha,\alpha)}(x)\right|^p\left(1-|x|^{1/l}\right)^l dx,$$
$$\int_{\Delta_l}|W_n(x,y)|^p\,dxdy = \frac{2}{p+1}\int_{-1}^{1}\left|P_n^{(\alpha,\alpha)}(x)\right|^p\left(1-|x|^{1/l}\right)^{(p+1)l} dx.$$

Then the symmetry relation (see [26, Chap. IV])

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$$

yields that

$$\int_{\Delta_l}\left|\frac{\partial W_n}{\partial y}(x,y)\right|^p dxdy = 4\int_{0}^{1}\left|P_n^{(\alpha,\alpha)}(x)\right|^p\left(1-x^{1/l}\right)^l dx,$$
$$\int_{\Delta_l}|W_n(x,y)|^p\,dxdy = \frac{4}{p+1}\int_{0}^{1}\left|P_n^{(\alpha,\alpha)}(x)\right|^p\left(1-x^{1/l}\right)^{(p+1)l} dx.$$

We now apply Bernoulli's inequality to deduce that

$$\left(\frac{1-x}{l}\right)^l \le \left(1-x^{1/l}\right)^l \le (1-x)^l$$

for each positive integer l and $x\in[0,1]$. Therefore, if $n\to\infty$, then

$$\frac{\int_{\Delta_l}\left|\frac{\partial W_n}{\partial y}(x,y)\right|^p dxdy}{\int_{\Delta_l}|W_n(x,y)|^p\,dxdy} \sim \frac{\int_0^1\left|P_n^{(\alpha,\alpha)}(x)\right|^p(1-x)^l\,dx}{\int_0^1\left|P_n^{(\alpha,\alpha)}(x)\right|^p(1-x)^{(p+1)l}\,dx}.$$

Now a result proved by Szegö (see [26, Chap. VII]) comes into play. With $\mu_{\alpha,p}=\alpha p-2+p/2$, we have

$$\int_0^1\left|P_n^{(\alpha,\alpha)}(x)\right|^p(1-x)^l\,dx \sim n^{\alpha p-2l-2} \quad \text{whenever} \quad 2l<\mu_{\alpha,p}, \tag{34}$$
$$\int_0^1\left|P_n^{(\alpha,\alpha)}(x)\right|^p(1-x)^{(p+1)l}\,dx \sim n^{\alpha p-2(p+1)l-2} \quad \text{whenever} \quad 2(p+1)l<\mu_{\alpha,p}. \tag{35}$$

If $2(p+1)l<\mu_{\alpha,p}$, then we can combine (34) and (35) to see that

$$\frac{\left\|\frac{\partial W_n}{\partial y}\right\|_{L^p(\Delta_l)}}{\|W_n\|_{L^p(\Delta_l)}} \sim n^{2l}. \tag{36}$$

As a consequence of (33) and (36), we then find that $\mu_p(\Delta_l)=2l$.

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