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Rate of convergence of multinode Shepard operators

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Abstract

The triangular Shepard method, introduced by Little in 1983 [7], is a convex combination of triangular basis functions with linear polynomials, based on the vertices of the triangles, that locally interpolate the given data at the vertices. The method has linear precision and reaches quadratic approximation order [3]. As specified by Little, the triangular Shepard method can be generalized to higher dimensions and to sets of more than three points. In this paper we introduce the multinode Shepard method as a generalization of the triangular Shepard method in the case of scattered points in $\mathbb{R}^s$, $s \in \mathbb{N}$, and we study the remainder term and its asymptotic behavior.

1 Introduction

In 1968 D. Shepard [12] introduced an approximation method for the interpolation of scattered data which consists in a weighted average of functional values at the data points. The method is easy to implement (indeed it is the fastest method for the interpolation of scattered data [13]) but it reproduces exactly only constant polynomials and has flat spots in the neighbourhood of all data points. With the aim of improving the performance of the Shepard method, in relation with the accuracy of approximation, the efficiency and the local behaviour, several methods have been proposed in the years, that use global or local (i.e. compactly supported) basis functions which are the normalization of the inverse distance from the scattered points, as in the Shepard method (see [4] and the references therein). All these methods make use of additional derivative data in order to better approximate the unknown function in the neighborhood of the data sites; if these data are not available they approximate them by means of different techniques (for example least squares approximation). In 1983 F. Little [7] considers weighted average of local linear interpolants based on triples of data sites and takes as basis functions the normalization of the product of inverse distances from the points of the triples. This method overcomes the drawbacks of the Shepard method and, at the same time, maintains its features of simplicity of implementation and speed. In fact, the use of a searching technique to detect and select the nearest neighbor points [1] to determine the best local linear interpolant on compact triangulations [3], allows to consider the triangular Shepard method a fast meshfree method with an adequate order and a good accuracy of approximation. As Little suggests, his method can be generalized to higher dimensions and to sets of more than three points. Consequently, there is the need to analyze the asymptotic behavior of the remainder term of interpolation operators whose basis functions are the normalization of the product of inverse distances from the points of σ-tuples in $\mathbb{R}^s$, $s \in \mathbb{N}$. The case $s = 1$ has been already studied in connection with the problem of the reconstruction of a function from Hermite-Birkhoff data [6]. Here we generalize that result to general dimensions and to a number of points which allows the unisolvence of local polynomial interpolation. The paper is organized as follows: in Section 2 we introduce and analyze some properties of the multinode Shepard operator. In Section 3 we study the approximation order of the multinode Shepard operator and in Section 4 we provide numerical evidences which confirm the theoretical results on its approximation order.

2 Multinode Shepard operators

Let be $X = \{x_0, x_1, \ldots, x_n\}$ a set of scattered nodes of $\mathbb{R}^s$ and $T = \{t_1, \ldots, t_m\}$ a covering of X by subsets constituted by σ points, that is $t_j = \left\{x_{j_k}\right\}_{k=1,\ldots,\sigma}$ is a set of pairwise distinct nodes $x_{j_1}, \ldots, x_{j_\sigma} \in X$ and

$$\bigcup_{j=1}^{m} \{j_1, \ldots, j_\sigma\} = \{1, 2, \ldots, n\}. \tag{1}$$

With $T = \{t_1, \ldots, t_m\}$ we denote also the set of σ-tuples $t_j = \left(x_{j_k}\right)_{k=1,\ldots,\sigma}$ that we identify with the polygon (possibly with self-intersections) bounded by the finite chain of straight line segments $\left[x_{j_k}, x_{j_{k+1}}\right]$, $k = 1, \ldots, \sigma$, $x_{j_{\sigma+1}} = x_{j_1}$, respectively. It will be clear from the context if we are dealing with subsets or σ-tuples, depending on the need of the order of the nodes in the

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subsets. The multinode basis function with respect to T is defined by

$$B_{\mu,j}(x)=\frac{\prod_{\ell=1}^{\sigma}|x-x_{j_\ell}|^{-\mu}}{\sum_{k=1}^{m}\prod_{\ell=1}^{\sigma}|x-x_{k_\ell}|^{-\mu}},\quad j=1,\dots,m,\quad \mu>0. \tag{2}$$

In analogy with the univariate and bivariate cases [5, 6], the multinode basis functions satisfy the following properties

Proposition 2.1. *The multinode basis function $B_{\mu,j}(x)$ and its derivatives up to the order $p-1$ vanish at all nodes $x_i\in X$ that are not a vertex of the corresponding polygon t_j. That is, for any $j=1,\dots,m$ and $i\notin\{j_1,\dots,j_\sigma\}$, we have*

$$B_{\mu,j}(x_i)=0, \tag{3}$$

$$D^{\ell}B_{\mu,j}(x_i)=0,\quad \mu>\ell, \tag{4}$$

where $D^k g$ denotes the vector of all partial derivative of g. Moreover, they form a partition of unity, that is

$$\sum_{j=1}^{m}B_{\mu,j}(x)=1 \tag{5}$$

and consequently, for each $i=1,\dots,n$,

$$\sum_{j\in J_i}B_{\mu,j}(x_i)=1, \tag{6}$$

$$\sum_{j\in J_i}D^{\ell}B_{\mu,j}(x_i)=0,\quad \mu>\ell \tag{7}$$

where $J_i=\left\{k\in\{1,\dots,m\}: i\in\{k_1,\dots,k_\sigma\}\right\}$ is the set of polygon which have x_i as a vertex.

Proof. Let $x_i\in X$. By (1) it follows that the set J_i is non-empty. By multiplying both the numerator and the denominator of $B_{\mu,j}(x)$ with $|x-x_i|^{\mu}$ we have

$$B_{\mu,j}(x)=\frac{C_j(x)}{\sum_{k=1}^{m}C_k(x)}$$

where

$$C_k(x)=|x-x_i|^{\mu}\prod_{\ell=1}^{\sigma}\frac{1}{|x-x_{j_\ell}|^{\mu}},\qquad k=1,\dots,m$$

and the proof follows straightforward [6]. □

The multinode Shepard operator is defined by

$$\mathcal{M}_{\mu}[f](x)=\sum_{j=1}^{m}B_{\mu,j}(x)P_j[f](x) \tag{8}$$

where $P_j[f](x)$ is an interpolation polynomial on the σ-tuple t_j which reproduces polynomials up to the degree p. Let us remark that, thanks to the properties satisfied by the basis function $B_{\mu,j}(x)$, stated in Proposition 2.1, the multinode Shepard operator inherits the interpolation conditions satisfied by the polynomial $P_j[f](x)$.

With the aim to study the rate of convergence of the multinode Shepard operator, we need to give a bound to the interpolation polynomial $P_j[f](x)$.

2.1 Error bound for $P_j[f](x)$

Let $\Omega\subset\mathbb{R}^s$ be a non-empty compact convex domain containing X. By following Farwig's notations [8] we denote by $C^{p,1}(\Omega)$ of differentiable functions $f:\Omega\to\mathbb{R}$ whose partial derivatives are Lipschitz-continuous of order p, equipped by the seminorm

$$\|f\|_{p,1}=\sup\left\{\frac{|D^{\nu}f(u)-D^{\nu}f(v)|}{|u-v|}: u,v\in\Omega, u\neq v, |\nu|=p\right\}. \tag{9}$$

Proposition 2.2. *Let $f\in C^{p,1}(\Omega)$ then, for any $x\in\Omega$; we have*

$$\left|f(x)-P_j[f](x)\right|\leq\left(1+\left\|P_j\right\|_{\infty}\right)\left(\frac{s^p}{(p-1)!}\left|x-x_{j_{\max}}\right|^{p+1}\right)\|f\|_{p,1},$$

where $\left|x-x_{j_{\max}}\right|=\max_{i=2,\dots,\sigma}\left|x-x_{j_i}\right|$.

Proof. By the reproduction property of $P_j[f](x)$ it follows that

$$\begin{aligned}\left|f(x)-P_j[f](x)\right| &\leq \left|f(x)-T_p\left[f,x_{j_1}\right](x)\right| + \left|T_p\left[f,x_{j_1}\right](x)-P_j[f](x)\right| \\ &\leq \left|f(x)-T_p\left[f,x_{j_1}\right](x)\right| + \left|P_j\left[T_p\left[f,x_{j_1}\right]\right](x)-P_j[f](x)\right| \\ &\leq \left(1+\left|\left|P_j\right|\right|_\infty\right)\left|f(x)-T_p\left[f,x_{j_1}\right](x)\right|\end{aligned}$$

where $T_p\left[f,x_{j_1}\right](x)$ is the p-th order multivariate Taylor polynomial for f centered at x_{j_1}. The thesis follows by bounding the remainder term in Taylor polynomial in standard way [8]

$$\left|f(x)-T_p\left[f,x_{j_1}\right](x)\right| \leq \frac{s^p}{(p-1)!}\,||f||_{p,1}\left|x-x_{j_{\max}}\right|^{p+1}.$$

□

Remark 1. Let us observe that the study of the remainder term of the interpolation polynomial $P_j[f](x)$, in each particular case, is the crucial point of the numerical algorithm since it gives a criteria for the selection of the optimal σ−tuples set which guarantees a good accuracy of approximation.

3 Approximation order of multinode Shepard operators

To study the approximation order of the multipoint Shepard operator (8) we need the following notations. Let $||\cdot||$ denote the maximum norm, $R_r(y)=\{x\in\mathbb{R}^s : ||x-y||\leq r\}$ the axis-aligned closed cube with centre y and edge length $2r$ and $Conv(t)$ the convex hull of $t\in T$. Let

$$h' = \inf\{r>0 : \forall x\in\Omega\ \exists t\in T : R_r(x)\cap t\neq\emptyset\}, \tag{10}$$

$$h'' = \inf\{r>0 : \forall t\in T\ \exists x\in\Omega : Conv(t)\subset R_r(x)\} \tag{11}$$

and finally

$$h = \max\{h',h''\}. \tag{12}$$

The positive real number h is then a measure of the fill distance of points in X and of the largeness of the polygons in T: h decreases if the number of a rather uniform distribution of scattered points increases and the polygons remain relatively small. We further let

$$M = \sup_{x\in\Omega}\#\{t\in T : R_h(x)\cap t\neq\emptyset\}, \tag{13}$$

the maximum number of polygons with at least one vertex in some square with edge length $2h$. Small values of M, in correspondence of small values of h, imply that there are no clusters of polygons.

Theorem 3.1. *Let Ω be a compact convex domain which contains X, $f\in C^{p,1}(\Omega)$ and $\mu>\frac{s+p+1}{\sigma}$. Then*

$$||f-\mathcal{M}_\mu[f]|| \leq CM||f||_{p,1}h^{p+1},$$

where C is a positive constant which depends only on S and μ.

Proof. Let

$$Q_r(y)=\{x=(x_1,\dots,x_s)\in\mathbb{R}^s : y_k-r<x_k\leq y_k+r, k=1,\dots,s\}$$

be the axis-aligned half-open cube with centre $y=(y_1,\dots,y_s)\in\mathbb{R}^s$ and edge length $2r$. For a fixed $x\in\Omega$ we consider the covering $\{U_k\}_{k\in\mathbb{N}_0}$ of Ω by disjoint half-open annuli with centre x, radius $2kh$, and width h

$$U_k = \bigcup_{\nu\in\mathbb{Z}^s,\,||\nu||=k} Q_h(x+2h\nu).$$

The compactness of Ω implies the existence of some $N\in\mathbb{N}$, independent of x and of order $O(1/h)$, such that

$$\Omega\subset\bigcup_{k=0}^{N}U_k.$$

By definition of M (13), the number of polygons with at least one vertex in U_k is bounded by

$$\#\{t\in T : U_k\cap t\neq\emptyset\}\leq 2sM(2k+1)^{s-1},\quad k=1,\dots,N. \tag{14}$$

Let now t be any polygon with at least one vertex x_i in U_k. By (11) all vertices of t lie in the half-open cube with center x_i and radius $2h$ and therefore only one of the following cases is possible:

$$\begin{array}{llcll} 1. & t\cap U_{k-1}\neq\emptyset & \Longrightarrow & (2k-3)h\leq||x-x_i||\leq(2k+1)h, & \forall x_i\in t,\\ 2. & t\subset U_k & \Longrightarrow & (2k-1)h\leq||x-x_i||\leq(2k+1)h, & \forall x_i\in t,\\ 3. & t\cap U_{k+1}\neq\emptyset & \Longrightarrow & (2k-1)h\leq||x-x_i||\leq(2k+3)h, & \forall x_i\in t.\end{array} \tag{15}$$

Let T_0 be the set of all hexagons with at least a vertex in U_0. The definitions of h' in (10) and M in (13) imply that T_0 contains at least one and at most M hexagons and for each hexagon $t_j \in T_0$ we have

$$\prod_{i=1}^{\sigma} \left|\left|x - x_{j_i}\right|\right| \leq h \cdot (3h)^{\sigma-1} = 3^{\sigma-1} h^{\sigma}, \tag{16}$$

because one vertex of t_j is inside U_0 and the other $\sigma - 1$ are in $U_0 \cup U_1$. For $k = 1, \ldots, N$ let T_k be the set of all polygons with at least a vertex in U_k and no vertex in U_{k-1}. By (14), this set contains at most $2sM\,(2k+1)^{s-1}$ polygons and by case 3 in (15) we have

$$((2k-1)h)^{\sigma} \leq \prod_{i=1}^{\sigma} \left|\left|x - x_{j_i}\right|\right| \leq ((2k+3)h)^{\sigma} \tag{17}$$

for each polygon $t_j \in T_k$. By construction,

$$\bigcup_{k=0}^{N} T_k = T, \qquad \bigcap_{k=0}^{N} T_k = \emptyset.$$

Let $e(x)$ denote the absolute value of the approximation error

$$e(x) = \left| f(x) - \mathcal{M}_{\mu}[f](x) \right|$$

of the Multipoint Shepard interpolant at x. By (8) and the fact that the basis function $B_{\mu,j}$ are non-negative and form a partition of unity,

$$e(x) = \left| \sum_{j=1}^{m} B_{\mu,j}(x) f(x) - \sum_{j=1}^{m} B_{\mu,j}(x) P_j[f](x) \right| \leq \sum_{j=1}^{m} \left| f(x) - P_j[f](x) \right| B_{\mu,j}(x).$$

By Proposition 2.2 and (2) we then get

$$\begin{aligned} e(x) &\leq ||f||_{p,1} \sum_{j=1}^{m} \left(\left(1 + \left|\left|P_j\right|\right|_{\infty}\right) \frac{s^p}{(p-1)!} \left|x - x_{j_{\max}}\right|^{p+1} \right) \frac{\prod_{\ell=1}^{\sigma} \left|x - x_{j_\ell}\right|^{-\mu}}{\sum_{k=1}^{m} \prod_{\ell=1}^{\sigma} \left|x - x_{k_\ell}\right|^{-\mu}}, \\ &\leq C' ||f||_{p,1} \sum_{j=1}^{m} \left(\left(1 + \left|\left|P_j\right|\right|_{\infty}\right) \frac{s^p}{(p-1)!} \left|\left|x - x_{j_{\max}}\right|\right|^{p+1} \right) \frac{\prod_{\ell=1}^{\sigma} \left|\left|x - x_{j_\ell}\right|\right|^{-\mu}}{\sum_{k=1}^{m} \prod_{\ell=1}^{\sigma} \left|\left|x - x_{k_\ell}\right|\right|^{-\mu}}, \end{aligned}$$

where $C' = \sqrt{s}^{(s+1)m\mu}$ is a constant which arises since we bound the Euclidean norm with the maximum norm. Now let $t_i \in T$ be an polygon such that

$$\prod_{\ell=1}^{\sigma} \left|\left|x - x_{i_\ell}\right|\right| = \min_{j=1,\ldots,m} \prod_{\ell=1}^{\sigma} \left|\left|x - x_{j_\ell}\right|\right|.$$

Since at least one polygon of T belongs to T_0, we know from (16) that

$$\prod_{\ell=1}^{\sigma} \left|\left|x - x_{j_\ell}\right|\right| \leq 3^{\sigma-1} h^{\sigma}.$$

For each $s_j \in S_0$ we then have

$$\prod_{\ell=1}^{\sigma} \frac{\left|\left|x - x_{i_\ell}\right|\right|}{\left|\left|x - x_{j_\ell}\right|\right|} \leq 1$$

and for each $s_j \in S_k$, $k = 1, \ldots, N$, using (17),

$$\prod_{\ell=1}^{\sigma} \frac{\left|\left|x - x_{i_\ell}\right|\right|}{\left|\left|x - x_{j_\ell}\right|\right|} \leq \frac{3^{\sigma-1} h^{\sigma}}{((2k-1)h)^{\sigma}} = \frac{3^{\sigma-1}}{(2k-1)^{\sigma}}.$$

Therefore,

$$\frac{\prod_{\ell=1}^{\sigma} \left|\left|x - x_{j_\ell}\right|\right|^{-\mu}}{\sum_{k=1}^{m} \prod_{\ell=1}^{\sigma} \left|\left|x - x_{k_\ell}\right|\right|^{-\mu}} \leq \prod_{\ell=1}^{\sigma} \frac{\left|\left|x - x_{j_\ell}\right|\right|^{-\mu}}{\left|\left|x - x_{i_\ell}\right|\right|^{-\mu}} \leq \begin{cases} 1, & \text{if } \ s_j \in S_0, \\ 3^{(\sigma-1)\mu}/(2k-1)^{\sigma\mu}, & \text{if } \ s_j \in S_k. \end{cases}$$

Without loss of generality, for $s_j \in S_k$ we can assume $x_{j_1} \in U_k$; then $\left|\left|x - x_{j_1}\right|\right| \leq h$ for each $s_j \in S_0$ and $\left|\left|x - x_{j_1}\right|\right| \leq (2k+1)h$ for each $s_j \in S_k$, $k = 1, \ldots, N$. Moreover, taking into account that $h_j \leq 3h$, we get

$$\begin{aligned} e(x) &\leq C' ||f||_{2,1} \left(1 + P_{\max}\right) \left(\sum_{s_j \in S_0} \frac{s^p}{(p-1)!} h^{p+1} \right| \\ &\left| + \sum_{k=1}^{N} \sum_{s_j \in S_k} \left(\frac{s^p}{(p-1)!} (2k+1)^{p+1} h^{p+1} \right) \frac{3^{(\sigma-1)\mu}}{(2k-1)^{\sigma\mu}} \right) \\ &\leq C' ||f||_{2,1} \left(1 + P_{\max}\right) \left(\sum_{s_j \in S_0} \left(\frac{s^p}{(p-1)!} \right) + 3^{(\sigma-1)\mu} \sum_{k=1}^{N} \sum_{s_j \in S_k} \frac{\frac{s^p}{(p-1)!} (2k+1)^{p+1}}{(2k-1)^{\sigma\mu}} \right) h^{p+1} \end{aligned}$$

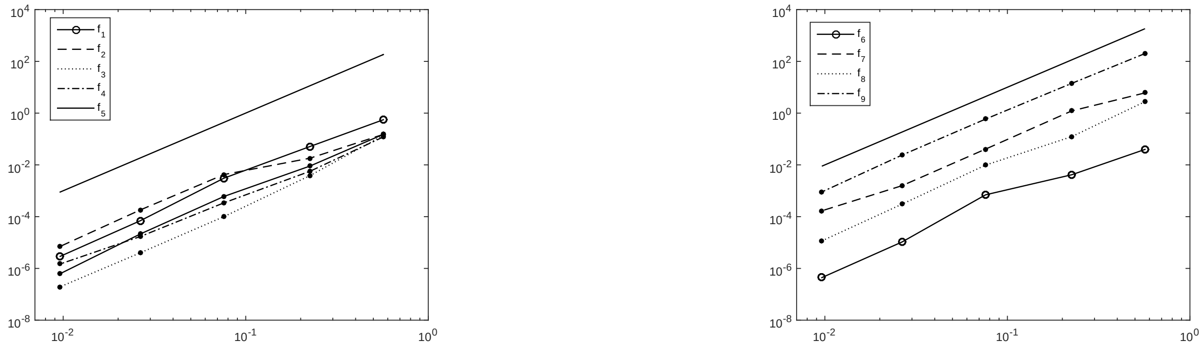

Figure 1: Log-log-plot of the approximation error of the hexagonal Shepard method. As reference, the solid line indicates a perfect cubic trend.

where $P_{\max} = \max_j \left|\left|P_j\right|\right|_\infty$. Using (14) we finally have

$$\begin{aligned} e(x) &\leq C'M\,||f||_{2,1}\,(1+P_{\max})\left(\left(\frac{2^p}{(p-1)!}\right)+3^{(\sigma-1)\mu}\sum_{k=1}^{N}2s\,(2k+1)^{s-1}\,\frac{\frac{2^p}{(p-1)!}\,(2k+1)^{p+1}}{(2k-1)^{\sigma\mu}}\right)h^{p+1}\\ &\leq C'M\,||f||_{2,1}\,(1+P_{\max})\left(\left(\frac{2^p}{(p-1)!}\right)+3^{(\sigma-1)\mu}\,\frac{2^{p+1}s}{(p-1)!}\sum_{k=1}^{N}\frac{(2k+1)^{s+p}}{(2k-1)^{\sigma\mu}}\right)h^{p+1}. \end{aligned}$$

As the serie $\sum_{k=1}^{\infty}\frac{(2k+1)^{s+p}}{(2k-1)^{\sigma\mu}}$ converges for $\mu > \frac{s+p+1}{\sigma}$, we conclude that the approximation order of $\mathcal{M}_\mu$ is $O(h^{p+1})$. □

4 Numerical evidences

4.1 Univariate case

The multinode Shepard operator in the univariate case has been studied in [6], in connection with the problem of the reconstruction of a function from Hermite-Birkhoff data. More precisely, the initial unsolvable problem is split up in subproblems which have a unique polynomial solution; the local polynomials are then blended with multinode basis functions to obtain a global interpolant. If, for simplicity, we assume that each basis function is defined on σ nodes and each interpolating polynomial reproduces polynomials up to the degree p, from [6, Theorem 4] follows that the approximation order $p+1$ is reached for $\mu > \frac{1+p+1}{\sigma}$, which is in line with the result demonstrated in Theorem 3.1.

4.2 Bivariate case

4.2.1 Triangular Shepard operator

The triangular Shepard operator has been introduced by Little in [7] and its properties and approximation order have been deeply studied in [3]. In this case the covering of X is realized by triangles t_j, i.e. $\sigma = 3$, the interpolation polynomial $P_j[f](x)$ is the linear Lagrange interpolation polynomial on t_j, i.e. $p = 1$. As specified in [3, Theorem 4.2], the quadratic approximation order is reached for $\mu > \frac{4}{3}$ which confirms the theoretical result shown in Theorem 3.1.

4.2.2 Quadratic triangular Shepard method

The quadratic triangular Shepard operator has been proposed in [5] in order to increase the approximation order of the triangular Shepard method. This operator is defined by combining the triangular Shepard basis functions with a modified version of the linear local interpolant on the vertices of the triangle which reproduces polynomials up to the degree 2. As specified in [5, Theorem 2], the cubic approximation order is reached for $\mu > \frac{5}{3}$ which is in line with Theorem 3.1.

4.2.3 Hexagonal Shepard operator

Another possibility to increase the approximation order of the triangular Shepard method is by introducing six-nodes basis functions combined with Lagrange polynomials on six-tuples. In this case the covering of X is realized by hexagons, i.e. $\sigma = 6$, the interpolation polynomial is the quadratic Lagrange interpolation polynomial on the six-tuple, i.e. $p = 2$. According to Theorem 3.1, the expected cubic approximation order will be reached for $\mu > \frac{5}{6}$ and the theoretical result is confirmed by the numerical tests shown in Figure 1, where we compare a perfect cubic trend with the approximation error of the hexagonal Shepard operator. The results are obtained by considering the first nine functions of the well known set of test functions for scattered data interpolation introduced in [11].

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4.3 Trivariate case

4.3.1 Tetrahedral Shepard operator

The tetrahedral Shepard operator is the generalization of the triangular Shepard method to the trivariate case. In this case $s = 3$, the covering of X is realized by means of tetrahedra, i.e. $\sigma = 4$ and the interpolation polynomial is the linear Lagrange polynomial on the vertices of each tetrahedron. According to Theorem 3.1, the expected quadratic approximation order will be reached for $\mu > \frac{5}{4}$.

5 Conclusions and future work

In this paper we introduce the multinode Shepard operator as a generalization of the triangular Shepard method [7] to any dimension s and to sets of $s+1$ or more points. The point sets have the same cardinality σ and we assume the unisolvence of all local interpolation problems by polynomials of degree not greater than p relative to sets of $\binom{p+s}{s}$ interpolation conditions. The main result of the paper regards the rate of convergence of the multinode Shepard operator in this general situation. This result is in line with previous studies on particular cases, which are reported at the end of the paper. It can be used as a reference for upcoming studies on this topic.

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