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Approximation results by multivariate sampling Kantorovich series in Musielak-Orlicz spaces

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Abstract

In this paper we study the theory of the so-called multivariate sampling Kantorovich operators in the general frame of the Musielak-Orlicz spaces. The main result in this context is a modular convergence theorem, that can be proved by density arguments. Several concrete cases of Musielak-Orlicz spaces and of kernel functions are presented and discussed.

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1 Introduction

In the present paper, we study the so-called multivariate sampling Kantorovich operators ([31]), defined by:

$$(S_w^\chi f)(\underline{x}) \;:=\; \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x}-t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} f(\underline{u})\, d\underline{u} \right] \qquad (\underline{x}\in\mathbb{R}^n),$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x}\in\mathbb{R}^n$, $\chi:\mathbb{R}^n\to\mathbb{R}$ is a suitable kernel which satisfies the usual assumptions of the discrete approximate identities ([21, 12, 14, 6, 7]), $\Delta_{k_i} := t_{k_i+1}-t_{k_i} > 0$, for every $i=1,...,n$ and $\underline{k}=(k_1,...,k_n)\in\mathbb{Z}^n$, with $(t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}\subset\mathbb{R}^n$,

$$R_{w,t_{\underline{k}}} := \left[\frac{t_{k_1}}{w},\frac{t_{k_1+1}}{w}\right]\times\left[\frac{t_{k_2}}{w},\frac{t_{k_2+1}}{w}\right]\times...\times\left[\frac{t_{k_n}}{w},\frac{t_{k_n+1}}{w}\right] \qquad (w>0),$$

is the n-dimensional interval of $\mathbb{R}^n$, and finally $A_{\underline{k}}=\Delta_{k_1}\ldots\Delta_{k_n}$.

In the lasts years the above operators have been widely studied in view of their connections with the application to Signal and Digital Image Processing ([10]).

From the theoretical point of view, the operators S_w^χ represents the L^1-extension of the generalized sampling operators introduced by Butzer ([22, 52]), defined by means of the mean values of f instead of the sample values $f(t_{\underline{k}}/w)$ (see also [3]); it turns out that these operators reduce "time-jitter" errors, that is very useful in signal processing ([12]).

Both the generalized and Kantorovich sampling operators represent approximate versions of the classical Whittaker-Kotelnikov-Shannon sampling theorem, see e.g. [53, 20, 18, 42, 43, 44].

The behavior of the sampling Kantorovich series has been studied pointwise at the continuity points of a given bounded signal f, and uniformly for uniformly continuous and bounded signals. Further, the behavior of S_w^χ at the discontinuity points of f has been studied in [27]. Note that, even if the above averages make the sampling Kantorovich operators more regular, for what concerns the convergence at the discontinuity points of f, due to technical reasons, the situation become rather delicate. This happens also when the convergence of the above operators are studied with respect to the Jordan variation in case of functions with bounded variation ([6, 7]).

Then discontinuous signals have been studied in some general contexts; for instance in [12, 31] approximation results have been obtained in the setting of Orlicz spaces ([49, 2, 28]), which are very general spaces that include, for instance, the L^p-spaces. Results concerning the order of approximation have been achieved in [32, 51, 25, 26, 4]; the saturation order has been studied in [37, 17].

The latter result allows us to apply the theory of the sampling Kantorovich operators to approximate and reconstruct images. In fact, static gray scale images are characterized by jumps of gray levels mainly concentrated in their contours or edges and this can be translated, from a mathematical point of view, by discontinuities (see e.g. [11]).

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For these reasons, multivariate sampling Kantorovich operators appear very appropriate for applications to image reconstruction. Moreover, some applications to civil and energy engineering have been presented in [10, 11].

In this paper, the theory of the multivariate sampling Kantorovich operators is extended to the frame of the Musielak-Orlicz spaces ([49, 47, 54, 41, 55]), the latter provide a further generalization of the above mentioned Orlicz spaces. In this context, our main result is a modular convergence theorem, that can be proved by density arguments. More precisely, we firstly prove that the above family of operators converges with respect to the Luxemburg norm (which is a notion of convergence stronger than the modular convergence) in case of continuous functions with compact support, then we prove a modular inequality for the operators involved, and finally we exploit the density of such functions in $L^{\varphi}(\mathbb{R}^n)$ in order to establish the above claim.

The Musielak-Orlicz spaces include as special cases, the weighted L^p-spaces, the weighted Zygmund spaces (also known as weighted interpolation spaces), and others ([19, 40, 39]).

Moreover, several examples of kernels for which the above theory holds have been presented and discussed at the end of the paper.

2 Notations and basic assumptions

In this paper, we will denote by $\mathbb{N}^n$ and $\mathbb{Z}^n$ the sets of vectors $\underline{k} = (k_1, ..., k_n)$, where k_i belongs respectively to $\mathbb{N}$ and $\mathbb{Z}$, for each $i = 1, ..., n$; $\mathbb{R}^n$ is defined analogously.

We will consider on $\mathbb{R}^n$ the usual Euclidean norm $\|\cdot\|_2$, defined by $\|\underline{u}\|_2 = (u_1^2 + ... + u_n^2)^{1/2}$, where $\underline{u} = (u_1, ..., u_n)$, $u_i \in \mathbb{R}$, for every $i = 1, ..., n$. Moreover, $B(\underline{x}, r) \subset \mathbb{R}^n$ represents the closed ball of center $\underline{x} \in \mathbb{R}^n$ and radius $r > 0$, i.e., the set of all the elements $\underline{u} \in \mathbb{R}^n$ such that $\|\underline{x} - \underline{u}\|_2 \leq r$.

We denote by $C(\mathbb{R}^n)$ (resp. $C^0(\mathbb{R}^n)$) the space of all uniformly continuous and bounded (resp. continuous and bounded) functions $f : \mathbb{R}^n \to \mathbb{R}$ endowed with the norm $\|f\|_\infty := \sup_{\underline{u}\in\mathbb{R}^n} |f(\underline{u})|$. $C_c(\mathbb{R}^n)$ is the subspace of $C(\mathbb{R}^n)$ consisting of functions with compact support and $M(\mathbb{R}^n)$ is the space of all (Lebesgue) measurable functions.

Let now $\Pi = (t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$ be a sequence defined by $t_{\underline{k}} = (t_{k_1}, ..., t_{k_n})$, where each $(t_{k_i})_{k_i\in\mathbb{Z}}$, $i = 1, ..., n$ is a sequence of real numbers with $-\infty < t_{k_i} < t_{k_{i+1}} < +\infty$, and such that $\lim_{k_i\to\pm\infty} t_{k_i} = \pm\infty$, for every $i = 1, ..., n$ and such that there exist Δ, $\delta > 0$ for which $\delta \leq \Delta_{k_i} := t_{k_i+1} - t_{k_i} \leq \Delta$, for every $i = 1, ..., n$. Moreover, we denote by

$$R_{w,t_{\underline{k}}} := \left[\frac{t_{k_1}}{w}, \frac{t_{k_1+1}}{w}\right] \times \left[\frac{t_{k_2}}{w}, \frac{t_{k_2+1}}{w}\right] \times ... \times \left[\frac{t_{k_n}}{w}, \frac{t_{k_n+1}}{w}\right] \qquad (w > 0),$$

the n-dimensional interval of $\mathbb{R}^n$ identified by the sequence $\Pi = (t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$. Note that the Lebesgue measure of $R_{w,t_{\underline{k}}}$ is given by $A_{\underline{k}}/w^n$, where $A_{\underline{k}} := \Delta_{k_1} \cdot \Delta_{k_2} \cdot ... \cdot \Delta_{k_n}$.

From now on, we define as *kernel* a function $\chi : \mathbb{R}^n \to \mathbb{R}$ which satisfies the following conditions:

(χ1) χ belongs to $L^1(\mathbb{R}^n)$, and it is bounded in a ball containing the origin of $\mathbb{R}^n$;

(χ2) for every $\underline{x} \in \mathbb{R}^n$, there holds:

$$\sum_{\underline{k}\in\mathbb{Z}^n} \chi(\underline{x} - t_{\underline{k}}) = 1;$$

(χ3) for some $\beta > 0$, we assume that the discrete absolute moment of order β of χ is finite, i.e.,

$$m_{\Pi,\beta}(\chi) := \sup_{\underline{u}\in\mathbb{R}} \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi(\underline{u} - t_{\underline{k}})\right| \cdot \|\underline{u} - t_{\underline{k}}\|_2^{\beta} < +\infty.$$

Now, by $(S_w^{\chi})_{w>0}$ we denote the family of the multivariate sampling Kantorovich operators, of the form:

$$(S_w^{\chi} f)(\underline{x}) := \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} f(\underline{u})\, d\underline{u}\right] \qquad (\underline{x} \in \mathbb{R}^n), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$.

We now recall the following lemma.

Lemma 2.1. *Let χ be a kernel, as above. We have:*

(i) $m_{\Pi,0}(\chi) < +\infty$;

(ii) For every $\gamma > 0$

$$\lim_{w\to+\infty} \sum_{\|w\underline{x}-t_{\underline{k}}\|_2>\gamma w} |\chi(w\underline{x} - t_{\underline{k}})| = 0,$$

uniformly with respect to $\underline{x} \in \mathbb{R}^n$.

(iii) For every $\gamma > 0$ *and* $\varepsilon > 0$ *there exists a constant* $\widetilde{M} > 0$ *such that*

$$\int_{\|\underline{x}\|_2>\widetilde{M}} w^n |\chi(w\underline{x} - t_{\underline{k}})|\, d\underline{x} < \varepsilon,$$

for sufficiently large $w > 0$ *and* $t_{\underline{k}}$ *such that* $\|t_{\underline{k}}\|_2 \leq \gamma w$.

For a proof of Lemma 2.1, see [31].

Remark 1. In case of $f \in L^\infty(\mathbb{R}^n)$, by Lemma 2.1 (i), it turns out that $S_w^\chi f$ are well-defined for every $w > 0$. Indeed,

$$\left|(S_w^\chi f)(\underline{x})\right| \leq m_{\Pi,0}(\chi)\, \|f\|_\infty < +\infty,$$

for every $\underline{x} \in \mathbb{R}^n$ and $w > 0$, i.e., $S_w : L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$.

We recall the following convergence theorem.

Theorem 2.2 ([31]). *Let $f \in C^0(\mathbb{R}^n)$ be fixed. For every $\underline{x} \in \mathbb{R}^n$ we have:*

$$\lim_{w\to+\infty} (S_w^\chi f)(\underline{x}) = f(\underline{x}).$$

In particular, if $f \in C(\mathbb{R}^n)$ there holds:

$$\lim_{w\to+\infty} \|S_w^\chi f - f\|_\infty = 0.$$

Now, we recall some basic fact concerning the Musielak-Orlicz spaces.

Let $\varphi : \mathbb{R}^n \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function of $(n+1)$-variables satisfying the following conditions:

(φ1) the function φ is ψ-bounded, i.e., there exists a function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and a constant $M \geq 1$ such that $\varphi(\underline{s}, u) \leq M\,\psi(u)$, for every $\underline{s} \in \mathbb{R}^n$ and $u \in \mathbb{R}_0^+$;

(φ2) for every $\underline{s} \in \mathbb{R}^n$, $\varphi(\underline{s}, \cdot)$ is convex on $\mathbb{R}_0^+$ with $\varphi(\underline{s}, 0) = 0$ and $\varphi(\underline{s}, u) > 0$ for $u > 0$;

(φ3) φ is τ-bounded, i.e., there exist a constant $C \geq 1$ and a measurable function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ such that, for every $\underline{t}, \underline{s} \in \mathbb{R}$ and $u \geq 0$ there holds:

$$\varphi(\underline{s} - \underline{t}, u) \leq \varphi(\underline{s}, Cu) + F(\underline{s}, \underline{t}). \tag{2}$$

In general, a function φ satisfying (φ2) is said a *φ-function*; assumption (φ2) also implies that the function φ is non-decreasing and continuous with respect the variable u. From now on, for the sake of simplicity, we will denote by a φ-function, a function satisfying (φ1), (φ2) and (φ3).

The definition of $\tau - boundedness$ can be found in the monograph of J. Musielak [49].

Now, we define the non-negative integral functional:

$$I^\varphi(f) := \int_{\mathbb{R}} \varphi(\underline{u}, |f(\underline{u})|)\, d\underline{u}, \tag{3}$$

where $f \in M(\mathbb{R}^n)$. It is easy to check that I^φ is a modular on $M(\mathbb{R}^n)$ according with the definition given, e.g., in [19, 15].

The modular space generated by I^φ is called the Musielak-Orlicz space (see e.g., [49, 15, 40]) and it is defined as follows:

$$L^\varphi(\mathbb{R}^n) := \left\{ f \in M(\mathbb{R}^n) : \lim_{\lambda\to 0} I^\varphi(\lambda f) = 0 \right\}.$$

In particular, in view of the convexity of $\varphi(\underline{s}, \cdot)$ it is possible to prove that the above definition is equivalent to the following:

$$L^\varphi(\mathbb{R}^n) = \{ f \in M(\mathbb{R}^n) : \text{ there exists } \lambda > 0 \text{ such that } I^\varphi(\lambda f) < +\infty \}.$$

Further, it is also possible to define a useful subspace of $L^\varphi(\mathbb{R}^n)$, the so-called space of all *finite elements* of $L^\varphi(\mathbb{R}^n)$, defined by:

$$E^\varphi(\mathbb{R}^n) := \{ f \in M(\mathbb{R}^n) : I^\varphi(\lambda f) < +\infty \text{ for every } \lambda > 0 \}.$$

A notion of convergence in Musielak-Orlicz spaces, called *modular convergence*, was introduced in [50], which induces a topology in $L^\varphi(\mathbb{R}^n)$, called *modular topology*.

A family $(f_w)_{w>0} \subset L^\varphi(\mathbb{R}^n)$ is said to be modularly convergent to $f \in L^\varphi(\mathbb{R}^n)$, if there exists $\lambda > 0$ such that:

$$I^\varphi(\lambda(f_w - f)) \longrightarrow 0, \quad as \quad w \to +\infty.$$

Moreover, in $L^\varphi(\mathbb{R}^n)$ can be also given a stronger notion of convergence, i.e., the Luxemburg norm convergence, see e.g. [15, 49].

We will say that a net of functions $(f_w)_{w>0} \subset L^\varphi(\mathbb{R}^n)$ converges with respect to the Luxemburg norm to $f \in L^\varphi(\mathbb{R}^n)$ if

$$\lim_{w\to\infty} I^\varphi(\lambda(f_w - f)) = 0,$$

for every $\lambda > 0$.

In conclusion of this section, we recall the following useful density result that can be deduced from Theorem 1 of [48].

Theorem 2.3. *Let φ be a fixed φ-function as above. Then:*

$$\overline{C_c(\mathbb{R}^n)} = L^\varphi(\mathbb{R}^n),$$

where the symbol $\overline{C_c(\mathbb{R}^n)}$ denotes the closure of $C_c(\mathbb{R}^n)$ with respect to the modular topology on $L^\varphi(\mathbb{R}^n)$, and it means that for every $f \in L^\varphi(\mathbb{R}^n)$ there exists $\lambda > 0$ such that, for every $\varepsilon > 0$ there exists $g \in C_c(\mathbb{R}^n)$ with $I^\varphi(\lambda(f - g)) < \varepsilon$.

3 The main results

We begin with the following lemma.

Lemma 3.1. *For every $f \in C_c(\mathbb{R}^n)$ the operators $S_w f$ belong to $E^\varphi(\mathbb{R}^n) \subset L^\varphi(\mathbb{R}^n)$, for every $w > 0$.*

Proof. First of all we can observe that, if $f \in C_c(\mathbb{R}^n)$ and we denote by $supp\ f$ the support of the function f, we have that $supp\ f \subset B(\underline{0}, \gamma)$, for a fixed $\gamma > 0$, and it turns out that $R_{w,t_{\underline{k}}} \cap B(\underline{0}, \gamma) = \emptyset$ for every $t_{\underline{k}} \notin B(\underline{0}, w\gamma)$; therefore:

$$\int_{R_{w,t_{\underline{k}}}} f(\underline{u})\, d\underline{u} \ = \ 0.$$

It follows that:

$$(S_w^\chi f)(\underline{x}) = \sum_{\|\frac{1}{w} t_{\underline{k}}\|_2 \leq \gamma} \left|\chi(w\underline{x} - t_{\underline{k}})\right| \left[\frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} f(\underline{u})\, d\underline{u} \right]$$

is well-defined for every $\underline{x} \in \mathbb{R}^n$. Now, we will show that $S_w^\chi f \in E^\varphi(\mathbb{R}^n) \subset L^\varphi(\mathbb{R}^n)$, for every $w > 0$.

Indeed, for every $\lambda > 0$, by using Jensen's inequality (see e.g., [30]), assumption (φ1), and the change of variable $\underline{t} = w\underline{x} - t_{\underline{k}}$, we have:

$$\begin{aligned}
I^\varphi\left(\lambda S_w^\chi f\right) \ &\leq \ \int_{\mathbb{R}^n} \varphi(\underline{x},\ \lambda \sum_{\|\frac{1}{w} t_{\underline{k}}\|_2 \leq \gamma} \left|\chi(w\underline{x} - t_{\underline{k}})\right| \|f\|_\infty)\, d\underline{x} \\
&\leq \ \sum_{\|\frac{1}{w} t_{\underline{k}}\|_2 \leq \gamma} \frac{1}{m_{\Pi,0}(\chi)} \int_{\mathbb{R}^n} \varphi(\underline{x},\ \lambda m_{\Pi,0}(\chi) \|f\|_\infty) \left|\chi(w\underline{x} - t_{\underline{k}})\right|\ d\underline{x} \\
&\leq \ \sum_{\|\frac{1}{w} t_{\underline{k}}\|_2 \leq \gamma} \frac{M\, \psi\left(\lambda m_{\Pi,0}(\chi) \|f\|_\infty\right)}{m_{\Pi,0}(\chi)} \int_{\mathbb{R}^n} \left|\chi(w\underline{x} - t_{\underline{k}})\right|\ d\underline{x} \\
&= \ \sum_{\|\frac{1}{w} t_{\underline{k}}\|_2 \leq \gamma} \frac{M\, \psi\left(\lambda m_{\Pi,0}(\chi) \|f\|_\infty\right)}{w^n\, m_{\Pi,0}(\chi)} \int_{\mathbb{R}^n} \left|\chi(\underline{t})\right|\ d\underline{t}.
\end{aligned}$$

Now, denoting by $\mathcal{T}$ the number of the indexes $\underline{k}$ in the above sum, for every $w \geq 1$, we can write:

$$\begin{aligned}
\mathcal{T} \ &\leq \ \left[2\left(\left[\frac{\gamma w}{\delta}\right] + 1\right)\right]^n \ = \ 2^n \sum_{i=0}^{n} \binom{n}{i} \left(\left[\frac{\gamma w}{\delta}\right]\right)^{n-i} \ = \ 2^n \left[\left(\left[\frac{\gamma w}{\delta}\right]\right)^n + \right. \\
&\left. + n\left(\left[\frac{\gamma w}{\delta}\right]\right)^{n-1} + \ \ldots + 1\right] = 2^n w^n \cdot \left[\left(\left[\frac{\gamma}{\delta}\right]\right)^n + n\left(\left[\frac{\gamma}{\delta}\right]\right)^{n-1} \cdot \frac{1}{w} + \ldots + \frac{1}{w^n}\right] \\
&\leq \ w^n \cdot \left\{2^n \left[\left(\left[\frac{\gamma}{\delta}\right]\right)^n + n\left(\left[\frac{\gamma}{\delta}\right]\right)^{n-1} + \ldots + 1\right]\right\} \ =: \ w^n \cdot K. \qquad (4)
\end{aligned}$$

Hence, we finally have:

$$I^\varphi\left(\lambda S_w^\chi f\right) \ \leq \ \frac{M\, \psi\left(\lambda m_{\Pi,0}(\chi) \|f\|_\infty\right)}{m_{\Pi,0}(\chi)} \cdot \|\chi\|_1 \cdot K,$$

for every $w > 0$, from which the above claim holds. □

Now, the following convergence theorem can be established.

Theorem 3.2. *For every $f \in C_c(\mathbb{R}^n)$ there holds:*

$$\lim_{w \to +\infty} I^\varphi\left(\lambda\left[S_w^\chi f - f\right]\right) \ = \ 0,$$

for every $\lambda > 0$, where φ is a fixed φ-function.

Proof. First of all, we have that $S_w f \in E^\varphi(\mathbb{R}^n) \subset L^\varphi(\mathbb{R}^n)$, for every $w > 0$, in view of Lemma 3.1. Now, we will prove that:

$$\lim_{w \to +\infty} I^\varphi(\lambda(S_w^\chi f - f)) \ = \ \lim_{w \to +\infty} \int_{\mathbb{R}^n} \varphi(\underline{x},\ \lambda \left|\left(S_w^\chi f\right)(\underline{x}) - f(\underline{x})\right|)\, d\underline{x} \ = \ 0,$$

for every $\lambda > 0$, that is equivalent to show that the family:

$$(\varphi(\cdot,\ \lambda \left|(S_w^\chi f)(\cdot) - f(\cdot)\right|))_{w>0}$$

converges to zero in $L^1(\mathbb{R}^n)$, for every $\lambda > 0$. In order to prove this, we will use the Vitali convergence theorem in $L^1(\mathbb{R}^n)$.

Let now $\lambda > 0$ and $\varepsilon > 0$ be fixed. First of all we know that, by Theorem 2.2, the continuity of $\varphi(\underline{s}, \cdot)$, and since $\varphi(\underline{s}, 0) = 0$:

$$\lim_{w \to +\infty} \varphi(\underline{x},\ \lambda\left[(S_w^\chi f)(\underline{x}) - f(\underline{x})\right]) \ \leq\ \lim_{w \to +\infty} \varphi(\underline{x},\ \lambda \left\| S_w^\chi f - f \right\|_\infty) \ =\ 0,$$

for every $\underline{x} \in \mathbb{R}^n$. Moreover, in correspondence to $\gamma > 0$ and $\varepsilon > 0$ above fixed, by Lemma 2.1 (iii), there exists a sufficiently large $\widetilde{M} = \widetilde{M}(\varepsilon, \gamma) > 0$ such that:

$$\int_{\|\underline{x}\|_2 > \widetilde{M}} w^n\, |\chi(w\underline{x} - t_{\underline{k}})|\, d\underline{x} \ <\ \varepsilon,$$

for sufficiently large $w > 0$ and $t_{\underline{k}}$ such that $\left\| t_{\underline{k}} \right\|_2 \leq \gamma w$. Now, we can estimate what follows by using the same procedure used in the proof of Lemma 3.1:

$$\begin{aligned}
&\int_{\|\underline{x}\|_2 > \widetilde{M}} \varphi(\underline{x},\ \lambda (S_w^\chi f)(\underline{x})) \quad d\underline{x} \\
&\quad\leq\ \int_{\|\underline{x}\|_2 > M} \varphi\left(\underline{x},\ \lambda \sum_{\|\frac{1}{w} t_{\underline{k}}\|_2 \leq \gamma} \left| \chi(w\underline{x} - t_{\underline{k}}) \right| \|f\|_\infty \right) d\underline{x} \\
&\quad\leq\ \sum_{\|\frac{1}{w} t_{\underline{k}}\|_2 \leq \gamma} \frac{M\, \psi\left(\lambda m_{\Pi,0}(\chi)\, \|f\|_\infty\right)}{w^n\, m_{\Pi,0}(\chi)}\ w^n \int_{\|\underline{x}\|_2 > \widetilde{M}} \left| \chi(w\underline{x} - t_{\underline{k}}) \right|\, d\underline{x}. \\
&\quad<\ \varepsilon \sum_{\|\frac{1}{w} t_{\underline{k}}\|_2 \leq \gamma} \frac{M\, \psi\left(\lambda m_{\Pi,0}(\chi)\, \|f\|_\infty\right)}{w^n\, m_{\Pi,0}(\chi)} \ \leq\ \varepsilon\ \frac{K\, M\, \psi\left(\lambda m_{\Pi,0}(\chi)\, \|f\|_\infty\right)}{m_{\Pi,0}(\chi)},
\end{aligned}$$

for $w > 0$ sufficiently large.

Therefore, for $\varepsilon > 0$ there exists a set $E_\varepsilon = B(\underline{0}, M)$ such that for every measurable set F, with $F \cap E_\varepsilon = \emptyset$, we have

$$\begin{aligned}
\int_F \varphi(\underline{x},\ \lambda \left|(S_w^\chi f)(\underline{x}) - f(\underline{x})\right|)\, d\underline{x} \ &=\ \int_F \varphi(\underline{x},\ \lambda \left|(S_w^\chi f)(\underline{x})\right|)\, d\underline{x} \\
&\leq\ \int_{\|\underline{x}\|_2 > \widetilde{M}} \varphi(\underline{x},\ \lambda \left|(S_w^\chi f)(\underline{x})\right|)\, d\underline{x} \ <\ \varepsilon \cdot \mathcal{D},
\end{aligned}$$

with $\mathcal{D} := K\, M\, \psi\left(\lambda m_{\Pi,0}(\chi)\, \|f\|_\infty\right) / m_{\Pi,0}(\chi)$.

Finally, since $m_{\Pi,0}(\chi) \geq 1$, and by the absolute continuity of the Lebesgue integral, in correspondence of $\varepsilon > 0$ there exits $\bar{\delta} > 0$ such that, for any measurable set $B \subset \mathbb{R}^n$ with $|B| \ <\ \bar{\delta}$, we have:

$$\begin{aligned}
&\int_B \varphi(\underline{x},\ \lambda \left|(S_w^\chi f)(\underline{x}) - f(\underline{x})\right|) d\underline{x} \\
&\quad\leq\ \frac{1}{2} \int_B \varphi(\underline{x},\ 2\lambda \left|(S_w^\chi f)(\underline{x})\right|)\, d\underline{x} \ +\ \frac{1}{2} \int_B \varphi(\underline{x},\ 2\lambda \left|f(\underline{x})\right|)\, d\underline{x} \\
&\quad\leq\ \frac{1}{2} \int_B \varphi(\underline{x},\ 2\lambda m_{\Pi,0}(\chi)\, \|f\|_\infty)\, d\underline{x} \ +\ \frac{1}{2} \int_B \varphi(\underline{x},\ 2\lambda \|f\|_\infty)\, d\underline{x} \\
&\quad\leq\ \int_B \varphi(\underline{x},\ 2\lambda m_{\Pi,0}(\chi)\, \|f\|_\infty)\, d\underline{x} \ <\ \varepsilon.
\end{aligned}$$

Thus, the integrals

$$\int_{(\cdot)} \varphi(\lambda \left|(S_w^\chi f)(\underline{x}) - f(\underline{x})\right|)\, d\underline{x}$$

are equi-absolutely continuous. In conclusion, since all the assumptions of the Vitali convergence theorem are satisfied and observing that $\lambda > 0$ is arbitrary, the proof follows. □

Now, we can prove the following modular-type inequality for the multivariate sampling Kantorovich operators.

Theorem 3.3. *For every $f \in L^\varphi(\mathbb{R}^n)$, where φ is a fixed φ-function, there holds:*

$$I^\varphi(\lambda S_w^\chi f) \ \leq\ \frac{\delta^{-n} \|\chi\|_1}{m_{\Pi,0}(\chi)}\, I^\varphi\left(\lambda\, C\, m_{\Pi,0}(\chi)\, f\right) \ +\ \mathcal{A}_w, \quad \lambda > 0,$$

$w > 0$, with:

$$\mathcal{A}_w \ :=\ \frac{\delta^{-n}}{m_{\Pi,0}(\chi)} \sum_{\underline{k} \in \mathbb{Z}^n} \int_{R_{w, t_{\underline{k}}}} \left\{ \int_{\mathbb{R}^n} F\left(\underline{u},\ \underline{u} - \frac{\underline{s} + t_{\underline{k}}}{w} \right) |\chi(\underline{s})|\, d\underline{s} \right\} d\underline{u},$$

where the function F and the constant $C > 0$ are those of assumption ($\varphi 3$).

In particular, if $\mathcal{A}_w$ is uniformly bounded with respect $w > 0$, then the above inequality implies that S_w^χ maps $L^\varphi(\mathbb{R}^n)$ in $L^\varphi(\mathbb{R}^n)$.

Proof. Let $\lambda > 0$ be fixed. Proceeding as in the proof of Theorem 3.2, by applying the Jensen inequality twice, and Fubini-Tonelli theorem, we have:

$$\begin{aligned} I^{\varphi}(\lambda S_w^{\chi} f) &\leq \int_{\mathbb{R}^n} \varphi\left(\underline{x},\, \lambda \sum_{\underline{k}\in\mathbb{Z}^n} \frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} |f(\underline{u})|\, d\underline{u}\, |\chi(w\underline{x} - t_{\underline{k}})|\right) d\underline{x} \\ &\leq \frac{1}{m_{\Pi,0}(\chi)} \sum_{\underline{k}\in\mathbb{Z}^n} \frac{w^n}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} \left\{ \int_{\mathbb{R}^n} \varphi\left(\underline{x},\, \lambda\, m_{\Pi,0}(\chi)\, |f(\underline{u})|\right) |\chi(w\underline{x} - t_{\underline{k}})|\, d\underline{x} \right\} d\underline{u}. \end{aligned}$$

Now, by using the change of variable $w\underline{x} - t_{\underline{k}} = \underline{s}$, and assumption $(\varphi 3)$, we obtain:

$$\begin{aligned} I^{\varphi}(\lambda S_w^{\chi} f) &\leq \frac{1}{m_{\Pi,0}(\chi)} \sum_{\underline{k}\in\mathbb{Z}^n} \frac{1}{A_{\underline{k}}} \int_{R_{w,t_{\underline{k}}}} \left\{ \int_{\mathbb{R}^n} \varphi\left(\frac{\underline{s} + t_{\underline{k}}}{w},\, \lambda\, m_{\Pi,0}(\chi)\, |f(\underline{u})|\right) |\chi(\underline{s})|\, d\underline{s} \right\} d\underline{u} \\ &\leq \frac{\delta^{-n}}{m_{\Pi,0}(\chi)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{R_{w,t_{\underline{k}}}} \left\{ \int_{\mathbb{R}^n} \varphi\left(\frac{\underline{s} + t_{\underline{k}}}{w},\, \lambda\, m_{\Pi,0}(\chi)\, |f(\underline{u})|\right) |\chi(\underline{s})|\, d\underline{s} \right\} d\underline{u} \\ &\leq \frac{\delta^{-n}}{m_{\Pi,0}(\chi)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{R_{w,t_{\underline{k}}}} \left\{ \int_{\mathbb{R}^n} \varphi\left(\underline{u},\, \lambda\, C\, m_{\Pi,0}(\chi)\, |f(\underline{u})|\right) |\chi(\underline{s})|\, d\underline{s} \right\} d\underline{u} \\ &+ \frac{\delta^{-n}}{m_{\Pi,0}(\chi)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{R_{w,t_{\underline{k}}}} \left\{ \int_{\mathbb{R}^n} F\left(\underline{u},\, \underline{u} - \frac{\underline{s} + t_{\underline{k}}}{w}\right) |\chi(\underline{s})|\, d\underline{s} \right\} d\underline{u} \\ &= \frac{\delta^{-n}}{m_{\Pi,0}(\chi)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{R_{w,t_{\underline{k}}}} \varphi\left(\underline{u},\, \lambda\, C\, m_{\Pi,0}(\chi)\, |f(\underline{u})|\right) d\underline{u} \left\{ \int_{\mathbb{R}^n} |\chi(\underline{s})|\, d\underline{s} \right\} \\ &+ \frac{\delta^{-n}}{m_{\Pi,0}(\chi)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{R_{w,t_{\underline{k}}}} \left\{ \int_{\mathbb{R}^n} F\left(\underline{u},\, \underline{u} - \frac{\underline{s} + t_{\underline{k}}}{w}\right) |\chi(\underline{s})|\, d\underline{s} \right\} d\underline{u} \\ &= \frac{\delta^{-n}\|\chi\|_1}{m_{\Pi,0}(\chi)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{R_{w,t_{\underline{k}}}} \varphi\left(\underline{u},\, \lambda\, C\, m_{\Pi,0}(\chi)\, |f(\underline{u})|\right) d\underline{u} \\ &+ \frac{\delta^{-n}}{m_{\Pi,0}(\chi)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{R_{w,t_{\underline{k}}}} \left\{ \int_{\mathbb{R}^n} F\left(\underline{u},\, \underline{u} - \frac{\underline{s} + t_{\underline{k}}}{w}\right) |\chi(\underline{s})|\, d\underline{s} \right\} d\underline{u} \\ &=: \frac{\delta^{-n}\|\chi\|_1}{m_{\Pi,0}(\chi)} \int_{\mathbb{R}^n} \varphi\left(\underline{u},\, \lambda\, C\, m_{\Pi,0}(\chi)\, |f(\underline{u})|\right) d\underline{u} + \mathcal{A}_w \\ &= \frac{\delta^{-n}\|\chi\|_1}{m_{\Pi,0}(\chi)} I^{\varphi}\left(\lambda\, C\, m_{\Pi,0}(\chi)\, f\right) + \mathcal{A}_w, \end{aligned}$$

for every $w > 0$. This completes the proof. □

Then we are able to prove the following modular convergence result.

Theorem 3.4. *Let φ be a fixed φ-function with the function F of assumption $(\varphi 3)$ such that the series $\mathcal{A}_w$ defined in the statement of Theorem 3.3 satisfies:*

$$\mathcal{A}_w \to 0, \qquad \textit{as} \qquad w \to +\infty. \tag{5}$$

Then for every $f \in L^{\varphi}(\mathbb{R}^n)$ there exists $\lambda > 0$ such that:

$$\lim_{w\to+\infty} I^{\varphi}\left(\lambda\left(S_w^{\chi} f - f\right)\right) = 0.$$

Proof. First of all, by Theorem 2.3 there exists $\bar{\lambda} > 0$ such that, for every $\varepsilon > 0$ there is $g \in C_c(\mathbb{R}^n)$ with $I^{\varphi}(\bar{\lambda}(f - g)) < \varepsilon$.

Let now $\varepsilon > 0$ be fixed. We choose $\lambda > 0$ such that:

$$\lambda \leq \min\left\{ \frac{\bar{\lambda}}{3},\, \frac{\bar{\lambda}}{3\, C\, m_{\Pi,0}} \right\},$$

where $\bar{\lambda} > 0$ is fixed as above. Now, using the convexity of φ and Theorem 3.3, we can write:

$$\begin{aligned} I^{\varphi}\left(\lambda\left(S_w^{\chi} f - f\right)\right) &\leq \frac{1}{3} I^{\varphi}\left(3\lambda\left(S_w^{\chi} f - S_w^{\chi} g\right)\right) + \frac{1}{3} I^{\varphi}\left(3\lambda\left(S_w^{\chi} g - g\right)\right) \\ &+ \frac{1}{3} I^{\varphi}\left(3\lambda(g - f)\right) \leq \frac{\delta^{-n}\|\chi\|_1}{m_{\Pi,0}(\chi)} I^{\varphi}\left(3\lambda\, C\, m_{\Pi,0}(\chi)\, (f - g)\right) \end{aligned}$$

$$+ \mathcal{A}_w + I^\varphi\left(3\lambda\left(S_w^\chi g - g\right)\right) + I^\varphi\left(3\lambda(g-f)\right)$$
$$\leq \left(1 + \frac{\delta^{-n}\|\chi\|_1}{m_{\Pi,0}(\chi)}\right) I^\varphi\left(\bar{\lambda}(g-f)\right) + I^\varphi\left(3\lambda\left(S_w^\chi g - g\right)\right) + \mathcal{A}_w$$
$$\leq \left(1 + \frac{\delta^{-n}\|\chi\|_1}{m_{\Pi,0}(\chi)}\right) \varepsilon + I^\varphi\left(3\lambda\left(S_w^\chi g - g\right)\right) + \mathcal{A}_w,$$

and passing to the limsup as $w \to +\infty$, we obtain the assertion in view of Theorem 3.2, assumption (5), and since ε is arbitrary. □

Remark 2. Note that, it is easy to find examples of τ-bounded φ-functions which satisfy condition (φ3) with $F = 0$ (see Section 4) and therefore $A_w = 0$. For further examples of Musielak-Orlicz spaces, generated by φ-function which satisfy the τ-boundedness with $F \neq 0$, see e.g., [49, 16, 15].

4 Example of kernels and applications to concrete spaces

Now, we show some well-known and important class of kernels which satisfy the above assumptions $(\chi 1)-(\chi 3)$, and for which the above results hold. For more details, see e.g., [31].

First of all, we recall the definition of the one-dimensional central B-spline of order N (see e.g., [5]):

$$\beta^N(x) := \frac{1}{(N-1)!}\sum_{i=0}^{N}(-1)^i\binom{N}{i}\left(\frac{N}{2}+x-i\right)_+^{N-1}, \qquad x \in \mathbb{R}. \tag{6}$$

The corresponding multivariate version of central B-spline of order N is given by:

$$\mathcal{B}_n^N(\underline{x}) := \prod_{i=1}^{n}\beta^N(x_i), \qquad \underline{x} = (x_1, ..., x_n) \in \mathbb{R}^n. \tag{7}$$

The multivariate kernels $\mathcal{B}_n^N$ satisfy assumptions $(\chi 1)-(\chi 3)$.

Other important kernels are given by the so-called Jackson type kernels of order N, defined in the univariate case by:

$$J_N(x) := c_N \operatorname{sinc}^{2N}\left(\frac{x}{2N\pi\alpha}\right), \qquad x \in \mathbb{R}, \tag{8}$$

with $N \in \mathbb{N}$, $\alpha \geq 1$, and c_N is a non-zero normalization coefficient, given by:

$$c_N := \left[\int_{\mathbb{R}} \operatorname{sinc}^{2N}\left(\frac{u}{2N\pi\alpha}\right) du\right]^{-1}.$$

For the sake of completeness, we recall that the well-known (above mentioned) $sinc$-function is defined as $\sin(\pi x)/\pi x$, if $x \neq 0$, and 1 if $x = 0$, see e.g., [45, 46]. As in case of the central B-splines, multivariate Jackson type kernels of order N are defined by:

$$\mathcal{J}_N^n(\underline{x}) := \prod_{i=1}^{n} J_N(x_i), \qquad \underline{x} = (x_1, ..., x_n) \in \mathbb{R}^n. \tag{9}$$

In particular, Jackson type kernels revealed to be very useful, e.g., for applications to the engineering field, [10, 11].

Finally, as a last important class of (radial) kernels we can mention the so called Bochner-Riesz kernels of order $N > 0$, defined as follows:

$$r_N(\underline{x}) := \frac{2^N}{\sqrt{2\pi}}\Gamma(N+1)\,\|\underline{x}\|_2^{-N-1/2} J_{N+1/2}(\|\underline{x}\|_2), \qquad \underline{x} \in \mathbb{R}^n, \tag{10}$$

where J_λ is the Bessel function of order λ ([23]), and Γ is the usual Euler gamma function. Note that, even if the Bochner-Riesz kernels do not have the tensor-product form (as instead has, e.g., the multivariate Jackson-type kernels) they work well in the applications to imaging (see e.g., [29] for a detailed investigation and a comparison with some classical interpolation and quasi-interpolation methods for digital image processing). For several examples of kernels, see, e.g., [8, 13, 9, 31, 33, 35, 34, 36, 38, 24].

Now, we will show how to construct non-trivial examples of multivariate Musielak-Orlicz spaces.

For instance, one can consider φ-functions of the product-type, of the form:

$$\varphi(\underline{s}, u) := \xi(\underline{s})\,\widetilde{\varphi}(u), \quad \underline{s} \in \mathbb{R}^n, \quad u \in \mathbb{R}_0^+, \tag{11}$$

which satisfy the following conditions:

($\mathcal{F}$1) $\xi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, such that there exist $M \geq m > 0$ such that $m \leq \xi(\underline{s}) \leq M$, for every $\underline{s} \in \mathbb{R}^n$;

($\mathcal{F}$2) $\widetilde{\varphi} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a convex function such that $\widetilde{\varphi}(0) = 0$, and $\widetilde{\varphi}(u) > 0$ for $u > 0$;

($\mathcal{F}$3) for every $\lambda_1 > 0$ there exists $\lambda_2 \geq 1$ such that:

$$\lambda_1\,\widetilde{\varphi}(u) \leq \widetilde{\varphi}(\lambda_2 u), \quad u \in \mathbb{R}_0^+.$$

In this way, it is easy to see that assumptions (φ1) and (φ2) are satisfied. Further, concerning (φ3) we can write what follows:

$$\varphi(\underline{s}-\underline{t},u) \;=\; \xi(\underline{s}-\underline{t})\,\widetilde{\varphi}(u) \;\leq\; \frac{M}{m}\,\xi(\underline{s})\,\widetilde{\varphi}(u) \;\leq\; \xi(\underline{s})\,\widetilde{\varphi}(\lambda_2 u) \;=\; \varphi(\underline{s},\lambda_2 u),$$

for every $u \geq 0$, where $\lambda_2 \geq 1$ is the parameter of condition ($\mathcal{F}$3) corresponding to $\lambda_1 = M/m$. The above inequality shows that φ-functions of the form as in (11) are τ-bounded with $F \equiv 0$ and $C = \lambda_2$. In fact, by the above construction we can consider to have obtained some examples of weighted Orlicz spaces.

Some concrete examples of weighted Orlicz spaces generated by φ-function of the form (11) can be obtained by choosing, for instance,

$$\xi(\underline{s}) \;:=\; \frac{5}{\|\underline{s}\|_2^2+1} + 1, \qquad \underline{s} \in \mathbb{R}^n, \tag{12}$$

and as function $\widetilde{\varphi}$ one of the following:

$$\widetilde{\varphi}_1(u) \;:=\; u^p, \qquad \widetilde{\varphi}_2(u) \;:=\; u^\alpha \log^\beta(u+e), \tag{13}$$

for every $u \geq 0$, where $1 \leq p < +\infty$, $\alpha \geq 1$, and $\beta > 0$. Obviously, it is easy to show that the above product-type φ-functions satisfy conditions ($\mathcal{F}$1), ($\mathcal{F}$2), and ($\mathcal{F}$3). The Musielak-Orlicz spaces generated by $\varphi = \xi\,\widetilde{\varphi}_1$ are the so-called *weighted L^p-spaces*, and that ones generated by $\varphi = \xi\,\widetilde{\varphi}_2$ are the *weighted Zygmund spaces*.

Furthermore, if $\xi \equiv 1$ in the above examples, i.e., $\varphi = \widetilde{\varphi}$ and does not depend on the parameter s, we found the case of the Orlicz spaces considered in [12] for functions of one-variable, and in [31] for the multivariate case. In particular, the functions $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ generate the well-known L^p-spaces, and Zygmung spaces, respectively.

Further, we can also consider the Musielak-Orlicz space generated by the following φ-function:

$$\varphi_3(\underline{s},u) \;:=\; e^{\Psi(\underline{s})u^\gamma} - 1, \qquad \underline{s} \in \mathbb{R}^n, \qquad u \in \mathbb{R}_0^+, \tag{14}$$

$\gamma > 0$, where the function Ψ satisfies the inequality of condition ($\mathcal{F}$1) for suitable $0 < m \leq M$. By simple computations, it can be shown that also φ-functions of the form as in (14) are τ-bounded with $F \equiv 0$ and $C = (M/m)^{1/\gamma}$. As an example of function Ψ one can consider again, e.g., the function $\Psi(\underline{s}) = \xi(\underline{s})$ defined in (12).

In the Orlicz-case, when $\Psi(\underline{s}) = 1$, for every $\underline{s} \in \mathbb{R}^n$, we have:

$$\widetilde{\varphi}_3(u) \;:=\; e^{u^\gamma} - 1, \quad u \in \mathbb{R}_0^+,$$

$\gamma > 0$, which generates the well-known *exponential spaces*.

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