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On the approximation of multivariate entire functions by Lagrange interpolation polynomials

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Abstract

We show that the intertwining of sequences of good Lagrange interpolation points for approximating entire functions is still a good sequence of interpolation points. We give examples of such sequences.

1 Introduction

Let $\mathcal{P}_d(\mathbb{C}^n)$ be the space of polynomials of degree at most d in $\mathbb{C}^n$. Its dimension N_d^n equals $\binom{n+d}{n}$. A subset X of N_d^n distinct points in $\mathbb{C}^n$ is said to be *unisolvent* of degree d if, for every function f defined on X, there exists a unique polynomial $P \in \mathcal{P}_d(\mathbb{C}^n)$ such that $P(\mathbf{z}) = f(\mathbf{z})$ for all $\mathbf{z} \in X$. This polynomial is called the *Lagrange interpolation polynomial* of f at X and is denoted by $\mathbf{L}[X; f]$. In fact, A is unisolvent if and only if it is not included in an algebraic hyper-surface of degree at most d. There is a natural way of constructing a unisolvent set, denoted by $A \oplus B$, of degree d in $\mathbb{C}^{n+m}$ by suitably combining (ordered) unisolvent sets of same degree A in $\mathbb{C}^n$ and B in $\mathbb{C}^m$. This new set, called the intertwining of A and B, preserves many properties of its factors. For example, in [10], Siciak considered the intertwining of univariate Leja sequences and showed that they provide excellent interpolations points for holomorphic functions on a neighbourhood of a Cartesian product of plane compact sets. Further sequences were considered in [1]. In [5], the authors gave a sequence in a compact set in $\mathbb{C}$ whose Lebesgue constant grows like a polynomials. Using a result from [4], they showed that an analogous property holds for the intertwining of these sequences.

In this paper, we are concerned with the problem of approximating multivariate entire functions by Lagrange interpolation polynomials. Given a sequence X_d of unisolvent sets of degree d in $\mathbb{C}^n$ for $d = 0, 1, \ldots$, we want to find conditions ensuring uniform convergence of $\mathbf{L}[X_d; f]$ to f on every compact subset of $\mathbb{C}^n$ for all $f \in H(\mathbb{C}^n)$. Here $H(\mathbb{C}^n)$ stands for the space of all entire functions in $\mathbb{C}^n$. It is well-known that, if $n = 1$, a sufficient condition is the boundedness of $\cup_{d=0}^{\infty} X_d$ and this is an immediate consequence of the classical Hermite Remainder Formula. Bloom and Levenberg showed in [3] that this property no longer holds true in the several variables case. In fact, it is not easy to construct specific unisolvent sets X_d in $\mathbb{C}^n$ with $n \geq 2$ such that the above convergence property holds. New examples formed of natural lattices satisfying certain conditions were recently given in [8]. Here we show (Theorem 3.1) that the above convergence property is preserved under the process of intertwining. Then, we present examples of sequences satisfying the convergence property and show that their intertwining provides essentially new sequences.

2 The intertwining sequences

The length of the n-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is defined by $|\alpha| := \sum_{j=1}^{n} \alpha_j$. Note that N_d^n is equal to the cardinality of the set $\{\alpha \in \mathbb{N}^n : |\alpha| \leq d\}$. Suppose that $\mathbb{N}^n$ is ordered by the graded lexicographic order. A tuple of points $A_d = (\mathbf{a}_\alpha : |\alpha| \leq d)$ in $\mathbb{C}^n$ is said to be *block unisolvent* of degree d if, for every $0 \leq j \leq d$, (the underlying set of) $A_j = (\mathbf{a}_\alpha : |\alpha| \leq j)$ is unisolvent of degree j. A sequence $A = (\mathbf{a}_\alpha : \alpha \in \mathbb{N}^n)$ is said to be block unisolvent if $A_d = (\mathbf{a}_\alpha : |\alpha| \leq d)$ is unisolvent of degree d for all $d \geq 0$.

Definition 2.1. ([4]) Let $A_d = (\mathbf{a}_\alpha : |\alpha| \leq d)$ (resp. $B_d = (\mathbf{b}_\beta : |\beta| \leq d)$) be a block unisolvent tuple of degree d in $\mathbb{C}^n$ (resp. $\mathbb{C}^m$). The intertwining of A_d and B_d, denoted by $A_d \oplus B_d$, is defined by

$$A_d \oplus B_d = \big((\mathbf{a}_\alpha, \mathbf{b}_\beta) : |\alpha| + |\beta| \leq d\big).$$

The intertwining of two block unisolvent sequences A and B is defined in the same way. That is, the blocks of $A \oplus B$ are given by

$$(A \oplus B)_d = A_d \oplus B_d, \quad \forall d \geq 0.$$

Note that the intertwining of two tuples depends on the ordering of the corresponding sets. Hence, from now on, we will work with unisolvent tuples rather than unisolvent sets. If C is another unisolvent tuple of degree d then $(A \oplus B) \oplus C = A \oplus (B \oplus C)$. Thus we may consider the intertwining of any number of tuples without using parenthesis.

Theorem 2.1. *([4]) The tuple $A_d \oplus B_d$ is block unisolvent of degree d in $\mathbb{C}^{n+m}$ with $(A_d \oplus B_d)_j = A_j \oplus B_j$ for $0 \leq j \leq d$*

The Lagrange interpolation polynomial at $A_d \oplus B_d$ of a product function is given in the following result which is proved in [4, Theorem 4.3].

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Theorem 2.2. *Let $A_d = (\mathbf{a}_\alpha : |\alpha| \le d)$ (resp. $B_d = (\mathbf{b}_\beta : |\beta| \le d)$) be a block unisolvent tuple of degree d in $\mathbb{C}^n$ (resp. $\mathbb{C}^m$). If $F(\mathbf{z},\mathbf{w}) = f(\mathbf{z})g(\mathbf{w})$ with $f : A_d \to \mathbb{C}$ and $g : B_d \to \mathbb{C}$ then*

$$\mathbf{L}[A_d \oplus B_d; F](\mathbf{z},\mathbf{w}) = \sum_{j+k\le d} \big(\mathbf{L}[A_j; f](\mathbf{z}) - \mathbf{L}[A_{j-1}; f](\mathbf{z})\big)\cdot\big(\mathbf{L}[B_k; g](\mathbf{w}) - \mathbf{L}[B_{k-1}; g](\mathbf{w})\big),$$

where $\mathbf{L}[A_{-1}; f] = \mathbf{L}[B_{-1}; g] = 0$.

3 Intertwining extremal sequences

We will say that a block unisolvent sequence $A \subset \mathbb{C}^n$ is *extremal* if, for all $f \in H(\mathbb{C}^n)$, the sequence $\mathbf{L}[A_d; f]$ converges to f uniformly on every compact subset of $\mathbb{C}^n$ as $d \to \infty$.

Theorem 3.1. *Let $A = (\mathbf{a}_\alpha : \alpha \in \mathbb{N}^n)$ (resp. $B = (\mathbf{b}_\beta : \beta \in \mathbb{N}^m)$) be an extremal sequence in $\mathbb{C}^n$ (resp. $\mathbb{C}^m$). Then $A \oplus B$ is also an extremal sequence.*

Proof. Let us denote by $\overline{\mathbb{D}}^n(R) = \{\mathbf{z} \in \mathbb{C}^n \ : \ |z_k| \le R, k = 1,\dots,n\}$ the closed ball of center the origin and radius $R > 0$ (with respect to the norm $|\mathbf{z}| = \max\{|z_k|, k = 1,\dots,n\}$). The space of entire functions $H(\mathbb{C}^n)$ is endowed with the topology of uniform convergence on every compact subset of $\mathbb{C}^n$. This topology (of Fréchet space) is denoted by τ and can be defined by the following family of semi-norms

$$f \mapsto \|f\|_{\overline{\mathbb{D}}^n(R)} = \sup\{|f(\mathbf{z})| : \mathbf{z} \in \overline{\mathbb{D}}^n(R)\}, \quad f \in H(\mathbb{C}^n), \quad R > 0.$$

Observe that $\mathbf{L}[A_d; \cdot] : H(\mathbb{C}^n) \to H(\mathbb{C}^n)$ is a continuous linear operator with respect to the topology τ. Since $\mathbf{L}[A_d; f]$ converges to f in τ for all $f \in H(\mathbb{C}^n)$, we can use the Banach-Steinhaus theorem for the sequence of operators $\{\mathbf{L}[A_d; \cdot]\}_{d=0}^{\infty}$ (see e.g. [9, Chapter 2]). It follows that, for each $R_1 > 0$, there exists $R_2 > 0$ and $C_1 > 0$ depending only on R_1 such that

$$\|\mathbf{L}[A_d; f]\|_{\overline{\mathbb{D}}^n(R_1)} \le C_1 \|f\|_{\overline{\mathbb{D}}^n(R_2)}, \quad \forall f \in H(\mathbb{C}^n), \quad d \ge 0. \tag{1}$$

Consequently,

$$\|\mathbf{L}[A_d; f] - \mathbf{L}[A_{d-1}; f]\|_{\overline{\mathbb{D}}^n(R_1)} \le 2C_1 \|f\|_{\overline{\mathbb{D}}^n(R_2)} \quad \forall f \in H(\mathbb{C}^n), \quad d \ge 0. \tag{2}$$

Using the same arguments we can also find $R_3 > R_2 > 0$ and $C_2 > 0$ depending only on R_1 such that

$$\|\mathbf{L}[B_d; g] - \mathbf{L}[B_{d-1}; g]\|_{\overline{\mathbb{D}}^m(R_1)} \le 2C_2 \|g\|_{\overline{\mathbb{D}}^m(R_3)}, \quad \forall g \in H(\mathbb{C}^m), \quad d \ge 0. \tag{3}$$

Of course, we may assume that $R_3 \ge 1$. For $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^m$, we set $f_\alpha(\mathbf{z}) = \mathbf{z}^\alpha$ and $g_\beta(\mathbf{w}) = \mathbf{w}^\beta$. Let us consider two sequences of polynomials defined by

$$p_0 = \mathbf{L}[A_0; f_\alpha], \quad p_j = \mathbf{L}[A_j; f_\alpha] - \mathbf{L}[A_{j-1}; f_\alpha] \quad \text{for } j \ge 1;$$
$$q_0 = \mathbf{L}[B_0; g_\beta], \quad q_k = \mathbf{L}[B_k; g_\beta] - \mathbf{L}[B_{k-1}; g_\beta] \quad \text{for } k \ge 1.$$

Using (2) and (3) we obtain

$$\|p_j\|_{\overline{\mathbb{D}}^n(R_1)} \le 2C_1 R_2^{|\alpha|} \le 2C_1 R_3^{|\alpha|}, \quad \forall j \ge 0,$$

and

$$\|q_k\|_{\overline{\mathbb{D}}^m(R_1)} \le 2C_2 R_3^{|\beta|}, \quad \forall k \ge 0.$$

Since $\mathbf{L}[A_j; \cdot]$ is a projector onto $\mathcal{P}_j(\mathbb{C}^n)$, $p_j = 0$ for all $j \ge |\alpha| + 1$. By the same reason we also get $q_k = 0$ for all $k \ge |\beta| + 1$. Therefore we can write

$$f_\alpha = \mathbf{L}[A_{|\alpha|}; f_\alpha] = \sum_{j=0}^{|\alpha|} p_j = \sum_{j=0}^{\infty} p_j \quad \text{and} \quad g_\beta = \mathbf{L}[B_{|\beta|}; g_\beta] = \sum_{k=0}^{|\beta|} q_k = \sum_{k=0}^{\infty} q_k.$$

Set $P_{\alpha\beta}(\mathbf{z},\mathbf{w}) = f_\alpha(\mathbf{z}) g_\beta(\mathbf{w}) = \mathbf{z}^\alpha \mathbf{w}^\beta$. By Theorem 2.2, we have

$$\begin{aligned} P_{\alpha\beta} - \mathbf{L}[A_d \oplus B_d; P_{\alpha\beta}] &= \sum_{j=0}^{\infty} p_j \sum_{k=0}^{\infty} q_k - \sum_{j+k\le d} p_j q_k \\ &= \sum_{j+k\ge d+1} p_j q_k = \sum_{j+k\ge d+1, j\le|\alpha|, k\le|\beta|} p_j q_k. \end{aligned}$$

Thus, for $(\mathbf{z},\mathbf{w}) \in \overline{\mathbb{D}}^n(R_1) \times \overline{\mathbb{D}}^m(R_1)$ and $d \ge 0$, since $R_3 \ge 1$, we have

$$\begin{aligned} |P_{\alpha\beta}(\mathbf{z},\mathbf{w}) - \mathbf{L}[A_d \oplus B_d; P_{\alpha\beta}](\mathbf{z},\mathbf{w})| &\le \sum_{j+k\ge d+1, j\le|\alpha|, k\le|\beta|} 2C_1 R_3^{|\alpha|} 2C_2 R_3^{|\beta|} \\ &\le \sum_{j\le|\alpha|, k\le|\beta|} 4C_1 C_2 R_3^{|\alpha|+|\beta|} \\ &= 4C_1 C_2 (|\alpha|+1)(|\beta|+1) R_3^{|\alpha|+|\beta|} \\ &\le C_1 C_2 (|\alpha|+|\beta|+2)^2 R_3^{|\alpha|+|\beta|}. \end{aligned} \tag{4}$$

Given $F \in H(\mathbb{C}^{n+m})$. The Taylor expansion of F is of the following form,

$$F(\mathbf{z},\mathbf{w}) = \sum_{(\alpha,\beta)\in\mathbb{N}^{n+m}} c_{\alpha\beta} P_{\alpha\beta}(\mathbf{z},\mathbf{w}). \tag{5}$$

Since Lagrange operators are linear continuous and $\mathbf{L}[A_d \oplus B_d; P_{\alpha\beta}] = P_{\alpha\beta}$ when $|\alpha| + |\beta| \le d$, we have

$$F - \mathbf{L}[A_d \oplus B_d; F] = \sum_{|\alpha|+|\beta|\ge d+1} c_{\alpha\beta}\big(P_{\alpha\beta} - \mathbf{L}[A_d \oplus B_d; P_{\alpha\beta}]\big). \tag{6}$$

Take $R_4 > 2R_3$, for all $(\alpha,\beta) \in \mathbb{N}^{n+m}$, Cauchy's inequalities [7, Theorem 2.27] give the following estimates for the coefficients of the Taylor series,

$$|c_{\alpha\beta}| \le \frac{\|F\|_{\overline{\mathbb{D}}^{n+m}(R_4)}}{R_4^{|\alpha|+|\beta|}}.$$

Combining this with (4) we obtain

$$\begin{aligned}
\|F - \mathbf{L}[A_d \oplus B_d; F]\|_{\overline{\mathbb{D}}^{n+m}(R_1)} &\le \sum_{|\alpha|+|\beta|\ge d+1} \frac{\|F\|_{\overline{\mathbb{D}}^{n+m}(R_4)}}{R_4^{|\alpha|+|\beta|}} C_1 C_2 (|\alpha| + |\beta| + 2)^2 R_3^{|\alpha|+|\beta|} \\
&= C_3 \|F\|_{\overline{\mathbb{D}}^{n+m}(R_4)} \sum_{|\gamma|\ge d+1} (|\gamma| + 2)^2 \big(R_3/R_4\big)^{|\gamma|} \\
&= C_3 \|F\|_{\overline{\mathbb{D}}^{n+m}(R_4)} \sum_{j=d+1}^{\infty} (j+2)^2 \binom{j+n+m-1}{j} \big(R_3/R_4\big)^j,
\end{aligned}$$

where $\gamma = (\alpha,\beta)$, $C_3 = C_1 C_2$. Since $2R_3 < R_4$ the series $\sum_{j=1}^{\infty}(j+2)^2\binom{j+n+m-1}{j}\big(R_3/R_4\big)^j$ is convergent. Thus $\sum_{j=d+1}^{\infty}(j+2)^2\binom{j+n+m-1}{j}\big(\frac{R_3}{R_4}\big)^j \to 0$ as $d \to \infty$. It follows that

$$\|F - \mathbf{L}[A_d \oplus B_d; F]\|_{\overline{\mathbb{D}}^{n+m}(R_1)} \to 0 \quad \text{as} \quad d \to \infty.$$

This completes the proof. □

Since any bounded sequence of complex numbers is extremal, the following result, already observed in [3], follows directly from Theorem 3.1.

Corollary 3.2. *If $A_k = \{a_{kj}\}_{j=0}^{\infty}$ is bounded sequences in $\mathbb{C}$ for $k = 1, \dots, n$, then $A := A_1 \oplus A_2 \oplus \cdots \oplus A_n$ is an extremal sequence in $\mathbb{C}^n$.*

Remark 1. As pointed out in [8, Remark 3.10] if $\cup_{d=0}^{\infty} A_d$ has no limit points then (A_d) cannot be an extremal sequence. Indeed, in that case, see [7, Theorem 7.2.11], there exists an entire function $f \not\equiv 0$ such that $f(\mathbf{a}) = 0$ for all $\mathbf{a} \in \cup_{d=0}^{\infty} A_d$. It follows that $\mathbf{L}[A_d; f] = 0$ for all $d \ge 0$. Hence $\mathbf{L}[A_d; f](\mathbf{z})$ does not tend to $f(\mathbf{z})$ whenever $f(\mathbf{z}) \ne 0$. We conjecture that if $\cup_{d=0}^{\infty} A_d$ is unbounded then there exists an entire function f such that $\mathbf{L}[A_d; f]$ does not converge to f uniformly on every compact subset of $\mathbb{C}^n$.

4 Construction of extremal sequences

We briefly present the construction of extremal sequences in $\mathbb{C}^n$ recently introduced in [8]. A hyperplane ℓ in $\mathbb{C}^n$ is defined by an equation $\ell = \{\mathbf{z} \in \mathbb{C}^n : \langle \mathbf{n}, \mathbf{z}\rangle + c = 0\}$. From now on, we will assume that the (normal) vector in the definition of the hyperplane is a unit vector, i.e. $\|\mathbf{n}\| = 1$ where $\|\cdot\|$ denotes the euclidean norm. For convenience, we write $\ell(\mathbf{z}) = \langle \mathbf{n}, \mathbf{z}\rangle + c$ and $\tilde{\ell}(\mathbf{z}) = \langle \mathbf{n}, \mathbf{z}\rangle$. A set H of n hyperplanes in $\mathbb{C}^n$ is said to be in general position if their intersection is a singleton, that is $\cap_{j=1}^{n} \ell_j = \{\vartheta_H\}$. If $\ell_j(\mathbf{z}) = \langle \mathbf{n}_j, \mathbf{z}\rangle + c_j$, then H is in general position if and only if $\det(\mathbf{n}_1, \dots, \mathbf{n}_n) \ne 0$. Here and subsequently, we confuse $\{\vartheta_H\}$ with $\boldsymbol{\vartheta}_H$. More generally, a family $\mathbb{H}_d = \{\ell_1, \dots, \ell_d\}$ of $d \ge n$ hyperplanes in $\mathbb{C}^n$ is said to be in general position if

1. Every $H \in \binom{\mathbb{H}_d}{n}$ (H a subset of n hyperplanes of $\mathbb{H}_d$) is in general position;
2. The map $H \in \binom{\mathbb{H}_d}{n} \mapsto \vartheta_H = \cap_{\ell\in H}\ell$ is one-to-one, i.e., $\vartheta_H \ne \vartheta_{H'}$ for $H \ne H'$.

The set $\Theta_{\mathbb{H}_d} = \{\vartheta_H : H \in \binom{\mathbb{H}_d}{n}\}$, formed of $\binom{d}{n}$ points, is called a *natural lattice* of order d. Chung and Yao in [6] showed that $\Theta_{\mathbb{H}_d}$ is unisolvent of degree $d - n$. Let us set $\mathbb{H}_k = \{\ell_1, \dots, \ell_k\}$ for $n \le k \le d$. Then $\Theta_{\mathbb{H}_k} \subset \Theta_{\mathbb{H}_{k+1}}$. Therefore, we can arrange $\Theta_{\mathbb{H}_d}$ in order to obtain a block unisolvent tuple of degree $d - n$. From now on, it is called a block natural tuple. According to this ordering, we can write

$$\Theta_{\mathbb{H}_k} = (\theta_\alpha : \alpha \in \mathbb{N}^n, |\alpha| \le k - n).$$

Note that there are exactly $\binom{k-1}{n-1}$ points θ_α with $|\alpha| = k - n$ and

$$\{\theta_\alpha : |\alpha| = k - n\} \subset \ell_k, \quad n \le k \le d.$$

Finally observe that

$$|\alpha| < k - n \implies \theta_\alpha \notin \ell_k.$$

Theorem 4.1 ([6]). *Let $\mathbb{H}_d = \{\ell_1, \dots, \ell_d\}$ be a family of $d \geq n$ hyperplanes in general position in $\mathbb{C}^n$. If ℓ_j is given by $\ell_j = \{\mathbf{z} \in \mathbb{C}^n : \langle \mathbf{n}_j, \mathbf{z} \rangle + c_j = 0\}$ and f is a function defined on $\Theta_{\mathbb{H}_d}$, then*

$$\mathbf{L}[\Theta_{\mathbb{H}_d}; f](\mathbf{z}) = \sum_{H \in \binom{\mathbb{H}_d}{n}} f(\vartheta_H) \mathbf{L}(\Theta_{\mathbb{H}_d}, \vartheta_H; \mathbf{z}), \tag{7}$$

where the fundamental Lagrange interpolation polynomial (FLIP) is given by

$$\mathbf{L}(\Theta_{\mathbb{H}_d}, \vartheta_H; \mathbf{z}) = \prod_{\ell \in \mathbb{H}_d \setminus H} \frac{\ell(\mathbf{z})}{\ell(\vartheta_H)}, \quad H \in \binom{\mathbb{H}_d}{n}. \tag{8}$$

Definition 4.1. ([8]) A sequence $\mathbb{H} = (\ell_k \ : \ k = 1, 2, \dots)$ of hyperplanes in $\mathbb{C}^n$ is said to be regular if the following assumptions hold.

1. $\mathbb{H}_d := \{\ell_1, \dots, \ell_d\}$ is in general position in $\mathbb{C}^n$ for all $d \geq n$.
2. $\liminf_{d \to \infty} h_{\mathbb{H}_d} > 0$, where
$$h_{\mathbb{H}_d} = \Big(\min \Big\{ \prod_{\ell \in \mathbb{H}_d \setminus K} \frac{|\tilde{\ell}(\mathbf{n}_K)|}{\|\mathbf{n}_K\|} : K \in \binom{\mathbb{H}_d}{n-1} \Big\} \Big)^{\frac{1}{d-n+1}}$$
and $\mathbf{n}_K$ is a non-zero vector that is parallel to the complex line $\cap_{\ell \in K} \ell$.

Theorem 4.2 ([8]). *Let $\mathbb{H} = (\ell_k \ : \ k = 1, \dots, \infty)$ be a regular sequence of hyperplanes in $\mathbb{C}^n$ such that $\Theta_{\mathbb{H}} := \cup_{d=n}^{\infty} \Theta_{\mathbb{H}_d}$ is bounded. Then the sequence $(\Theta_{\mathbb{H}_d} \ : \ d = n, n+1, \dots)$ is extremal.*

Here the sequence $(\Theta_{\mathbb{H}_d} \ : \ d = n, n+1, \dots)$ is a result of arranging the set $\Theta_{\mathbb{H}}$ such that the first $\binom{d}{n}$ elements of the sequence form the set $\Theta_{\mathbb{H}_d}$. The following result gives an explicit regular sequence.

Theorem 4.3 ([8]). *Let $n \geq 2$ and $a > 0$. Let $\{t_k\}_{k=1}^{\infty}$ be a sequence in $[-a, a]$ such that $|t_j - t_k| \geq M/d$ for $j \neq k$ and $1 \leq j, k \leq d$, where M is a positive constant. Set $T_d = \{t_1, \dots, t_d\}$ for $d \geq n$. We denote by Θ_d the subset of $\mathbb{C}^n$ consisting of $\binom{d}{n}$ points and defined by*

$$\Theta_d = \Big\{ \vartheta_U = \big((-1)^{n-1} \sigma_n(U), (-1)^{n-2} \sigma_{n-1}(U), \cdots, \sigma_1(U) \big) : U \in \binom{T_d}{n} \Big\},$$

where $\sigma_j(U)$ is the j-th elementary symmetric polynomial of n elements in U. Then $\Theta_d \subset \Theta_{d+1}$ and we can arrange the set $\cup_{d=n}^{\infty} \Theta_d$ to obtain an extremal sequence in $\mathbb{C}^n$.

It is not difficult to construct sequences (t_k) satisfying the above assumption. Here is an example. Choose $t_1 = 1/2$ and, for $k \geq 0$, having constructed $t_1, \dots, t_{2^k}$ we choose $t_{2^k+1}, \dots, t_{2^{k+1}}$ in

$$\Big\{ \frac{j}{2^{k+1}} \ : \ j = 1, \dots, 2^{k+1} \Big\} \setminus \{t_1, \dots, t_{2^k}\}.$$

Such a sequence is included in $[0, 1]$ and satisfies,

$$\{t_1, t_2, \dots, t_{2^k}\} = \Big\{ \frac{j}{2^k} : j = 1, \dots, 2^k \Big\}, \quad k = 1, 2, \dots.$$

Fix $d \in \mathbb{N}^*$ and take k such that $2^{k-1} < d \leq 2^k$. By construction, for $1 \leq i < j \leq d$, there are distinct integers n_i and n_j such that $t_i = n_i/2^k$ and $t_j = n_j/2^k$ so that

$$|t_i - t_j| = \frac{|n_i - n_j|}{2^k} \geq \frac{1}{2^k} \geq \frac{1}{2d}.$$

5 Bivariate extremal sequences in a prescribed convex set

The second condition in Definition 4.1 depends only on the normal vectors for the hyperplanes. The following theorem implies that, having a sequence of normal vectors satisfying 2 in Definition 4.1, we can construct a sequence of corresponding hyperplanes producing a natural lattices whose points belong to a given convex set.

Theorem 5.1. *Let $\mathbb{H}_d = \{\ell_1, \dots, \ell_d\}$ be d complex lines in general position in $\mathbb{C}^2$ such that normal vectors of the ℓ_k's belong to $\mathbb{R}^2$. Let $\mathbf{n} \in \mathbb{R}^2$ be a (unit) vector which normal to none of the ℓ_k, $k = 1, \dots, d$. Let further Ω be an open convex subset of $\mathbb{C}^2$ (or $\mathbb{R}^2$) containing $\Theta_{\mathbb{H}_d}$. Then there exists a complex line ℓ whose normal vector is $\mathbf{n}$ such that $\mathbb{H}_d \cup \{\ell\}$ in general position and $\Theta_{\mathbb{H}_d \cup \{\ell\}} \subset \Omega$.*

Proof. We may assumed that $\mathbf{n} = (1, 0)$, hence no ℓ_j is parrallel to the y-axis and their equation are of the form,

$$\ell_j = \{\mathbf{z} : a_j z_1 - z_2 + c_j = 0\}, \ \ j = 1, \dots, d. \tag{9}$$

Since the hyperplanes are in general position, the coefficients $a_1, \dots, a_d$ are pairwise distinct. We will assume that $a_1 < a_2 < \cdots < a_d$. For $1 \leq j < k \leq d$, let ϑ_{jk} denote the intersection of ℓ_j and ℓ_k. It is readily seen that

$$\vartheta_{jk} = (z_1^{(jk)}, z_2^{(jk)}) = \Big(-\frac{c_j - c_k}{a_j - a_k}, \frac{a_j c_k - a_k c_j}{a_j - a_k} \Big).$$

Note that for every $1 < j < d$ we have

$$z_1^{(1j)} \neq z_1^{(1d)} \quad \text{and} \quad z_1^{(jd)} \neq z_1^{(1d)}.$$

Indeed, if, for instance, $z_1^{(1j)} = z_1^{(1d)}$ then ℓ_1 passes through two points whose first coordinates are identity which implies that ℓ_1 is parallel to the vertical axis and this contradicts (9). We will prove that we may take $\ell(z) = z_1 - z_1^{(1d)} + \epsilon$ with ϵ small.

For $1 < j < d$, a direct computation gives

$$(z_1^{(1j)} - z_1^{(1d)})(z_1^{(jd)} - z_1^{(1d)}) = \frac{-(c_d a_j - c_d a_1 - c_1 a_j - c_j a_d + c_j a_1 + c_1 a_d)^2}{(a_d - a_1)^2 (a_j - a_1)(a_d - a_j)} < 0, \tag{10}$$

where the inequality follows from the assumption on the ordering of the a_i. Now, (10) implies that $z_1^{(1d)}$ lies in the open segment $(z_1^{(1j)}, z_1^{(jd)})$. Hence the line $\ell(\mathbf{z}) = z_1 - z_1^{(1d)}$ intersects the line segment $[\vartheta_{1j}, \vartheta_{jd}]$ at a point ϑ_j lying in the relative interior of the segment. By hypothesis $[\vartheta_{1j}, \vartheta_{jd}] \subset \ell_j \cap \Omega$ and $\vartheta_j = \ell \cap \ell_j \in \Omega$ for $1 < j < d$. Observe that if $\ell \cap \ell_j = \vartheta_j = \vartheta_i = \ell \cap \ell_i$ then the common value coincide with $\vartheta_{ij} = \ell_i \cap \ell_j$. We take $\ell(z) = z_1 - z_1^{(1d)} + \epsilon$ where ϵ is chose small enough so that $z_1^{1d} + \epsilon$ still lies in $(z_1^{(1j)}, z_1^{(jd)})$ (for every j) and so that ℓ does not meet any point in $\Theta_{\mathbb{H}_d}$. There are at most N_{d-2}^2 values of ϵ that fail to meet the criterion. The theorem is proved. □

Observe that, in the above construction, the new points (on the hyperplane ℓ) are included in the convex hull of the previous points. By repeating the process we obtain the following corollary in which the points are included in the triangle (in Ω) formed by the first three points.

Corollary 5.2. *Let $\mathcal{V} = \{\mathbf{n}_j\}_{j=1}^{\infty}$ be a sequence of unit vectors in $\mathbb{R}^2$ such that any two vectors of $\mathcal{V}$ are not parallel. Then for any given open convex set $\Omega \subset \mathbb{C}^2$ we can construct a sequence $\mathbb{H} = \{\ell_j\}_{j=1}^{\infty}$ of lines in general position in $\mathbb{C}^2$ such that $\mathbf{n}_j$ is a normal vector of ℓ_j for $j \geq 1$, and $\Theta_{\mathbb{H}} = \{\ell_j \cap \ell_k : j \neq k\} \subset \Omega$.*

The following example shows that a similar property as in Proposition 5.1 does not hold in general as soon as $n \geq 3$. For convenience we work with $\mathbb{R}^3$.

Let $\ell_j = \{\mathbf{x} \in \mathbb{R}^3 : x_j = 0\}$ for $j = 1, 2, 3$, $\ell_4 = \{\mathbf{x} \in \mathbb{R}^3 : x_1 + x_2 + x_3 - 1 = 0\}$ and $\ell_5^c = \{\mathbf{x} \in \mathbb{R}^3 : 2x_1 + x_2 + 3x_3 - c = 0\}$. We denote by $\mathbf{n}$ its normal vector, $\mathbf{n} = (2, 1, 3)$. Suppose that $\{\vartheta_{ijk}\} = \ell_i \cap \ell_j \cap \ell_k$ with $1 \leq i < j < k \leq 4$ and $\{\vartheta_{ij5}\} = \ell_i \cap \ell_j \cap \ell_5^c$ with $1 \leq i < j \leq 4$. It is easily seen that $\vartheta_{123}, \vartheta_{124}, \vartheta_{134}$ and ϑ_{234} are four vertices of the standard simplex in $\mathbb{R}^3$. Direct computations give $\vartheta_{135} = (0, c, 0)$, $\vartheta_{245} = (3-c, 0, c-2)$. For $\epsilon > 0$, the open ball $B_\epsilon := B((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \sqrt{\frac{3}{4} + \epsilon})$ contains ϑ_{ijk} with $1 \leq i < j < k \leq 4$. We show that when ϵ is small enough (for example $\epsilon = \frac{1}{10}$) then ϑ_{135} or ϑ_{245} does not lie in B_ϵ. Indeed, set $\vartheta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then $\|\vartheta - \vartheta_{135}\|^2 = (c - \frac{1}{2})^2 + \frac{1}{2}$ and $\|\vartheta - \vartheta_{245}\|^2 = 2(c - \frac{5}{2})^2 + \frac{1}{4}$. For every $c \in \mathbb{R}$, one of the two numbers $(c - \frac{1}{2})^2 + \frac{1}{2}, 2(c - \frac{5}{2})^2 + \frac{1}{4}$ is greater than $\frac{3}{4} + \epsilon$ with $\epsilon = 1/10$, which proves the claim.

6 The intertwining of natural lattices

We now show that the intertwining $A \oplus B$ of two arrays of degree $d \geq 2$ is never a natural lattice. In particular the intertwining of two natural lattice is not a natural lattice and this shows that the application of Theorem 3.1 to the sequences introduced in Section 4 provides essentially new extremal sequences.

Theorem 6.1. *Let $A_d = (a_\alpha : \alpha \in \mathbb{N}^n, |\alpha| \leq d) \subset \mathbb{C}^n$ (resp. $B_d = (b_\beta : \beta \in \mathbb{N}^m, |\beta| \leq d) \subset \mathbb{C}^m$) be block unisolvent tuples of degree $d \geq 2$. Then $A \oplus B$ is not a natural lattice.*

Proof. We assume the contrary, that is, confusing $A \oplus B$ with its underlying set,

$$A \oplus B = \Theta_{\mathbb{H}} \quad \text{with} \quad \mathbb{H} = \{\ell_1, \ldots, \ell_{m+n+d}\}$$

where the ℓ_i's are hyperplanes in general position in $\mathbb{C}^{n+m}$, and look for a contradiction. As before, we denote by $(\mathbf{z}, \mathbf{w})$ an element of $\mathbb{C}^{n+m}$ with $\mathbf{z} \in \mathbb{C}^n$ and $\mathbf{w} \in \mathbb{C}^m$. First, observe that if f is a function that depends only on $\mathbf{z}$ then

$$\mathbf{L}[A \oplus B; f](\mathbf{z}, \mathbf{w}) = \mathbf{L}[A; f](\mathbf{z}). \tag{11}$$

Indeed, the right-hand side is a polynomial of degree at most d in $\mathbf{z}$, hence in $(\mathbf{z}, \mathbf{w})$ and it matches f at all points $(\mathbf{a}_\alpha, \mathbf{b}_\beta)$ with $|\alpha| + |\beta| \leq d$. Let us apply the above relation with $f = \mathbf{L}[A, \mathbf{a}_\alpha; \mathbf{z}]$ the FLIP for $\mathbf{a}_\alpha \in A$, $|\alpha| = d$. The right-hand side of (11) equals to $\mathbf{L}[A, \mathbf{a}_\alpha; \mathbf{z}]$. Since $(\mathbf{a}_\alpha, \mathbf{b}_0) \in A \oplus B = \Theta_{\mathbb{H}}$, for some $S = S(\alpha) \in \binom{\{1,2,\ldots,m+n+d\}}{n+m}$ we have

$$(\mathbf{a}_\alpha, \mathbf{b}_0) = \vartheta_S = \cap_{i \in S} \ell_i.$$

Using Theorem 4.1 we see at once that the left-hand side of (11) is equal to $\prod_{i \notin S} \dfrac{\ell_i(\mathbf{z}, \mathbf{w})}{\ell_i(\vartheta_S)}$. Hence

$$\mathbf{L}[A, \mathbf{a}_\alpha; \mathbf{z}] = \prod_{i \notin S} \frac{\ell_i(\mathbf{z}, \mathbf{w})}{\ell_i(\vartheta_S)}.$$

Let us write $S'(\alpha) = \{1, \ldots, n+m\} \setminus S(\alpha)$. Hence for every $i \in S'(\alpha)$, $\ell_i(\mathbf{z}, \mathbf{w})$ is a divisor of $\mathbf{L}[A, \mathbf{a}_\alpha; z]$, this implies that ℓ_i does not depend on $\mathbf{w}$. Thus, for any $i \in S'(\alpha)$, the normal vector $\mathbf{n}_i$ to ℓ_i belongs to $\mathbb{C}^n \times \{0\}$. The same reasoning works with any α of length d. Now, we have one-to-one map

$$\{(\mathbf{a}_\alpha, \mathbf{b}_0) \,:\, |\alpha| = d\} \subset \Theta_{\mathbb{H}} \longrightarrow S'(\alpha) \in \binom{\{1, \ldots, m+n+d\}}{d}.$$

Let U denote the union of the $S'(\alpha)$. The above reasoning shows that for every i in U, $\mathbf{n}_i \in \mathbb{C}^n \times \{0\}$. Now, if the cardinality of U is N then there at most $\binom{N}{d}$ sets $S'(\alpha)$, while, on the other hand, we know that there are exactly $\binom{n+d-1}{d}$ such sets (as this is the number of the $(\mathbf{a}_\alpha, \mathbf{b}_0)$). It follows that N cannot be smaller than $n+d-1$. So, since $d \geq 2$, $N \geq n+1$. Hence, we have at least $n+1$ vectors $\mathbf{n}_i$ in $\mathbb{C}^n \times \{0\}$ and this contradicts the fact that the hyperplanes are in general position. □

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