A unifying framework for interpolation on general Lissajous-Chebyshev points

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Outline of the talk

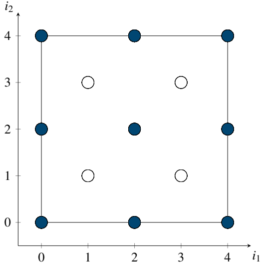
- Introduction
  - What are Lissajous-Chebyshev points?
  - Preliminary questions towards a unified theory
- Interpolation on Lissajous-Chebyshev nodes $\underline{\mathbf{LC}}^{(\boldsymbol{m})}_{\boldsymbol{\kappa}}$
  - Some description of the involved Lissajous curves
  - Interpolation and quadrature on $\underline{\mathbf{LC}}^{(\boldsymbol{m})}_{\boldsymbol{\kappa}}$
  - Convergence and fast algorithms of the interpolation schemes

Definition of Lissajous-Chebyshev points $\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}$

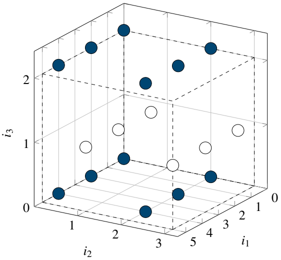
We define the sets $\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}$ with help of the index sets

$$\underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})} = \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}},0}^{(\boldsymbol{m})} \cup \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}},1}^{(\boldsymbol{m})} \text{ with the sets } \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}},\mathfrak{r}}^{(\boldsymbol{m})},\ \mathfrak{r} \in \{0,1\}, \text{ given by}$$

$$\underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}},\mathfrak{r}}^{(\boldsymbol{m})} = \left\{ \underline{\boldsymbol{i}} \in \mathbb{N}_0^{\mathsf{d}} \mid \forall\, \mathsf{j} :\ 0 \leq i_{\mathsf{j}} \leq m_{\mathsf{j}} \text{ and } i_{\mathsf{j}} \equiv \mathfrak{r} + \kappa_{\mathsf{j}} \mod 2 \right\}.$$



(a) Index set $\mathbf{I}_{(0,0)}^{(4,4)}$



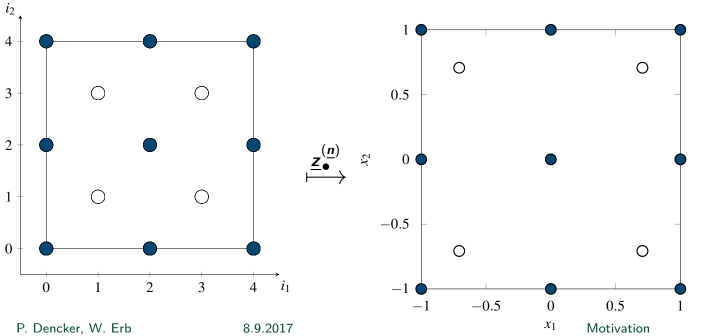
(b) Index set $\mathbf{I}_{(0,0,0)}^{(5,3,2)}$

With the Chebyshev-Gauss-Lobatto points given by

$$\underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{m}})} = \left(z_{i_1}^{(m_1)}, \ldots, z_{i_\mathrm{d}}^{(m_\mathrm{d})}\right), \quad z_i^{(m)} = \cos\left(i\pi/m\right).$$

we then define the Lissajous-Chebyshev points as

$$\boxed{\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})} = \left\{ \underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{m}})} \;\middle|\; \underline{\boldsymbol{i}} \in \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})} \right\}.}$$



Cardinalities of the node sets

We have

$$\#\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})} = \#\underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})} = \#\underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}},0}^{(\underline{\boldsymbol{m}})} + \#\underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}},1}^{(\underline{\boldsymbol{m}})}$$

with

$$\#\underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}},\mathfrak{r}}^{(\underline{\boldsymbol{m}})} = \prod_{\substack{\mathsf{i}\in\{1,\ldots,\mathsf{d}\} \\ m_\mathsf{i}\equiv 0 \mod 2 \\ \kappa_\mathsf{i}\equiv\mathfrak{r} \mod 2}} \frac{m_\mathsf{i}+2}{2} \quad\times \prod_{\substack{\mathsf{i}\in\{1,\ldots,\mathsf{d}\} \\ m_\mathsf{i}\equiv 0 \mod 2 \\ \kappa_\mathsf{i}\not\equiv\mathfrak{r} \mod 2}} \frac{m_\mathsf{i}}{2} \quad\times \prod_{\substack{\mathsf{i}\in\{1,\ldots,\mathsf{d}\} \\ m_\mathsf{i}\equiv 1 \mod 2}} \frac{m_\mathsf{i}+1}{2}.$$

Examples

The interpolation nodes $\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$ are well-known in the literature

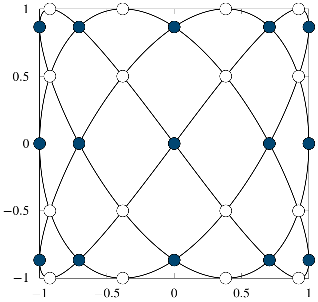
- Morrow-Patterson-Xu points 2D: $\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(m,m)}$ [10, 11].
- Morrow-Patterson-Xu points 3D: $\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(m,m,m)}$ [5].
- Padua points: $\underline{\mathbf{LC}}_{(0,0)}^{(\underline{\boldsymbol{m}})}$ for $\underline{\boldsymbol{m}} = (m, m+1)$ or $\underline{\boldsymbol{m}} = (m+1, m)$ [3, 4].
- Lissajous nodes in MPI: $\underline{\mathbf{LC}}_{(0,1)}^{(2m_1,2m_2)}$ with m_1, m_2 relatively prime [9].
- Degenerate Lissajous curves: $\underline{\mathbf{LC}}_{\underline{\mathbf{0}}}^{(\underline{\boldsymbol{m}})}$, in which $\underline{\boldsymbol{m}}$ consists of relatively prime numbers [6].

$\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$ are also well-known nodes for multivariate quadrature [1].

Observation 1:

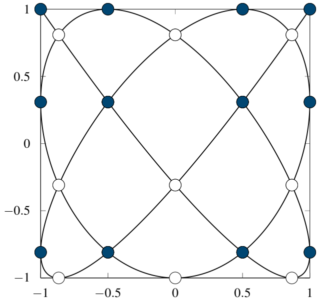
- Polynomial interpolation on all of these point sets is very similar.
- Many of these points have a generating Lissajous curve:

$\underline{\ell}^{(8,6)}_{(4,3)}(t) = (\sin 3t, \sin 4t)$



Non-degenerate Lissajous curve used in Magnetic Particle Imaging [9].

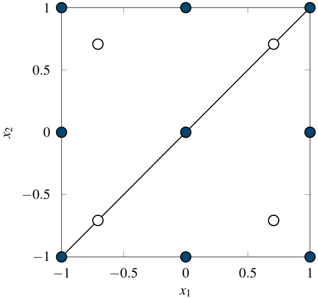
$\underline{\ell}^{(6,5)}_{(0,0)}(t) = (\cos 5t, \cos 6t)$



Degenerate Lissajous curve generating the Padua points [3, 4].

Observation 2:

- Morrow-Patterson-Xu (MPX) points are more symmetric compared to Padua points. In the literature, there is however no generating curve given for MPX points.
- Interpolation spaces have a slightly more complicated structure [11].



Is there a way to get a single Lissajous curve that connects these points?

Questions considered in this tutorial

- Is there a unified interpolation framework including Padua points, MPX points and Lissajous curves?
- Is there a single generating curve for the MPX points? What are the alternatives?
- Are there fundamental differences in the convergence and the implementation of the different schemes?

Definition of d-dimensional Lissajous curves

We will consider d-dimensional Lissajous curves

$$\underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{u}}} : \mathbb{R} \to \mathbb{R}^{\mathsf{d}}$$

in the parametrized form

$$\underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{u}}}(t) = \left(u_1 \cos\left(\frac{\operatorname{lcm}[\underline{\boldsymbol{m}}] \cdot t - \kappa_1 \pi}{m_1} \right), \cdots, u_{\mathsf{d}} \cos\left(\frac{\operatorname{lcm}[\underline{\boldsymbol{m}}] \cdot t - \kappa_{\mathsf{d}} \pi}{m_{\mathsf{d}}} \right) \right),$$

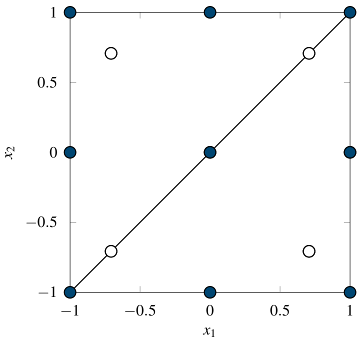
where

- $\underline{\boldsymbol{m}} = (m_1, \ldots, m_{\mathsf{d}}) \in \mathbb{N}^{\mathsf{d}}$ are 'frequency dividers',
- $\underline{\boldsymbol{u}} \in \{-1, 1\}^{\mathsf{d}}$ are 'reflection parameters',
- $\operatorname{lcm}[\underline{\boldsymbol{m}}]$ is the least common multiple of $m_1, \ldots, m_{\mathsf{d}}$,
- $\underline{\boldsymbol{\kappa}} = (\kappa_1, \ldots, \kappa_{\mathsf{d}}) \in \mathbb{R}^{\mathsf{d}}$ specifies additional phase shifts.

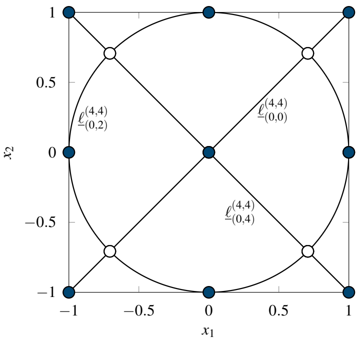
The definition guarantees that in any case the minimal period of $\underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{u}}}$ is 2π.

We know: If the entries m_{i} are pairwise relatively prime, then the Lissajous curve $\underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}$ generates the points $\underline{\mathbf{LC}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}$ [6].

If we try to use Lissajous curves to generate the MPX points we get



Using $\underline{\boldsymbol{\ell}}^{(4,4)}_{(0,0)}(t) = (\cos t, \cos t)$ as generating curve.



Using $\underline{\boldsymbol{\ell}}^{(4,4)}_{(0,0)}(t)$, $\underline{\boldsymbol{\ell}}^{(4,4)}_{(0,2)}(t)$ and $\underline{\boldsymbol{\ell}}^{(4,4)}_{(0,4)}(t)$ as generating curves.

Observation: For MPX points in general more than 1 generating curve is needed. The number depends on $\underline{\boldsymbol{m}}$ and $\underline{\boldsymbol{\kappa}}$.

The union of all generating Lissajous curves forms an algebraic variety

$$\mathcal{C}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})} = \left\{ \, \underline{\boldsymbol{x}} \in [-1,1]^{\mathrm{d}} \,\middle|\, (-1)^{\kappa_1} T_{m_1}(x_1) = \ldots = (-1)^{\kappa_{\mathrm{d}}} T_{m_{\mathrm{d}}}(x_{\mathrm{d}}) \, \right\},$$

where T_m denote the Chebyshev polynomial of first kind of degree m. The variety $\mathcal{C}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}$ is called Chebyshev variety.

Theorem

We have

$$\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})} = \left\{ \, \underline{\boldsymbol{x}} \in [-1,1]^{\mathrm{d}} \,\middle|\, (-1)^{\kappa_1} T_{m_1}(x_1) = \ldots = (-1)^{\kappa_{\mathrm{d}}} T_{m_{\mathrm{d}}}(x_{\mathrm{d}}) \in \{\pm 1\} \right\}.$$

Note: the elements of $\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}$ in the interior of the hypercube $[-1,1]^{\mathrm{d}}$ are exactly the singular points of the variety $\mathcal{C}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}$.

Characterize the Lissajous curves inside $\mathcal{C}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$

Proposition

Let $\underline{\boldsymbol{m}} \in \mathbb{N}^{\mathsf{d}}$. There exist (not necessarily uniquely determined) integer vectors $\underline{\boldsymbol{m}}^{\sharp}, \underline{\boldsymbol{m}}^{\flat} \in \mathbb{N}^{\mathsf{d}}$ such that the following properties are satisfied:

$$\text{For all } \mathsf{i} \in \{1, \ldots, \mathsf{d}\}\text{: } m_{\mathsf{i}} = m_{\mathsf{i}}^{\flat} m_{\mathsf{i}}^{\sharp} \tag{1a}$$

$$\text{For all } \mathsf{i} \in \{1, \ldots, \mathsf{d}\}\text{: } m_{\mathsf{i}}^{\flat} \text{ and } m_{\mathsf{i}}^{\sharp} \text{ are relatively prime.} \tag{1b}$$

$$\text{The numbers } m_1^{\sharp}, \ldots, m_{\mathsf{d}}^{\sharp} \text{ are pairwise relatively prime.} \tag{1c}$$

$$\text{We have } \mathrm{lcm}[\underline{\boldsymbol{m}}] = \mathsf{p}[\underline{\boldsymbol{m}}^{\sharp}] = \prod_{\mathsf{i}=1}^{\mathsf{d}} m_{\mathsf{i}}^{\sharp}. \tag{1d}$$

Define the sets

$$H^{(\underline{\boldsymbol{m}}^{\sharp})} = \{0, \ldots, 2\mathsf{p}[\underline{\boldsymbol{m}}^{\sharp}] - 1\} \quad \text{and} \quad \underline{\boldsymbol{R}}^{(\underline{\boldsymbol{m}}^{\flat})} = \bigtimes_{\mathsf{i}=1}^{\mathsf{d}} \{0, \ldots, m_{\mathsf{i}}^{\flat} - 1\}.$$

Proposition

Let $\underline{\boldsymbol{m}}, \underline{\boldsymbol{m}}^\sharp, \underline{\boldsymbol{m}}^\flat \in \mathbb{N}^{\mathrm{d}}$ satisfy the conditions (1a)-(1d), then

a) For all $(l, \underline{\boldsymbol{\rho}}) \in H^{(\underline{\boldsymbol{m}}^\sharp)} \times \underline{\boldsymbol{R}}^{(\underline{\boldsymbol{m}}^\flat)}$, there exists a uniquely determined $\underline{\boldsymbol{i}} \in \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$ and a (not necessarily unique) $\underline{\boldsymbol{v}} \in \{-1, 1\}^{\mathrm{d}}$ such that

$$\forall\, \mathrm{i} \in \{1, \ldots, \mathrm{d}\}: \quad i_{\mathrm{i}} \equiv v_{\mathrm{i}} \left(l - 2\rho_{\mathrm{i}} m_{\mathrm{i}}^\sharp - \kappa_{\mathrm{i}} \right) \mod 2m_{\mathrm{i}}.$$

Thus, a function $\underline{\boldsymbol{j}} : H^{(\underline{\boldsymbol{m}}^\sharp)} \times \underline{\boldsymbol{R}}^{(\underline{\boldsymbol{m}}^\flat)} \to \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$ is well defined by

$$\underline{\boldsymbol{j}}(l, \underline{\boldsymbol{\rho}}) = \underline{\boldsymbol{i}}.$$

b) Let $\mathrm{M} \subseteq \{1, \ldots, \mathrm{d}\}$. If $\underline{\boldsymbol{i}} \in \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$ and $\underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{m}})} \in \underline{\boldsymbol{F}}_{\mathrm{M}}$, then

$$\#\{\, (l, \underline{\boldsymbol{\rho}}) \in H^{(\underline{\boldsymbol{m}}^\sharp)} \times \underline{\boldsymbol{R}}^{(\underline{\boldsymbol{m}}^\flat)} \,|\, \underline{\boldsymbol{j}}(l, \underline{\boldsymbol{\rho}}) = \underline{\boldsymbol{i}} \,\} = 2^{\#\mathrm{M}}.$$

We consider the following set of Lissajous curves

$$\underline{\mathfrak{L}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}}^\sharp,\,\underline{\boldsymbol{m}}^\flat)} = \left\{ \underline{\boldsymbol{\ell}}^{(\underline{\boldsymbol{m}})}_{(\kappa_1+2\rho_1 m_1^\sharp,\ldots,\kappa_\mathsf{d}+2\rho_\mathsf{d} m_\mathsf{d}^\sharp)} \;\middle|\; \underline{\boldsymbol{\rho}} \in \underline{\boldsymbol{R}}^{(\underline{\boldsymbol{m}}^\flat)} \right\}.$$

Theorem

Let $\underline{\boldsymbol{m}}, \underline{\boldsymbol{m}}^\sharp, \underline{\boldsymbol{m}}^\flat \in \mathbb{N}^\mathsf{d}$ satisfy the conditions (1a)-(1d).

a) *Using the sampling points $t_l^{(\underline{\boldsymbol{m}})}$, we have*

$$\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})} = \left\{ \underline{\boldsymbol{\ell}}(t_l^{(\underline{\boldsymbol{m}})}) \;\middle|\; \underline{\boldsymbol{\ell}} \in \underline{\mathfrak{L}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}}^\sharp,\,\underline{\boldsymbol{m}}^\flat)},\ l \in H^{(\underline{\boldsymbol{m}}^\sharp)} \right\}.$$

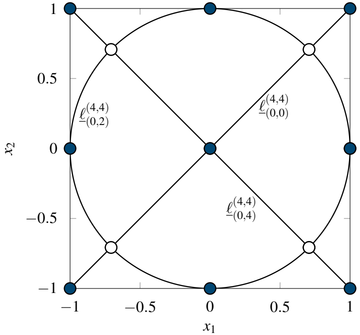
b) *The affine Chebyshev variety $\mathcal{C}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$ can be decomposed as*

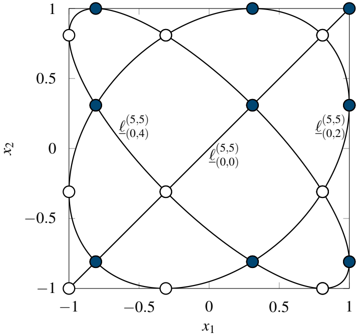
$$\mathcal{C}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})} = \bigcup_{\underline{\boldsymbol{\ell}} \in \underline{\mathfrak{L}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}}^\sharp,\,\underline{\boldsymbol{m}}^\flat)}} \underline{\boldsymbol{\ell}}([0, 2\pi)).$$

Example: MPX points in 2D

One possible decomposition of $\underline{\boldsymbol{m}}$ is given by $\underline{\boldsymbol{m}}^\sharp = (m, 1)$, $\underline{\boldsymbol{m}}^\flat = (1, m)$. The respective sets $H^{(\underline{\boldsymbol{m}}^\sharp)}$ and $R^{(\underline{\boldsymbol{m}}^\flat)}$ are given by

$$H^{(\underline{\boldsymbol{m}}^\sharp)} = \{0, \ldots, 2m-1\} \quad \text{and} \quad \underline{\boldsymbol{R}}^{(\underline{\boldsymbol{m}}^\flat)} = \{0\} \times \{0, \ldots, m-1\}.$$

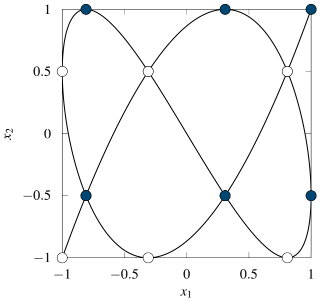




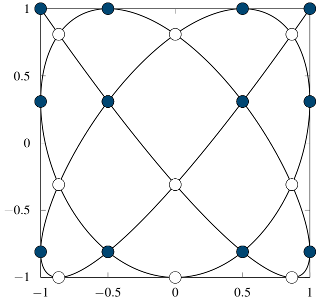
Example: Padua points and Lissajous curves

If m_1 and m_2 are relatively prime, the decomposition of $\underline{\boldsymbol{m}}$ is given by $\underline{\boldsymbol{m}}^\sharp = (m_1, m_2)$, $\underline{\boldsymbol{m}}^\flat = (1, 1)$. Then

$$H^{(\underline{\boldsymbol{m}}^\sharp)} = \{0, \dots, 2m_1m_2 - 1\} \quad \text{and} \quad \underline{\boldsymbol{R}}^{(\underline{\boldsymbol{m}}^\flat)} = \{0\} \times \{0\}.$$



$\underline{\boldsymbol{\ell}}^{(5,3)}_{(0,0)}(t) = (\cos 3t, \cos 5t)$



$\underline{\boldsymbol{\ell}}^{(6,5)}_{(0,0)}(t) = (\cos 5t, \cos 6t)$

Polynomial interpolation on $\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$

We are looking for polynomial interpolants of the form

$$P_{\underline{\boldsymbol{\kappa}},h}^{(\underline{\boldsymbol{m}})}(\underline{\boldsymbol{x}}) = \sum_{\underline{\gamma} \in \underline{\boldsymbol{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}} c_{\underline{\gamma}}(h)\, T_{\underline{\gamma}}(\underline{\boldsymbol{x}}),$$

$$\tilde{P}_{\underline{\boldsymbol{\kappa}},h}^{(\underline{\boldsymbol{m}})}(\underline{\boldsymbol{x}}) = \sum_{\underline{\gamma} \in \overline{\underline{\boldsymbol{\Gamma}}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}} \frac{c_{\underline{\gamma}}(h)}{\#[\underline{\gamma}]}\, T_{\underline{\gamma}}(\underline{\boldsymbol{x}}),$$

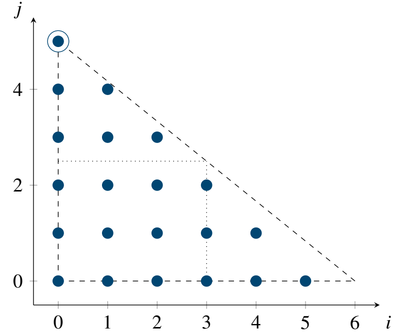
such that the following interpolation condition is satisfied:

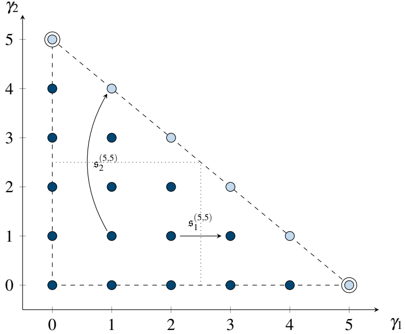
$$P_{\underline{\boldsymbol{\kappa}},h}^{(\underline{\boldsymbol{m}})}(\underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{m}})}) = \tilde{P}_{\underline{\boldsymbol{\kappa}},h}^{(\underline{\boldsymbol{m}})}(\underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{m}})}) = h(\underline{\boldsymbol{i}}), \qquad \underline{\boldsymbol{i}} \in \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}. \qquad \text{(IP)}$$

- $T_{\underline{\gamma}}(\underline{\boldsymbol{x}}) = \prod_{\mathrm{i}=1}^{\mathrm{d}} \cos(\gamma_{\mathrm{i}} \arccos x_{\mathrm{i}})$ are multivariate Chebyshev polynomials,
- $\underline{\boldsymbol{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$, $\overline{\underline{\boldsymbol{\Gamma}}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$ are appropriate spectral index sets.

Examples

For the Padua points $\underline{\mathbf{LC}}^{(6,5)}_{(0,0)}$ we use the index set $\underline{\boldsymbol{\Gamma}}^{(6,5)}_{(0,0)}$.





For the MPX points $\underline{\mathbf{LC}}^{(5,5)}_{(0,0)}$, we can use the index set $\underline{\overline{\boldsymbol{\Gamma}}}^{(5,5)}_{(0,0)}$.

General defintion of spectral index sets $\overline{\underline{\mathbf{\Gamma}}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}$

For $\underline{\boldsymbol{m}} \in \mathbb{N}^{\mathsf{d}}$, $\underline{\boldsymbol{\kappa}} \in \mathbb{N}^{\mathsf{d}}$, $\mathfrak{r} \in \{0,1\}$, we define the cubic index sets

$$\underline{\mathbf{\Gamma}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},\mathfrak{r}} = \left\{ \underline{\gamma} \in \mathbb{N}_0^{\mathsf{d}} \;\middle|\; \begin{array}{ll} \forall\, \mathsf{i} \text{ with } \kappa_{\mathsf{i}} \equiv \mathfrak{r} \mod 2: & 2\gamma_{\mathsf{i}} \leq m_{\mathsf{i}}, \\ \forall\, \mathsf{i} \text{ with } \kappa_{\mathsf{i}} \not\equiv \mathfrak{r} \mod 2: & 2\gamma_{\mathsf{i}} < m_{\mathsf{i}} \end{array} \right\},$$

and the spectral index sets

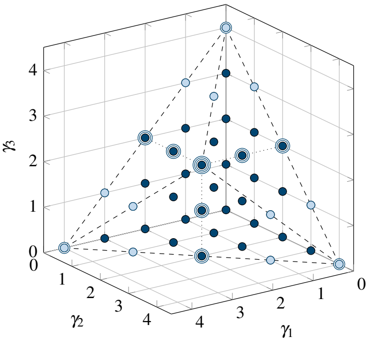
$$\overline{\underline{\mathbf{\Gamma}}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}} = \left\{ \underline{\gamma} \in \mathbb{N}_0^{\mathsf{d}} \;\middle|\; \begin{array}{ll} \forall\, \mathsf{i} \in \{1,\ldots,\mathsf{d}\}: & \gamma_{\mathsf{i}} \leq m_{\mathsf{i}}, \\ \forall\, \mathsf{i},\mathsf{j} \text{ with } \mathsf{i} \neq \mathsf{j}: & \gamma_{\mathsf{i}}/m_{\mathsf{i}} + \gamma_{\mathsf{j}}/m_{\mathsf{j}} \leq 1, \\ \forall\, \mathsf{i},\mathsf{j} \text{ with } \kappa_{\mathsf{i}} \not\equiv \kappa_{\mathsf{j}} \mod 2: & (\gamma_{\mathsf{i}},\gamma_{\mathsf{j}}) \neq (m_{\mathsf{i}}/2, m_{\mathsf{j}}/2) \end{array} \right\}.$$

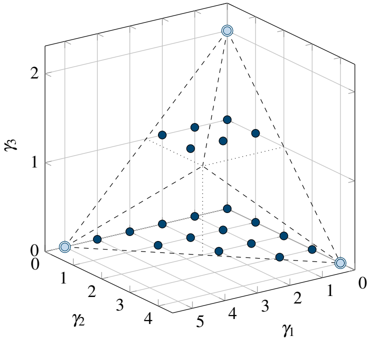
For $\mathsf{d} = 2$, $\overline{\underline{\mathbf{\Gamma}}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}$ contains all integer vectors inside a triangle.

If $\mathsf{d} > 2$, the spectral index set $\overline{\underline{\mathbf{\Gamma}}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}$ has a polyhedral structure.

Examples in 3D

The spectral index set $\underline{\overline{\Gamma}}^{(4,4,4)}_{(0,0,0)}$ for the MPX points.





The spectral index set $\underline{\overline{\Gamma}}^{(5,4,2)}_{(0,0,1)}$.

We introduce a class decomposition $\left[\overline{\mathbf{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}\right]$ of $\overline{\mathbf{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$. We define

$$\mathsf{K}^{(\underline{\boldsymbol{m}})}(\underline{\gamma}) = \left\{ \mathsf{j} \in \{1, \ldots, \mathsf{d}\} \;\middle|\; \gamma_{\mathsf{j}}/m_{\mathsf{j}} = \max{}^{(\underline{\boldsymbol{m}})}[\underline{\gamma}] \right\}$$

where $\max^{(\underline{\boldsymbol{m}})}[\underline{\gamma}] = \max \{ \gamma_{\mathsf{i}}/m_{\mathsf{i}} \mid \mathsf{i} \in \{1, \ldots, \mathsf{d}\} \}$.

Further, using the flip operator

$$\mathfrak{s}_{\mathsf{j}}^{(\underline{\boldsymbol{m}})}(\underline{\gamma}) = (\gamma_1, \ldots, \gamma_{\mathsf{j}-1}, m_{\mathsf{j}} - \gamma_{\mathsf{j}}, \gamma_{\mathsf{j}+1}, \ldots, \gamma_{\mathsf{d}})$$

we define the sets $\mathfrak{S}^{(\underline{\boldsymbol{m}})}(\underline{\gamma}) = \left\{ \mathfrak{s}_{\mathsf{j}}^{(\underline{\boldsymbol{m}})}(\underline{\gamma}) \mid \mathsf{j} \in \mathsf{K}^{(\underline{\boldsymbol{m}})}(\underline{\gamma}) \right\}$.

Now, we introduce the class decomposition $\left[\overline{\mathbf{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}\right]$ as

$$\boxed{\left[\overline{\mathbf{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}\right] = \left\{ \{\underline{\gamma}\} \;\middle|\; \underline{\gamma} \in \underline{\mathbf{\Gamma}}_{\underline{\boldsymbol{\kappa}},0}^{(\underline{\boldsymbol{m}})} \right\} \cup \left\{ \mathfrak{S}^{(\underline{\boldsymbol{m}})}(\underline{\gamma}) \;\middle|\; \underline{\gamma} \in \underline{\mathbf{\Gamma}}_{\underline{\boldsymbol{\kappa}},1}^{(\underline{\boldsymbol{m}})} \right\}.}$$

The set $\underline{\mathbf{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$ denotes a set of representatives of $\left[\overline{\mathbf{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}\right]$.

By this definition of the class decomposition $\left[\overline{\boldsymbol{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}\right]$ we get

$$\# \left[\overline{\boldsymbol{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}\right] = \# \underline{\boldsymbol{\Gamma}}_{\underline{\boldsymbol{\kappa}},0}^{(\boldsymbol{m})} + \# \underline{\boldsymbol{\Gamma}}_{\underline{\boldsymbol{\kappa}},1}^{(\boldsymbol{m})} = \# \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}},1}^{(\boldsymbol{m})} + \# \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}},0}^{(\boldsymbol{m})} = \# \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})} = \# \underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}$$

In special cases (as for instance the Padua points) the situation is simpler.

Proposition

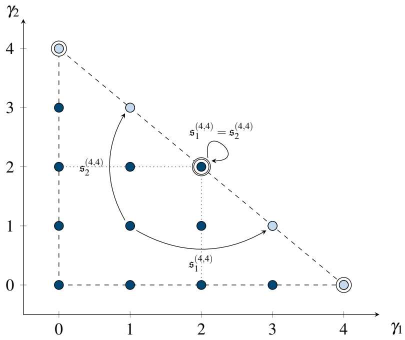
Let $\underline{\boldsymbol{\kappa}} \in \mathbb{Z}^{\mathrm{d}}$. The following statements are equivalent.

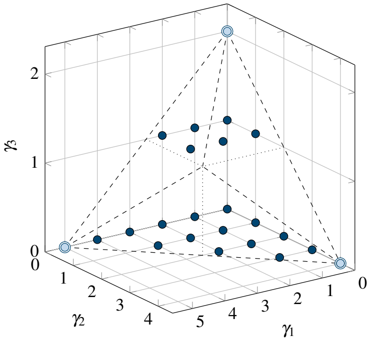
i) We have $\gcd\{m_{\mathrm{i}}, m_{\mathrm{j}}\} \leq 2$ for all $\mathrm{i}, \mathrm{j} \in \{1, \ldots, \mathrm{d}\}$ with $\mathrm{i} \neq \mathrm{j}$.

ii) All classes in $\left[\overline{\boldsymbol{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}\right] \setminus \{\mathfrak{S}^{(\boldsymbol{m})}(\underline{\mathbf{0}})\}$ consist of precisely one element.

Examples

The spectral index set $\underline{\overline{\mathbf{\Gamma}}}_{(0,0)}^{(4,4)}$ for MPX points.





The spectral index set $\underline{\overline{\mathbf{\Gamma}}}_{(0,0,1)}^{(5,4,2)}$.

Discrete orthogonality structure

For $\underline{\boldsymbol{i}} \in \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$, the weights are given by

$$\mathfrak{w}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{m}})} = 2^{\#\mathrm{M}}/(2\mathrm{p}[\underline{\boldsymbol{m}}]) \quad \text{if} \quad \underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{m}})} \in \underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})} \cap \underline{\boldsymbol{F}}_{\mathrm{M}}^{\mathrm{d}}.$$

and the measure $\omega_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$ on the power set of $\underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}$ by $\omega_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}\{\underline{\boldsymbol{i}}\}) = \mathfrak{w}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{m}})}$.

Denote by $\mathcal{L}(\underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})})$ the set of the functions $h : \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})} \to \mathbb{C}$.

To prove the unisolvence of the interpolation problem (IP), we show that

$$\chi_{\underline{\boldsymbol{\gamma}}}^{(\underline{\boldsymbol{m}})}(\underline{\boldsymbol{i}}) = T_{\underline{\boldsymbol{\gamma}}}(\underline{\boldsymbol{z}}_{\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{m}})}) = \prod_{\mathrm{i}=1}^{\mathrm{d}} \cos(\gamma_{\mathrm{i}} i_{\mathrm{i}} \pi / m_{\mathrm{i}}), \qquad \underline{\boldsymbol{\gamma}} \in \underline{\mathbf{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})},$$

is an orthogonal basis of the Hilbert space $\mathcal{L}(\underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})})$ with respect to

$$\langle f, g \rangle_{\omega_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}} = \sum_{\underline{\boldsymbol{i}} \in \underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}} f(\underline{\boldsymbol{i}}) \, \overline{g(\underline{\boldsymbol{i}})} \, \mathfrak{w}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{i}}}^{(\underline{\boldsymbol{m}})}.$$

Proposition

For $\underline{\boldsymbol{\gamma}} \in \mathbb{N}_0^{\mathrm{d}}$ and $\chi_{\underline{\boldsymbol{\gamma}}}^{(\underline{\boldsymbol{m}})} \in \mathcal{L}(\mathbf{I}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})})$ we have

$$\sum_{\underline{\boldsymbol{i}} \in \mathbf{I}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}} \chi_{\underline{\boldsymbol{\gamma}}}^{(\underline{\boldsymbol{m}})}(\underline{\boldsymbol{i}})\ \mathfrak{w}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})} \neq 0$$

if and only if

$$\text{there exists } \underline{\boldsymbol{h}} \in \mathbb{N}_0^{\mathrm{d}} \text{ with } \gamma_{\mathrm{i}} = h_{\mathrm{i}} m_{\mathrm{i}},\ \mathrm{i} = 1, \ldots, \mathrm{d}, \text{ and } \sum_{\mathrm{i}=1}^{\mathrm{d}} h_{\mathrm{i}} \in 2\mathbb{N}_0. \tag{2}$$

If (2) is satisfied, then $\displaystyle\sum_{\underline{\boldsymbol{i}} \in \mathbf{I}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})}} \chi_{\underline{\boldsymbol{\gamma}}}^{(\underline{\boldsymbol{m}})}(\underline{\boldsymbol{i}})\ \mathfrak{w}_{\underline{\boldsymbol{\kappa}}}^{(\underline{\boldsymbol{m}})} = (-1)^{\sum_{\mathrm{i}=1}^{\mathrm{d}} h_{\mathrm{i}} \kappa_{\mathrm{i}}}$.

For the proof of the orthogonality we further need the product formula

$$\chi_{\underline{\boldsymbol{\gamma}}}^{(\underline{\boldsymbol{m}})} \chi_{\underline{\boldsymbol{\gamma}}'}^{(\underline{\boldsymbol{m}})} = \frac{1}{2^{\mathrm{d}}} \sum_{\underline{\boldsymbol{v}} \in \{-1,1\}^{\mathrm{d}}} \chi_{(|\gamma_1 + v_1 \gamma_1'|, \ldots, |\gamma_{\mathrm{d}} + v_{\mathrm{d}} \gamma_{\mathrm{d}}'|)}^{(\underline{\boldsymbol{m}})}.$$

Main interpolation result

We consider $\Pi_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})} = \operatorname{span}\{\, T_{\underline{\gamma}} \,|\, \underline{\gamma} \in \underline{\boldsymbol{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})} \}$ and an appropriately defined space $\tilde{\Pi}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}$ regarding (anti-)symmetries on the classes $[\underline{\gamma}]$, see [7].

Theorem

For $h \in \mathcal{L}(\underline{\mathbf{I}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})})$, the unique coefficients $c_{\underline{\gamma}}(h)$ such that the polynomials

$$P_{\underline{\boldsymbol{\kappa}},h}^{(\boldsymbol{m})}(\underline{\boldsymbol{x}}) = \sum_{\underline{\gamma} \in \underline{\boldsymbol{\Gamma}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}} c_{\underline{\gamma}}(h)\, T_{\underline{\gamma}}(\underline{\boldsymbol{x}}), \quad \tilde{P}_{\underline{\boldsymbol{\kappa}},h}^{(\boldsymbol{m})}(\underline{\boldsymbol{x}}) = \sum_{\underline{\gamma} \in \overline{\underline{\boldsymbol{\Gamma}}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}} \frac{c_{\underline{\gamma}}(h)}{\#[\underline{\gamma}]}\, T_{\underline{\gamma}}(\underline{\boldsymbol{x}}),$$

solve the interpolation problem (IP) in $\Pi_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}$ or $\tilde{\Pi}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}$, respectively, are

$$c_{\underline{\gamma}}(h) = \frac{1}{\|\chi_{\underline{\gamma}}^{(\boldsymbol{m})}\|^2_{\omega_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}}} \,\langle\, h, \chi_{\underline{\gamma}}^{(\boldsymbol{m})} \rangle_{\omega_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}}.$$

Efficient computation of the interpolating polynomial

We introduce

$$g^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}(\underline{\boldsymbol{i}}) = \begin{cases} \mathfrak{w}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{i}}} h(\underline{\boldsymbol{i}}), & \text{if } \underline{\boldsymbol{i}} \in \underline{\mathbf{I}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}, \\ 0, & \text{if } \underline{\boldsymbol{i}} \in \underline{\mathbf{J}}^{(\underline{\boldsymbol{m}})} \setminus \underline{\mathbf{I}}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}, \end{cases} \qquad \underline{\mathbf{J}}^{(\underline{\boldsymbol{m}})} = \bigtimes_{\mathrm{i}=1}^{\mathrm{d}} \{0,\ldots,m_{\mathrm{i}}\},$$

and the d-dimensional discrete cosine transform $\hat{g}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},\underline{\boldsymbol{\gamma}}}$ of $g^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}$ starting with

$$\hat{g}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},(\gamma_1)}(i_2,\ldots,i_{\mathrm{d}}) = \sum_{i_{\mathrm{i}}=0}^{m_1} g^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}(\underline{\boldsymbol{i}}) \cos(\gamma_1 i_1 \pi / m_1).$$

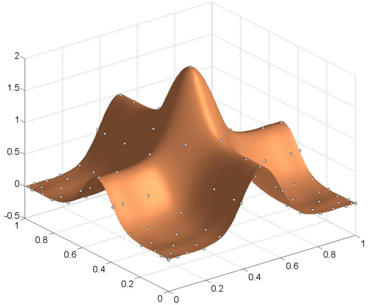
and, then proceeding recursively for $\mathrm{i} = 2,\ldots,\mathrm{d}$ with

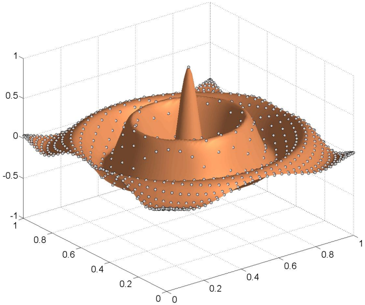
$$\hat{g}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},(\gamma_1,\ldots,\gamma_{\mathrm{i}})}(i_{\mathrm{i}+1},\ldots,i_{\mathrm{d}}) = \sum_{i_{\mathrm{i}}=0}^{m_{\mathrm{i}}} \hat{g}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}},(\gamma_1,\ldots,\gamma_{\mathrm{i}-1})}(i_{\mathrm{i}},\ldots,i_{\mathrm{d}}) \cos(\gamma_{\mathrm{i}} i_{\mathrm{i}} \pi / m_{\mathrm{i}}).$$

Then, we have

$$c_{\underline{\boldsymbol{\gamma}}}(h) = \|\chi^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\gamma}}}\|^{-2}_{\omega^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}} \; \hat{g}^{(\underline{\boldsymbol{m}})}_{\underline{\boldsymbol{\kappa}}}(\underline{\boldsymbol{\gamma}}).$$

Using FFT this can be done in $\mathcal{O}(\mathrm{p}[\underline{\boldsymbol{m}}] \log \mathrm{p}[\underline{\boldsymbol{m}}])$ steps.





Matlab toolboxes for interpolation at the nodes $\underline{\mathbf{LC}}_{\underline{\boldsymbol{\kappa}}}^{(\boldsymbol{m})}$ are available at

https://github.com/WolfgangErb/LC2Ditp
https://github.com/WolfgangErb/LC3Ditp

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