Newton-Okounkov bodies and transfinite diameter

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Abstract

We present an explicit calculation of a Newton-Okounkov body associated to an algebraic variety. This is used to derive a formula for transfinite diameter on the variety. We relate this formula to a recent result of Witt Nyström.

1 Introduction

This paper investigates Newton-Okounkov bodies and their relation to transfinite diameter. These bodies were first studied by Okounkov in [17]. Lazarsfeld and Mustata [15], as well as Kaveh and Khovanskii [12], introduced them into algebraic geometry as an important tool for the asymptotic study of linear series.[1] Newton-Okounkov bodies are also used to count solutions to systems of algebraic equations; in this context they play the same role as Newton polyhedra in a generalized version of the Bernstein-Kouchnirenko theorem. (See [14] for a very accessible exposition of the latter.) For convenience we will refer to Newton-Okounkov bodies in what follows as simply *Okounkov bodies*, which is common usage in much of the literature.

In [1], Berman and Boucksom used pluripotential theory to study the asymptotic properties of powers of big line bundles. These powers form a natural linear series; therefore, Okounkov bodies ought to be related to pluripotential theory. The connection was made in a theorem of Witt Nyström [19].

Witt Nyström's result on Okounkov bodies is closely related to a classical theorem of Zaharjuta [20]. The latter (reproduced in this paper as Theorem 5.1) gives an important capacity in pluripotential theory—the transfinite diameter of a compact set $K \subset \mathbb{C}^n$—as a real integral over an $(n-1)$-dimensional simplex in $\mathbb{R}^n$ involving so-called *directional Chebyshev constants*, which are quantities defined in terms of polynomials on K. Witt Nyström gives a similar looking integral formula that relates the Monge-Ampère energy of Hermitian metrics on a line bundle L over a compact complex manifold, to an integral over the Okounkov body associated to L of the Chebyshev transforms of these metrics. (We will explain the terms later.)

To relate the two results, classical objects in pluripotential theory need to be transferred into the modern theory on complex manifolds:

$$\text{polynomials} \quad \hookrightarrow \quad \text{sections of line bundles} \tag{1}$$

$$\text{compact set} \quad \hookrightarrow \quad \text{Hermitian metric} \tag{2}$$

$$\text{simplex} \quad \hookrightarrow \quad \text{Okounkov body} \tag{3}$$

where $A \hookrightarrow B$ means that all objects in A can be modelled (more or less) by objects in B.

The main aim of this paper is to prove a version of Witt Nyström's result on algebraic subvarieties of $\mathbb{C}^n$. A subvariety of $\mathbb{C}^n$ is an intermediate setting which is both a natural extension of the classical theory in $\mathbb{C}^n$ as well as a concrete illustration of the theory on complex manifolds. It provides a natural bridge from the classical to the modern point of view.

Zaharjuta's methods may be naturally adapted to this setting, with some additional tools from computational algebraic geometry and weighted pluripotential theory. The methods used in this paper are also similar to [7], especially the use of computational algebraic geometry to carry out computations on a variety. They do not use much pluripotential theory: no plurisubharmonic functions or Monge-Ampère integrals are required, only polynomials.

We begin with some background material and describe the relationships (1), (2), and (3) given above. Section 2 deals with the first two. First, (1) is elementary complex geometry; one may skip this part if one is familiar with relating polynomials on a variety in $\mathbb{C}^n$ to powers of the line bundle $\mathcal{O}(1)$ over the corresponding variety in $\mathbb{P}^n$. Next, to describe (2) we use the notion of a *(weakly) admissible weight*. The identification of Hermitian metrics and weights is reasonably familiar from the application of pluripotential theory to complex geometry (see e.g. [8], [9]). Capacities associated to sets can be defined in terms of weights supported on these sets.[2]

In section 3 we define the Okounkov body and study some of its properties. This definition depends on a choice of coordinates, and we use Noether normalization from computational algebraic geometry to choose coordinates which have good computational

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[1]Think of linear series as classes of polynomials.
[2]If the weight is identically 1 we recover the classical (unweighted) theory.

properties. It is easily seen by definition that the Okounkov for $\mathbb{C}^n$ is the region S bounded by the coordinate hyperplanes and the standard simplex in $\mathbb{R}^n$. (This provides, more or less, the connection (3).) We then present a fairly explicit algorithm for constructing an Okounkov body associated to a variety in $\mathbb{C}^n$. Although we do not give a rigorous proof of the method in general, we use it to compute the Okounkov body associated to a complexified unit sphere in $\mathbb{C}^3$.

In section 4, we define directional Chebyshev constants, the Chebyshev transform, and a notion of transfinite diameter. We then prove our main theorem (Theorem 4.5) that gives transfinite diameter on the variety as an integral over the Okounkov body. The relation with Witt Nyström's result is seen by translating things into the language of complex geometry.

In section 5 we make the explicit connection to Zaharjuta's classical theorem, as well as a homogeneous version of Jedrzejowski [11]. This involves making a projective change of coordinates. (It is good to have the complex geometric point of view here.)

Finally, in section 6 we investigate further properties of Chebyshev constants on the sphere in $\mathbb{C}^3$. In particular, we look at directional Chebyshev constants associated to so-called *locally circled sets* (Proposition 6.7). The notion of a locally circled set is adapted from the classical notion of a circled set in $\mathbb{C}^n$ (cf. [2], [4]).

Acknowledgement

2   Preliminaries

2.1   Varieties in $\mathbb{C}^n$

Let $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$ denote the ring of polynomials in n variables. Recall that an algebraic variety V in $\mathbb{C}^n$ ($n > 1$ is an integer), is the solution to a finite collection of polynomial equations

$$V = \{a \in \mathbb{C}^n : P_1(a) = \cdots = P_m(a) = 0, \; P_j \in \mathbb{C}[z] \forall j\};$$

Notation 2.1. Given an algebraic variety $V \subseteq \mathbb{C}^n$, define

$$\mathbf{I}(V) := \{p \in \mathbb{C}[z] : \; p(a) = 0 \text{ for all } a \in V\}.$$

It is easy to see that this is an ideal. Also, given an ideal $I \subseteq \mathbb{C}[z]$, define

$$\mathbf{V}(I) := \{a \in \mathbb{C}^n : \; p(a) = 0 \text{ for all } p \in \mathbb{C}[z]\}.$$

Theorem 2.2.   1. (Hilbert basis theorem) Any ideal I is finitely generated; consequently, $\mathbf{V}(I)$ is always an algebraic variety.

2. (Nullstellensatz) For any ideal $I \subseteq \mathbb{C}[z]$, we have $I \subseteq \mathbf{I}(\mathbf{V}(I))$, and if the property

$$p^m \in I \text{ for some } m \in \mathbb{N} \Longrightarrow p \in I, \tag{4}$$

holds, then $I = \mathbf{I}(\mathbf{V}(I))$.

3. For any algebraic variety V, $\mathbf{I}(V)$ satisfies (4) and $\mathbf{V}(\mathbf{I}(V)) = V$.

Suppose $V = \mathbf{V}(I)$ where $I = \langle P_1, \ldots, P_m \rangle$ is the ideal generated by the polynomials P_j. If I satisfies (4) then the above theorem implies that restricting the evaluation of polynomials to points of V is equivalent to working with elements of the factor ring $\mathbb{C}[z]/I$ via the correspondence

$$p = q \text{ on } V \iff p - q \in I.$$

2.2   Projective space

Algebraic subvarieties of $\mathbb{C}^n$ can be put into the complex geometric setting using projective space. Consider $\mathbb{C}^n \subset \mathbb{P}^n$ via the usual embedding

$$z = (z_1, \ldots, z_n) \hookrightarrow [1 : z_1 : \cdots : z_n] = [1 : z],$$

where we use homogeneous coordinates on the right-hand side: $\mathbb{P}^n = \mathbb{C}^{n+1}/\sim$ with the equivalence $(z_0, \cdots, z_n) \sim (w_0, \cdots, w_n)$ if there is a $\lambda \in \mathbb{C}$ such that $\lambda z_i = w_i$ for each i; we write $[z_0 : z] = [z_0 : \cdots : z_n] = [w_0 : \cdots : w_n] = [w_0 : w]$. We have $\mathbb{P}^n = \mathbb{C}^n \cup H_\infty$ where $H_\infty = \{[0 : z] \in \mathbb{P}^n : z \in \mathbb{C}^n\}$ is the hyperplane at infinity.

The standard affine charts of $\mathbb{P}^n$ as a complex manifold will be denoted by $\mathcal{U}_j$, $j = 0, \ldots, n$. These are given by $\mathcal{U}_0 = \mathbb{C}^n$ with the standard embedding described above, and for $j > 0$, $\mathcal{U}_j = \{[Z_0 : Z_1 : \cdots : Z_N] \in \mathbb{P}^n : Z_j \neq 0\}$ with the map

$$\mathbb{C}^n \ni (w_1, \ldots, \widehat{w}_j, \ldots, w_n) \hookrightarrow [w_1 : \cdots : w_{j-1} : 1 : w_{j+1} : \cdots : w_n] \in \mathcal{U}_j \tag{5}$$

giving local coordinates on $\mathcal{U}_j$ (here $\widehat{w}_j$ means that there is no w_j coordinate). Going the other way is *dehomogenization*:

$$\mathcal{U}_j \ni [Z_0 : \cdots : Z_n] \mapsto (Z_0/Z_j, \ldots, Z_n/Z_j) \in \mathbb{C}^n.$$

We also have the change of coordinates on the overlap $\mathcal{U}_j \cap \mathcal{U}_k$:

$$w_j = 1/v_k, \quad w_\ell = v_\ell/v_k \text{ for all } \ell \neq j,$$

with $(v_0, \ldots, \widehat{v}_j, \ldots, v_n) \in \mathcal{U}_j$ and $(w_0, \ldots, \widehat{w}_k, \ldots, w_n) \in \mathcal{U}_k$.

Let $V_\mathbb{P} \subset \mathbb{P}^n$ be the continuous extension of $V \subset \mathbb{C}^n$ across points of H_∞ under the above embedding (the *projective closure*). One can use homogeneous coordinates to characterize it:

$$V_\mathbb{P} = \{[Z_0 : \cdots : Z_n] \in \mathbb{P}^n : \; p(Z_0, \ldots, Z_n) = 0 \text{ for all } p \in \mathbf{I}_h(V)\}$$

where $\mathbf{I}_h(V) \subset \mathbb{C}[Z_0, \ldots, Z_n]$ is the collection of homogeneous polynomials $p(Z)$ such that $p(1, z_1, \ldots, z_n) = 0$ whenever $(z_1, \ldots, z_n) \in V$.

2.3 Sections of line bundles and polynomials

Let us recall the basic notions associated to a holomorphic line bundle L over a complex manifold M of dimension m, which is essentially a union of complex lines (i.e. complex vector spaces of dimension 1 or copies of $\mathbb{C}$) parametrized holomorphically by points of M.[3] Precisely, L is a manifold of dimension $m+1$ with a projection $\pi: L \to M$ such that $L_a := \pi^{-1}(a)$ is a complex line for each $a \in M$. A *(holomorphic) section of* L is a holomorphic map $s: M \to L$ with $(\pi \circ s)(a) = a$.

We review the details of the local product structure of L: any point $a \in M$ has a neighborhood U for which there is a holomorphic injection $U \times \mathbb{C} \ni (z,\zeta) \overset{\varphi}{\mapsto} v \in L$ such that for each $z \in U$, $\pi \circ \varphi(z,\zeta) = z$ and the map $\mathbb{C} \ni \zeta \mapsto \varphi(z,\zeta) \in L_a$ is a linear isomorphism. The pair (U, φ) is called a *local trivialization*.

Let $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ be a collection of local trivializations that cover L, i.e., $\bigcup_\alpha U_\alpha = M$, so that $\bigcup_\alpha \left(\bigcup_{a \in U_\alpha} L_a \right) = L$. Define $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{C}$ by $g_{\alpha\beta}(z) = \eta/\zeta$, where

$$U_\alpha \times \mathbb{C} \ni (z,\zeta) \overset{\varphi_\alpha}{\longmapsto} v \overset{\varphi_\beta}{\longleftarrow} (w,\eta) \in U_\beta \times \mathbb{C}$$

for some $v \in \bigcup_{a \in U_\alpha \cap U_\beta} L_a$. Clearly for any α, β, γ and $z \in U_\alpha \cap U_\beta \cap U_\gamma$,

$$g_{\alpha\beta}(z) g_{\beta\alpha}(z) = g_{\alpha\beta}(z) g_{\beta\gamma}(z) g_{\gamma\alpha}(z) = 1$$

(the *cocycle condition*.)

Given a section s of L, there is an associated collection $\{s_\alpha\}_\alpha$ of (local) functions $s_\alpha: U_\alpha \to \mathbb{C}$ such that

$$s_\beta(z) = g_{\alpha\beta}(z) s_\alpha(z), \quad \text{for all } z \in U_\alpha \cap U_\beta.$$

It is straightforward to verify that a collection $\{s_\alpha\}_\alpha$ of functions satisfying the above conditions characterizes a section s (since L is determined by $\{g_{\alpha\beta}\}$).

We now specialize to our context. Define $\mathcal{O}(1)$ as the collection of pairs

$$\mathcal{O}(1) = \{([Z_0 : \cdots : Z_n], a_0 Z_0 + \cdots + a_n Z_n) : [Z_0 : \cdots : Z_n] \in \mathbb{P}^n, \ a_j \in \mathbb{C}\}. \tag{6}$$

The line bundle structure of $\mathcal{O}(1)$ comes from function evaluation. Let us see how this works by fixing $Z \in \mathbb{P}^n$ and computing L_Z explicitly. First, pick $b(Z)$ such that $b(Z) \neq 0$; this is true (or not) independently of the homogeneous coordinates used to compute $b(Z)$. We claim that

$$L_Z = \{(Z, \lambda b(Z)) : \lambda \in \mathbb{C}\}. \tag{7}$$

For any $a(Z) = a_0 Z_0 + \cdots + a_n Z_n$, define $\lambda \in \mathbb{C}$ by $\lambda := \frac{a(Z)}{b(Z)}$; note that this computation of λ is independent of homogeneous coordinates. Rewrite this as $a(Z) = \lambda b(Z)$, and substitute into (6) to get (7). This also verifies that L_Z is indeed a complex line.

The sections of $\mathcal{O}(1)$ can be immediately read off from (6) as the objects $a(Z)$, identified with linear homogeneous polynomials in $n+1$ variables. They form a space of dimension $n+1$, usually denoted by $H^0(\mathbb{P}^n, \mathcal{O}(1))$. Now $\mathbb{C}[z_1, \ldots, z_n]_{\leq 1}$, the polynomials of degree at most 1 in n variables, can be mapped into $H^0(\mathbb{P}^n, \mathcal{O}(1))$ by homogenizing coordinates,

$$a_0 + a_1 z_1 + \cdots + a_n z_n = a(z) \longmapsto (Z, a(Z)) \in H^0(\mathbb{P}^n, \mathcal{O}(1))$$

where $Z = [Z_0 : \cdots : Z_n] = [1 : z_1 : \cdots : z_n]$ (see (5)).

For fixed $z \in \mathbb{C}^n$ (and associated $Z = [1 : z] \in \mathcal{U}_0 \subset \mathbb{P}^n$), it is an exercise to show that this identifies $a(z)$ as the local function on $\mathcal{U}_0$ of the section given by $a(Z)$ under the local trivialization

$$\mathbb{C}^n \times \mathbb{C} \ni (z, \lambda) \mapsto (Z, \lambda Z_0) \in \mathcal{O}(1),$$

where the right-hand side is as in (7), with $\lambda = a(Z)/Z_0$.

For other values of j, a similar formula holds; e.g. when $j = 1$, consider

$$a(w) = a_0 w_0 + a_1 + a_2 w_2 + \cdots + a_n w_n \in \mathbb{C}[w_0, w_2, \ldots, w_n]_{\leq 1}.$$

Form $a(Z)$ with $Z = [w_0 : 1 : w_2 : \cdots : w_n]$; then $a(w)$ corresponds to $a(Z)$ under $(w, \lambda) \mapsto (Z, \lambda Z_1)$.

We can also calculate the maps g_{jk} on $\mathcal{U}_j \cap \mathcal{U}_k$; let us do the case $j = 0, k = 1$. Suppose $Z \in \mathcal{U}_0 \cap \mathcal{U}_1$, with coordinates z on $\mathcal{U}_0$ and w on $\mathcal{U}_1$ given by

$$[1 : z_1 : \cdots : z_n] = Z = [w_0 : 1 : w_2 : \cdots : w_n],$$

so that $w_0 = 1/z_1$ and $w_j = z_j/z_1$ for $j \neq 0, 1$. Then with

$$\zeta = a_0 + a_1 z_1 + \cdots z_n z_n \longmapsto (Z, a(Z)) \longleftarrow a_0 w_0 + a_1 + a_2 w_2 + \cdots + a_n w_n = \eta,$$

we have

$$\begin{aligned} g_{01}(Z) = \frac{\eta}{\zeta} &= \frac{a_0 w_0 + a_1 + a_2 w_2 + \cdots + a_n w_n}{a_0 + a_1 z_1 + \cdots z_n z_n} \\ &= \frac{a_0/z_1 + a_1 + a_2 z_2/z_1 + \cdots + a_n z_n/z_1}{a_0 + a_1 z_1 + \cdots z_n z_n} = \frac{1}{z_1}. \end{aligned}$$

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[3] In what follows all notions will, unless otherwise stated, refer to their complex versions.

So $g_{01}(Z) = 1/z_1 = w_0$.

For any positive integer k, define $\mathcal{O}(k)$ to be the line bundle over $\mathbb{P}^n$ given by

$$\mathcal{O}(k) = \Big\{ (Z, P(Z)) : Z \in \mathbb{P}^n, P(Z) = \sum_{|\alpha|=k} a_\alpha Z^\alpha,\ a_\alpha \in \mathbb{C} \Big\}$$

where we use the standard multi-index notation $Z^\alpha = Z_0^{\alpha_0} \cdots Z_n^{\alpha_n}$. Similar calculations as above yield the following:

1. Given $Z \in \mathbb{P}^n$, the line L_Z is generated by any $P(Z)$ which evaluates to a nonzero complex number;

2. When $Z = [1 : z]$, the map

$$\mathbb{C}[z]_{\leq k} \ni p(z) \to (Z, P(Z)) \in H^0(\mathbb{P}^n, \mathcal{O}(k))$$

   identifies $\mathbb{C}[z]_{\leq k}$ with $H^0(\mathbb{P}^n, \mathcal{O}(k))$ under $(z, \lambda) \mapsto (Z, \lambda Z_0^k)$;

3. When $Z = [1 : z] = [w_0 : 1 : w_2 : \cdots : w_n]$, we have $g_{01}(Z) = \dfrac{1}{z_1^k} = w_0^k$.

Remark 1. The last item shows that $\mathcal{O}(k)$ can be identified with $\mathcal{O}(1)^{\otimes k}$, the k-th tensor power of $\mathcal{O}(1)$, where we take the tensor power fiberwise (over each line L_a). It is an exercise to show that the same transition functions are obtained, and hence these line bundles have the same structure.

The constructions of $\mathcal{O}(k)$ on $\mathbb{P}^n$ give line bundles on holomorphic submanifolds of $\mathbb{P}^n$, by restriction; in particular, when $V \subset \mathbb{C}^n$ is a smooth algebraic subvariety with extension $V_{\mathbb{P}} \subset \mathbb{P}^n$. The restriction of $\mathcal{O}(k)$ to points over $V_{\mathbb{P}}$ corresponds to the restriction to V of the polynomials $\mathbb{C}[z]_{\leq k} = \{p \in \mathbb{C}[z] : \deg(p) \leq k\}$.

2.4 Hermitian metrics and weights

Recall that a Hermitian inner product on a complex vector space V is a map $\langle \cdot, \cdot \rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ for which $z \mapsto \langle z, w \rangle$ is linear for each fixed w, and $w \mapsto \langle z, w \rangle$ is conjugate-linear for each fixed z, and $\langle z, w \rangle = \overline{\langle w, z \rangle}$.

A Hermitian metric on a line bundle L over M is a family of Hermitian inner products $\langle \cdot, \cdot \rangle_a$ on the fibers L_a varying continuously in a: for any sections s, t the map $a \mapsto \langle s(a), t(a) \rangle_a$ is continuous. (In what follows we will suppress the dependence of the inner product on $a \in M$.)

Let us specialize to the line bundles $\mathcal{O}(k)$ over $\mathbb{P}^n$, with sections given by homogeneous polynomials as above. Later, we will restrict to subvarieties.

Example 2.1. Consider the metric on $\mathcal{O}(k)$ given by

$$\langle p(Z), q(Z) \rangle = \frac{p(Z)\overline{q(Z)}}{|Z|^{2k}},$$

where $|Z|^2 = |Z_0|^2 + \cdots + |Z_N|^2$, and we evaluate the right-hand side in homogeneous coordinates. (Note that the value obtained is independent of homogeneous coordinates.) In local coordinates on $\mathcal{U}_0$, one can write this as

$$\langle p(z), q(z) \rangle = \frac{p(z)\overline{q(z)}}{(1 + |z|^2)^k}. \tag{8}$$

Example 2.2. Suppose $W : \mathbb{C}^{n+1} \to \mathbb{C}$ is a continuous function with the property that $|W(\lambda Z)| = |\lambda|^k |W(Z)|$. Then the formula

$$\langle p(Z), q(Z) \rangle_W := \frac{p(Z)}{W(Z)} \overline{\left(\frac{q(Z)}{W(Z)} \right)} \tag{9}$$

defines a Hermitian metric on $\mathcal{O}(k)$.

Replacing W by its absolute value $|W|$ makes no difference to the right-hand side, but if W is holomorphic in some region it might be useful to leave this structure intact.

As before, sections of $\mathcal{O}(k)$ may be identified with polynomials in $\mathbb{C}[z]_{\leq k}$ by transforming to affine coordinates on $\mathcal{U}_0$; we will see below that the Hermitian metric may be identified with a weight on $\mathbb{C}^n$. We now define what this is.

Definition 2.1. Let $K \subset \mathbb{C}^n$ be a set. An *weight function on K* is a function $w : K \to \mathbb{C}$ for which

1. the absolute value $z \mapsto |w(z)|$ is lower semicontinuous; and

2. there is a non-negative real number r such that $|w(z)|$ decays like $o(|z|^{-r})$ as $|z| \to \infty$. Let us denote by $r(w)$ the inf over all such r.

If $r(w) < 1$ then w is said to be an *admissible weight function*. If $r(w) \leq 1$, then w is *weakly admissible*.

Clearly $r(w^t) = t r(w)$ for any positive integer t.

Example 2.3. Consider a polynomial $p \in \mathbb{C}[z]$ of degree $d \in \mathbb{N}$. For any $\epsilon > 0$, the function $w = 1/p$ is a weight on any set $K \subseteq (\mathbb{C}^n \setminus \{z : |p(z)| > \epsilon\})$. If K is bounded then $r(w) = 0$, otherwise $r(w) = d$, and the weight given by $|p|^{-1/d}$ is weakly admissible.

An admissible weight function on $K \subset \mathbb{C}^n$ is used to evaluate polynomials.

Definition 2.2. Let $w : K \to \mathbb{C}$ be an admissible weight on K. For any $p \in \mathbb{C}[z]$ we define the *weighted polynomial evaluation*

$$p(z)_w := w(z)p(z), \quad p(z)_{w,k} := w(z)^k p(z) \quad (z \in K, \ k \in \mathbb{N}),$$

which we extend by zero: $p(z)_w = p(z)_{w,k} = 0$ if $z \notin K$. This also yields the *weighted sup norms*

$$\|p\|_{K,w,k} := \|w^k p\|_K = \sup_{z \in K} |w(z)^k p(z)| \quad (k \in \mathbb{N}).$$

Let us relate the Hermitian metric on $\mathcal{O}(k)$ given by Example 2.2 to a weight on $\mathbb{C}^n$. Writing equation (9) in terms of z coordinates, where $[1 : z] = Z \in \mathcal{U}_0$, we have

$$\langle p(Z), q(Z) \rangle_W := \frac{p(z)}{W([1:z])} \overline{\left(\frac{q(z)}{W([1:z])} \right)} = p(z)_{w,k} \overline{q(z)_{w,k}}$$

where $w(z) := |W([1:z])|^{-1/k}$ defines the weight function. If we compare the above to (8) in Example 2.1,

$$\left| \frac{\langle p(Z), q(Z) \rangle_W}{\langle p(Z), q(Z) \rangle} \right| = \frac{|w(z)|^{2k}}{(1 + |z|^2)^k}.$$

Using the above formula we can deduce that w must be continuous and weakly admissible. For fixed Z, we can choose p, q for which $\langle p(Z), q(Z) \rangle \neq 0$ in a neighbourhood of Z. Then the left-hand side is continuous. The right-hand side then shows that this quantity is independent of p and q on $\mathbb{C}^n = \mathbb{P}^n \setminus H_\infty$, and hence on $\mathbb{P}^n$ (extending by continuity). It is bounded as a function of Z since $\mathbb{P}^n$ is compact. Looking at the right-hand side again, this implies that w is continuous and weakly admissible. (Note that if w is admissible, then $\langle \cdot, \cdot \rangle_W$ vanishes on all lines over H_∞.)

We also have a notion of sup norm on sections of a line bundle.

Definition 2.3. Let $\langle \cdot, \cdot \rangle_W$ be a Hermitian metric on a line bundle L over $\mathbb{P}^n$. Then for each $s \in H^0(\mathbb{P}^n, L)$ we define

$$\|s\|_W^2 := \sup_{Z \in \mathbb{P}^n} |\langle s(Z), s(Z) \rangle_W|.$$

Remark 2. Formally, the sup norm $\|p\|_{K,w,k}$ of Definition 2.2 on $\mathbb{C}[z]_{\leq k}$ can be put into this geometric context by defining

$$W(Z) := \begin{cases} w(Z/Z_0)^k & \text{if } Z_0 \neq 0 \text{ and } z \in K \\ +\infty & \text{if } Z_0 = 0 \text{ or } z \notin K \end{cases}$$

and defining $\langle \cdot, \cdot \rangle_W$ on $\mathcal{O}(k)$ as in equation (9). Then $\langle p(Z), p(Z) \rangle_W = \|p\|_{K,w,k}$. Note that since W is not necessarily continuous, it is an instance of a more general object called a *singular Hermitian metric*. Such objects are important in the application of pluripotential theory to complex geometry [8].

All of the above goes through on a smooth subvariety $V \subset \mathbb{C}^n$. One can define a weight function on $K \subset V$, as well as a Hermitian metric on $\mathcal{O}(k)$ over $V_\mathbb{P}$. One simply restricts attention to points of V.

Weight functions are also convenient for doing local computations on a line bundle. In this paper, we are really only interested in the special case of projective space.

Example 2.4. Let $z = (z_1, \ldots, z_n)$ denote affine coordinate in $\mathbb{C}^n$ and suppose $K \subset V \setminus \{z_1 = 0\}$, where $V \subset \mathbb{C}^n$ is an algebraic subvariety, extended to $V_\mathbb{P} \subset \mathbb{P}^n$. Let $v = (v_0, v_2, \ldots, v_n)$ be the local coordinates at infinity given by dehomogenization at z_1. This is the holomorphic map $v = g_{10}(z)$ on $\mathbb{C}^n \cap \{z_1 \neq 0\}$ given explicitly by

$$v_0 = \frac{1}{z_1}, \ v_2 = \frac{z_2}{z_1}, \ \ldots, v_n = \frac{z_n}{z_1},$$

and $z = g_{01}(v) := g_{10}^{-1}(v)$ is given by a similar formula. A section in $H^0(V_\mathbb{P}, \mathcal{O}(k))$ is given by a homogeneous polynomial $p(Z_0, \ldots, Z_n)$, with local evaluations related by

$$p(1, z_1, \ldots, z_n) = v_0^k p(v_0, 1, v_2, \ldots, v_n).$$

Hence polynomial evaluation in affine coordinates with weight $w(z) = w(z_1, \ldots, z_n)$ on K transforms to a polynomial evaluation with weight $\tilde{w}(v) = v_0^k w(g_{01}(v))$. The transition function simply appears as an additional factor. This is why it is convenient to allow complex-valued weights.

Remark 3. Given a positive finite measure μ supported on K and $k \in \mathbb{N}$, we also have the *weighted L^2 inner product and norm* on $\mathbb{C}[z]_{\leq k}$,

$$\langle p, q \rangle_{\mu,w,k} := \int p(z)_{w,k} \overline{q(z)_{w,k}} d\mu(z), \quad \|p\|_{\mu,w,k}^2 = \langle p, p \rangle_{\mu,w,k}.$$

The triple (K, μ, w) is said to satisfy the *Bernstein-Markov property* if there is a sequence $M_1, M_2, \ldots$ of positive integers such that

$$\|p\|_{K,w,k} \leq M_k \|p\|_{\mu,w,k} \text{ for all } k \in \mathbb{N}, \ p \in \mathbb{C}[z]_{\leq k}, \quad \text{and } \limsup_{k \to \infty} (M_k)^{1/k} = 1.$$

We call μ a Bernstein-Markov measure for K (with weight w).

The Bernstein-Markov property is important because it means that certain asymptotic quantities in pluripotential theory associated to a set may be computed using the L^2 norm of a Bernstein-Markov measure rather than the sup norm. The additional tools provided by the L^2 theory are important in pluripotential theory, but we will not need them in this paper.

3 Computational algebraic geometry

We want to study Okounkov bodies associated to varieties using methods of computational algebraic geometry. We will work on an algebraic variety $V \subset \mathbb{C}^n$. We first review some background material on Noether normalization and normal forms of polynomials.

3.1 Normal forms and Noether normalization

By the Nullstellensatz, restricting the evaluation of $p \in \mathbb{C}[z]$ to points of V is equivalent to taking the quotient $\mathbb{C}[z]/\mathbf{I}(V)$, with associated equivalence relation $p \sim q$ if $p - q \in \mathbf{I}(V)$.

Notation 3.1. Given $k \in \mathbb{N}$, denote by $\mathbb{C}[V]_{\leq k}$ the quotient space $\mathbb{C}[z]_{\leq k}/\sim$ with $\sim$ as above. For $q \in \mathbb{C}[z]_{\leq k}$ we can identify equivalence classes containing q under the natural inclusion

$$\mathbb{C}[V]_{\leq k} \ni \{p \in \mathbb{C}[z]_{\leq k} : p \simeq q \in \mathbb{C}[V]_{\leq k}\} \hookrightarrow \{p \in \mathbb{C}[z] : p \simeq q \in \mathbb{C}[V]_{\leq k}\} \in \mathbb{C}[V].$$

Then under this identification, one can see that $\mathbb{C}[V] = \bigcup_k \mathbb{C}[V]_{\leq k}$. For a general polynomial p, put $\deg_V(p) = k$ if p is equivalent to a polynomial of degree k but not of degree $k-1$ (i.e., in $\mathbb{C}[V]_{\leq k} \setminus \mathbb{C}[V]_{\leq k-1}$).

Via dehomogenization in affine coordinates, $H^0(V_{\mathbb{P}}, \mathcal{O}(k))$ may be identified with $\mathbb{C}[V]_{\leq k}$.

Theorem 3.2 (Noether Normalization Theorem). *Suppose V is of dimension m. There is a complex linear change of coordinates on $\mathbb{C}^n$ such that, in the new coordinates (which we denote by $(x, y) := (x_1, \ldots, x_m, y_1, \ldots, y_{n-m})$),*

1. *The projection map $\pi : V \to \mathbb{C}^m$ given by $\pi(x, y) = x$ is onto, and $\pi^{-1}(x)$ is finite for each $x \in \mathbb{C}^m$;*

2. *We have an injection $\mathbb{C}[x] \hookrightarrow \mathbb{C}[V]$ that exhibits $\mathbb{C}[V]$ as a finite dimensional algebra over $\mathbb{C}[x]$.*

The map $\mathbb{C}[x] \hookrightarrow \mathbb{C}[V]$ given in the theorem is given by identifying p with its equivalence class in $\mathbb{C}[V]$. (See e.g. Chapter 5 §6 of [6] for a proof of this theorem.)

We turn to algebraic computation in $\mathbb{C}[V]$; this requires an ordering on monomials. First, we recall the *lexicographic (lex) ordering* on $\mathbb{Z}_{\geq 0}^n$ (denoted $\prec_l$). We have $\alpha \prec_l \beta$ if there exists a $j \in \{1, \ldots, n\}$ for which $\alpha_j < \beta_j$, and $\alpha_k = 0$ for all $k > j$. Monomials in $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$ are ordered accordingly: $z^\alpha \prec_l z^\beta$ if $\alpha \prec_l \beta$, so that $z_1 \prec_l z_2 \prec_l \cdots \prec_l z_n$. We will come back to lex ordering later.

We also recall the *grevlex ordering* which has good computational properties. This is the ordering $\prec_g$ for which $z^\alpha \prec_g z^{\alpha'}$ whenever

1. $|\alpha| < |\alpha'|$; or

2. $|\alpha| = |\alpha'|$ and $z^\alpha \prec_l z^{\alpha'}$.

Notation 3.3. For a polynomial $p \in \mathbb{C}[z]$l, let us denote by LT(p) the leading term of p with respect to grevlex, and for an ideal I of $\mathbb{C}[z]$, let LT$(I) := \{$LT$(p) : p \in I\}$.

We will use the grevlex ordering to compute normal forms. Let us recall what these are. First, a *Groebner basis* of an ideal I is a collection $\{g_1, \ldots, g_\ell\} \subset I$ for which

$$I = \langle g_1, \ldots, g_\ell \rangle \text{ and } \langle \text{LT}(I) \rangle = \langle \text{LT}(g_1), \ldots, \text{LT}(g_\ell) \rangle.$$

For each element of $\mathbb{C}[z]/I$ there is a unique polynomial representative, called the *normal form*, which contains no monomials in the ideal $\langle \text{LT}(I) \rangle$. The normal form of a polynomial p may be computed in practice as the remainder r upon dividing p by a Groebner basis of I:

$$p = q_1 g_1 + \cdots + q_\ell g_\ell + r,$$

where $q_1, \ldots, q_\ell \in \mathbb{C}[z]$ are the quotients. (See e.g. chapter 3 of [6] for a description of the associated division algorithm.)

Let $\mathbb{C}[z]_I$ be the collection of normal forms. This is an algebra over $\mathbb{C}[z]$ under the usual addition of polynomials, and with multiplication defined by

$$\mathbb{C}[z]_I \times \mathbb{C}[z]_I \ni (r_1, r_2) \longmapsto \text{"the normal form of } r_1 r_2 \text{"} \in \mathbb{C}[z]_I. \tag{10}$$

The following algebraic version of Noether normalization is given in [7].

Proposition 3.4. *Let V be of dimension m and let (x, y) be coordinates as in Theorem 3.2. Let $\mathbb{C}[x, y]_I$ be the algebra of normal forms for $I = \mathbf{I}(V)$. Then*

1. $\deg_V(p) = \deg(p)$ *(i.e. the usual degree) whenever p is a normal form.*

2. *We have the inclusion $\mathbb{C}[x] \subseteq \mathbb{C}[x, y]_I$, which exhibits $\mathbb{C}[x, y]_I$ as a finite dimensional algebra over $\mathbb{C}[x]$.*

Here, multiplication in $\mathbb{C}[x, y]_I$ is as in equation (10). The proposition says that any $p \in \mathbb{C}[x]$ is a normal form, and shows that grevlex has good computational properties.

In what follows, we will usually assume polynomials to be normal forms, and $\mathbb{C}[V]$ will be identified with $\mathbb{C}[x, y]_I$. The inclusion in item (2) of the proposition is called a *Noether normalization*; we will also write (via our identifications) $\mathbb{C}[x] \subseteq \mathbb{C}[V]$. Let us also refer to the coordinates (x, y) as *(Noether) normalized coordinates*.

Since $\mathbb{C}[V]$ is finite dimensional over $\mathbb{C}[x]$, and has a basis of monomials, there are only a finite number of monomials y^β for which $x^\alpha y^\beta$ is a normal form. Hence any normal form, being a linear combination of such monomials, can be expressed as a finite sum

$$p(z) = p(x, y) = \sum_\beta y^\beta p_\beta(x), \quad p_\beta \in \mathbb{C}[x]. \tag{11}$$

Example 3.1. The (complexified) sphere in $\mathbb{C}^3$ is given by

$$V = \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 1\}, \tag{12}$$

and $\langle z_1^2 + z_2^2 + z_3^2 - 1 \rangle = \mathbf{I}(V) =: I$. Any polynomial in I is of the form $q(z)(z_1^2 + z_2^2 + z_3^2 - 1)$, and

$$\text{LT}(q(z)(z_1^2 + z_2^2 + z_3^2 - 1)) = \text{LT}(q(z)z_3^2) \in \langle z_3^2 \rangle;$$

it follows easily that $\langle \text{LT}(I) \rangle = \langle z_3^2 \rangle$. Hence a normal form $p \in \mathbb{C}[V]$ is a polynomial given by

$$p(z) = p_1(z_1, z_2) + z_3 p_2(z_1, z_2), \quad p_1, p_2 \in \mathbb{C}[z_1, z_2]. \tag{13}$$

(Compare the above to (11).) As a 2-dimensional algebra over $\mathbb{C}[z_1, z_2]$, multiplication is given by

$$(p_1 + z_3 p_2) \cdot (q_1 + z_3 q_2) = p_1 q_1 + (1 - z_1^2 - z_2^2) p_2 q_2 + z_3 (p_1 q_2 + p_2 q_1).$$

Clearly, $x = (x_1, x_2) := (z_1, z_2)$ and $y = z_3$ give Noether normalized coordinates satisfying Theorem 3.2: $x \in \mathbb{C}^2$ lifts to at most 2 points $(x, y) \in V \subset \mathbb{C}^3$ given by the branches of the square root in the expression $y = (1 - x_1^2 - x_2^2)^{1/2}$.

3.2 Okounkov body computation

Following Witt Nyström [19], let us define the Okounkov body of a line bundle. Returning to the geometric setting, let L be a holomorphic line bundle over a complex manifold M of dimension n, and $p \in M$. In a local trivialization containing p, any $s \in H^0(M, L)$ is given by a holomorphic function (let us also denote the function by s). Hence it can be expressed as a power series

$$s = \sum_\alpha c_\alpha z^\alpha \tag{14}$$

where z is a local holomorphic coordinate centered at the point p, and we use multi-index notation: for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, we have $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

Definition 3.1. Suppose a local holomorphic coordinate z is fixed at $p \in M$. Given a section $s \in H^0(M, L)$, we define $v(s)$ to be the lowest exponent in the power series (14) with respect to the lex order, $\prec_l$. That is, if $v(s) = \gamma$ then $c_\alpha = 0$ whenever $\alpha \prec_l \gamma$. When $\gamma = v(s)$ we also define

    $\text{TT}(s) = c_\gamma z^\gamma$, the *trailing term*;

    $\text{TC}(s) = c_\gamma$, the *trailing coefficient*; and

    $\text{TM}(s) = z^\gamma$, the *trailing monomial*.

Definition 3.2. Fix a local holomorphic coordinate z at a point $a \in M$, and let $k \in \mathbb{N}$. Expand $s \in H^0(M, L^{\otimes k})$ as in (14), and define

$$\mathcal{N}_k := \{v(s) \in \mathbb{Z}_{\geq 0}^n : s \in H^0(M, L^{\otimes k})\}.$$

Let $\Delta_k \subset \mathbb{R}^m$ be the convex hull of the set $\frac{1}{k} \mathcal{N}_k$. The *Okounkov body of L (with respect to these coordinates)*, denoted by $\Delta = \Delta(L)$, is defined to be the convex hull of the set $\bigcup_{k \in \mathbb{N}} \Delta_k$.

In our concrete setting of an algebraic subvariety $V \subseteq \mathbb{C}^n$ of dimension m, we will use the sections $H^0(V_\mathbb{P}, \mathcal{O}(k))$, or equivalently, the polynomials $\mathbb{C}[V]_{\leq k}$. (For convenience, let us assume $V_\mathbb{P}$ is smooth, so that the theory on a complex manifolds can be transferred without any technicality.) We will use Noether normalized coordinates $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_{n-m})$ on V. Without loss of generality, assume that the point a at which the Okounkov body is calculated is of the form $(0, \ldots, 0, a_{m+1}, \ldots, a_n)$, i.e., all of the x coordinates are zero.[4] We will also assume that the point a is a regular point for the projection to the x coordinates, i.e., V satisfies the hypotheses of the holomorphic implicit function theorem:

$$\det \begin{bmatrix} \partial f_1/\partial y_1(a) & \cdots & \partial f_1/\partial y_{n-m}(a) \\ \vdots & \ddots & \vdots \\ \partial f_{n-m}/\partial y_1(a) & \cdots & \partial f_{n-m}/\partial y_{n-m}(a) \end{bmatrix} \neq 0, \tag{15}$$

where the polynomials $f_1, \ldots, f_{n-m}$ determine V in a neighborhood of a. Locally, we can write $y = Y(x)$ for some holomorphic function Y in a neighborhood of $x = (0, \ldots, 0)$. The x coordinates in the Noether normalization provide the local coordinates with which the Okounkov body will be calculated.

Notation 3.5. Suppose $Y(x) = (Y_1(x), \ldots, Y_{n-m}(x))$ in components. Then for a multi-index $\beta = (\beta_1, \ldots, \beta_{n-m})$ the holomorphic function Y^β is given by

$$Y^\beta(x) := Y_1(x)^{\beta_1} Y_2(x)^{\beta_2} \cdots Y_{n-m}(x)^{\beta_{n-m}}.$$

In a neighborhood of the origin, it may be expressed as a power series in x:

$$Y^\beta(x) = \sum_\alpha c_{\beta\alpha} x^\alpha. \tag{16}$$

---

[4] Any translation $x \mapsto x + c = \tilde{x}$ gives an isomorphism of normal forms $p(\tilde{x}, y) \mapsto p(x + c, y)$; the verification of this is left as an exercise.

Let $k \in \mathbb{N}$; we want to compute $\mathcal{N}_k \subset \mathbb{Z}_{\geq 0}^m$. A section of $H^0(V_{\mathbb{P}}, \mathcal{O}(k))$ can be identified with a polynomial in $\mathbb{C}[V]_{\leq k}$; by Proposition 3.4(1) it is given by a normal form p of degree $\leq k$. Let $\{y^\beta\}_\beta$ be the finite collection of y monomials as in (11). In a neighborhood of the origin, we rewrite $p = p(x, y)$ as the holomorphic function $x \mapsto p(x, Y(x))$. Write this as a power series in x, and compute the coefficients by multiplying out the terms of the polynomials p_β with the power series (16) for Y^β:

$$p(x, Y(x)) = \sum_\beta Y^\beta(x) p_\beta(x) = \sum_\beta \sum_\alpha p_\beta(x) c_{\beta\alpha} x^\alpha =: \sum_\alpha b_\alpha x^\alpha.$$

We can then read off $\nu(p)$ from the trailing term on the right-hand side.

We determine the possibilities for $\nu(p)$. For all $\alpha \in \mathbb{Z}_{\geq 0}^m$ with $|\alpha| \leq k$, we have, by Proposition 3.4,

$$x^\alpha \in \mathbb{C}[x]_{\leq k} \subseteq \mathbb{C}[V]_{\leq k}, \quad \text{and } \nu(x^\alpha) = \alpha.$$

The points α/k, with α as above, fill out a grid of rational points in the region bounded by the coordinate hyperplanes and the standard simplex in $\mathbb{R}^m$.[5] This accounts for all normal forms in $\mathbb{C}[x]_{\leq k}$, and so remaining points in $\mathcal{N}_k$ must be calculated from normal forms containing powers of y, which involve the analytic functions Y^β. We will use the notion of *S-polynomial* to do this systematically (Definition 3.3). Computing the points of $\frac{1}{k}\mathcal{N}_k$ for larger and larger values of k, we eventually fill in the Okounkov body.

We write out the details of some calculations on the complexified sphere in what follows. From these calculations, we derive a general method that can be applied to varieties given in Noether normalized coordinates.[6]

3.3 Explicit computation on the sphere

Let V be the complexified sphere as in (12), Example 3.1. For the Okounkov body, take the point $a = (0, 0, 1)$, and (z_1, z_2) as the local coordinates in which to expand polynomials on V. By (13) the normal forms in $\mathbb{C}[V]$ are linear combinations of monomials of the form $z_1^j z_2^k$ or $z_1^j z_2^k z_3$. At the point a,

$$\frac{\partial}{\partial z_3}(z_1^2 + z_2^2 + z_3^2 - 1)\big|_{(0,0,1)} = 2z_3\big|_{(0,0,1)} = 2 \neq 0,$$

so (15) holds and we can write $z_3 = Y(z_1, z_2)$, which is in fact the standard square root function:

$$z_3 = Y(z_1, z_2) = (1 - z_1^2 - z_2^2)^{1/2} =: \sum c_{jk} z_1^j z_2^k.$$

The coefficients $c_{jk} = \dfrac{\partial^{j+k}}{\partial z_1^j \partial z_2^k} Y(0,0)$ may be computed by implicit differentiation, for example,

$$0 = \frac{\partial}{\partial z_1}(z_1^2 + z_2^2 + Y^2)\bigg|_{(0,0,1)} = 2z_1 + 2Y \frac{\partial Y}{\partial z_1}\bigg|_{(0,0,1)} = 2 \cdot 0 + 2c_{10},$$

so that $c_{10} = 0$. Further differentiation gives more coefficients, e.g. $c_{20} = -\frac{1}{2}$, so that $z_3 = 1 - \frac{1}{2}z_1^2 + \cdots$. (Note that here it is faster just to read off the coefficients from the binomial series for the square root.)

The following properties are straightforward to verify.

Lemma 3.6. *For all nonzero $p, q \in \mathbb{C}[V]$,*

$$\begin{aligned} \nu(pq) &= \nu(p) + \nu(q), \text{ and} \\ \nu(p + q) &\succ_l \min\{\nu(p), \nu(q)\}. \end{aligned} \tag{17}$$

When $\nu(p) \neq \nu(q)$ then $\nu(p + q) = \min\{\nu(p), \nu(q)\}$. □

Hence (17) is strict only if $\nu(p) = \nu(q)$ and we have cancellations of lowest terms. This motivates the following definition.

Definition 3.3. Suppose $\nu(p) = \nu(q)$. We define the *S-polynomial*[7] of p and q by

$$S(p, q) := \text{TC}(q)p - \text{TC}(p)q.$$

From $p = a_0 + a_1 z_1 + a_2 z_2 + a_3 z_3 \in \mathbb{C}[V]_{\leq 1}$, the properties of ν under addition in Lemma 3.6 gives possible values for $\nu(p) \in \mathcal{N}_1$ as

$$\nu(1), \nu(z_1), \nu(z_2), \nu(z_3).$$

Now $\nu(1) = (0, 0) = \nu(z_3)$, so we can arrange a possible cancellation of the constant term. We compute $S(z_3, 1) = z_3 - 1$ and $\nu(z_3 - 1) = (2, 0)$, to get the remaining point of $\mathcal{N}_1$.

The following elementary observation is useful.

Lemma 3.7. *For any $k \in \mathbb{N}$, $dim\mathbb{C}[V]_{\leq k}$ is the number of points in $\mathcal{N}_k$.*

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[5]The standard simplex in $\mathbb{R}^m$ is the convex hull of the standard basis vectors $\{(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$.

[6]But we do not give a rigorous proof.

[7]Here S stands for *syzygy* (a pair of connected or corresponding things). The terminology is adapted from chapter 2 of [6].

$\mathcal{N}_2$: initial step

$\mathcal{N}_2$: first S-polynomial calculation

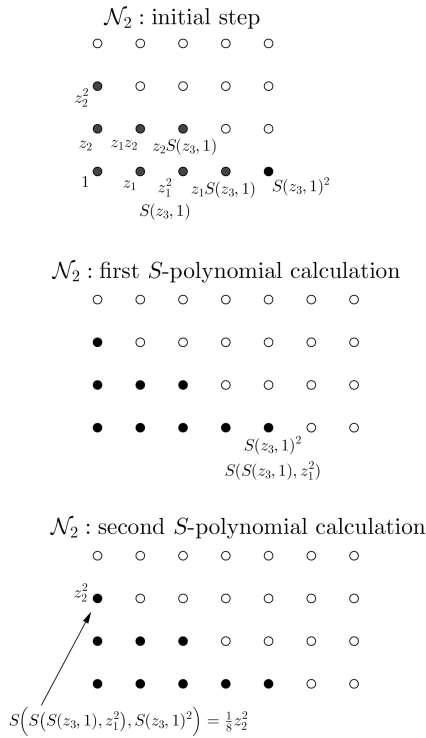$\mathcal{N}_2$: second S-polynomial calculation

Figure 1: Computing points of $\mathcal{N}_2$ and S-polynomials.

Proof. Let $M_k := \dim \mathbb{C}[V]_{\leq k}$ and let $\{\mathbf{e}_j\}_{j=1}^{M_k}$ be a basis. Then $\{\nu(\mathbf{e}_j)\}_{j=1}^{M_k} \subseteq \mathcal{N}_k$, so the number of points in $\mathcal{N}_k$ is at least M_k.

On the other hand, let p, q be nonzero polynomials in $\mathbb{C}[V]_{\leq k}$ with $\nu(p) \neq \nu(q)$; without loss of generality, $\nu(p) \prec_l \nu(q)$. Then

$$\nu(\lambda p + \mu q) = \begin{cases} \nu(p) & \text{if } \lambda \neq 0 \\ \nu(q) & \text{if } \lambda = 0 \neq \mu \end{cases}.$$

This implies $\lambda p + \mu q \neq 0$ unless $\lambda = \mu = 0$. Hence any set of polynomials $\{p_\alpha\}_{\alpha \in \mathcal{N}_k} \subset \mathbb{C}[V]_{\leq k}$ (for which $\nu(p_\alpha) = \alpha$) is linearly independent, so the size of this set is at most $\dim \mathbb{C}[V]_{\leq k}$. $\square$

Let us compute $\mathcal{N}_2$; this will motivate the general case. First, note that products of pairs of polynomials from $\mathcal{B}_1 := \{1, z_1, z_2, z_3, S(z_3, 1)\}$ span $\mathbb{C}[V]_{\leq 2}$; this follows easily from the fact that the map

$$\mathbb{C}[\nu_1, \nu_2, \nu_3, \nu_4]_{\leq 2} \longrightarrow \mathbb{C}[V]_{\leq 2} \tag{18}$$

given by making the substitutions $\nu_1 = z_1, \nu_2 = z_2, \nu_3 = z_3, \nu_4 = S(z_3, 1)$, followed by reduction to normal form, is well-defined and onto. From such products, we immediately obtain points of $\mathcal{N}_2$ given by

$$\nu(pq), \quad \text{where } p, q \in \mathcal{B}_1.$$

This gives nine points. Since $\dim \mathbb{C}[V]_{\leq 2} = 9$, then by the previous lemma, we are done.

Without knowing the dimension a priori, one can simply compute possible S-polynomials and check that no new points are obtained. For $\mathcal{N}_2$, we have $\nu(z_1^2) = \nu(z_3 - 1) = (2, 0)$, and

$$S(S(z_3 - 1), z_1^2)) = z_3 - 1 + \tfrac{1}{2} z_1^2$$

with $\nu(z_3 - 1 + \tfrac{1}{2} z_1^2) = (4, 0)$. But now, $\nu(S(z_3, 1)^2) = (4, 0)$ also, and provides another possibility for cancellation. We calculate $(z_3 - 1)^2 = z_3^2 - 2z_3 + 1 = 2z_3 - 2 - z_1^2 - z_2^2$, and then the S-polynomial

$$S(z_3 - 1 + \tfrac{1}{2} z_1^2, 2z_3 - 2 - z_1^2 - z_2^2) = \tfrac{1}{8} z_2^2,$$

whose ν value is $(2, 0)$. Finally, $S(\tfrac{1}{8} z_2^2, z_2^2) = 0$. There are no further S-polynomials to be calculated. See Figure 1 for a picture.

3.4   General algorithm

Let us use the same method inductively for $\mathcal{N}_k$. We begin by setting $\mathcal{N}_k$ to be $\mathcal{N}_{k-1}$. Step 1 below uses the fact that if the set $\{\mathbf{e}_j\}$ spans $\mathbb{C}[V]_{\leq k-1}$, then the normal forms of the products

$$\{\mathbf{e}_j \mathbf{e}_\ell : \deg(\mathbf{e}_j \mathbf{e}_\ell) \leq k\}$$

span $\mathbb{C}[V]_{\leq k}$.

Step 1 *Multiplication*. Let $\{\mathbf{e}_j\}$ be a collection of polynomials whose ν values give $\mathcal{N}_{k-1}$ (one polynomial for each ν value). Compute all products in which $\deg(\mathbf{e}_j\mathbf{e}_\ell) = k$, and denote this collection of products by $\mathcal{P}$.

Step 2 *Compute ν values*. Compute $\nu(p)$ for each $p \in \mathcal{P}$. Add any new ν values thus obtained to $\mathcal{N}_k$.

Step 3 *Find α for possible cancellation*. Let $\alpha \in \mathcal{N}_k$ be the smallest (according to $\prec_l$) point that is given by more than one element of $\{\mathbf{e}_j\} \cup \mathcal{P}$.

Step 4 *Compute S-polynomials at α*. Compute S-polynomials associated to distinct pairs of polynomials p in $\{\mathbf{e}_j\} \cup \mathcal{P}$ for which $\nu(p) = \alpha$.

Step 5 *Update $\mathcal{N}_k$ and $\mathcal{P}$*. Find the ν values of the nonzero S-polynomials in the previous step. Add the ν values to the set $\mathcal{N}_k$, and add the S-polynomials to the set $\mathcal{P}$.

Step 6 *Repeat*. Let α be the next point in $\mathcal{N}_k$ (according to $\prec_l$) that is the ν value of at least two different elements of $\mathcal{P} \cup \{\mathbf{e}_j\}$. But if no such α exists, or we know that the current size of $\mathcal{N}_k$ equals the dimension of $\mathbb{C}[V]_{\leq k}$, go to the next step. Otherwise, return to Step 4.

Step 7 *Stop*. Output $\mathcal{N}_k$.

To compute $\mathcal{N}_{k+1}$, expand the collection $\{\mathbf{e}_j\}$ by adding elements of $\mathcal{P}$ computed in the above algorithm so that there is one polynomial for each point of $\mathcal{N}_k$; then begin again from Step 1 with $k := k+1$.

The above method can be applied to a subvariety $V \subseteq \mathbb{C}^n$ of pure dimension m at a smooth point $a \in V$. Assume that we are working in coordinates $(x_1, \ldots, x_m, y_1, \ldots, y_{n-m}) = (x, y)$ that give a Noether normalization of V at a. Without loss of generality, we translate coordinates so that the x coordinates of a are zero.

Start with the monomials $\{1, x_1, \ldots, x_m, y_1, \ldots, y_{n-m}\}$ and compute $\mathcal{N}_1$ using the power series of the monomials y_j, $j = 1, \ldots, n-m$, as local functions of the x coordinates near a. Continue inductively, computing $\mathcal{N}_2, \mathcal{N}_3, \ldots$ exactly as above, by multiplying polynomials, and computing S-polynomials if required. Use the points of $\frac{1}{k}\mathcal{N}_k$ for sufficiently many values of k to fill in the Okounkov body.

Remark 4. The above algorithm is analogous to Gaussian elimination. Eliminating a lower order monomial by computing S-polynomials is essentially eliminating a variable in a linear system.

Using Lemma 3.7 it may be possible to deduce the Okounkov body exactly. Let us do this for our sphere example.

Proposition 3.8. *The Okounkov body of the complexified sphere is the triangle given by*

$$\Delta := \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1, \theta_2 \geq 0, \theta_1 + 2\theta_2 \leq 1\}.$$

Proof. It is sufficient to show that for each $k \in \mathbb{N}$, $\Delta \subseteq \Delta_k$. This will follow from the following claim:
$$\mathcal{N}_k \text{ consists of the integer points of } k\Delta, \text{ i.e., } \mathcal{N}_k = k\Delta \cap \mathbb{Z}^2.$$
Our calculations above have already established this claim for $k = 1, 2$. Working by induction on k, suppose $\mathcal{N}_{k-1} = (k-1)\Delta \cap \mathbb{Z}^2$. Then it is easy to see that

$$k\Delta \cap \mathbb{Z}^2 = \{\nu(pq) : p \in \mathbb{C}[V]_{\leq 1}, q \in \mathbb{C}[V]_{\leq k-1}\} \subseteq \mathcal{N}_k.$$

The claim will follow if the number of points in $k\Delta \cap \mathbb{Z}^2 =: \mathcal{M}_k$ equals $\dim \mathbb{C}[V]_{\leq k}$. By induction, this can be reduced to verifying that the number of new points, i.e., the size of $\mathcal{M}_k \setminus \mathcal{N}_{k-1}$, is equal to the number of elements of degree k in a basis for $\mathbb{C}[V]_{\leq k}$.

On the one hand, the elements of degree k in a monomial basis of $\mathbb{C}[V]_{\leq k}$ are

$$\{z_1^k, z_1^{k-1}z_2, \ldots, z_2^k, z_1^{k-1}z_3, z_1^{k-2}z_2z_3, \ldots, z_2^{k-1}z_3\},$$

which are $2k-1$ elements. On the other hand, one can see that $\mathcal{M}_k \setminus \mathcal{N}_{k-1}$ has $2k-1$ points by some elementary geometry in the plane.[8] $\square$

We close this section by looking a bit more closely at the limiting behaviour of the Okounkov body construction. Although $k < l$ does not imply $\Delta_k \subset \Delta_l$, it is easy to verify the weaker statement

$$\Delta_k \subseteq \Delta_{rk}, \quad \text{for all } k, r \in \mathbb{N}. \tag{19}$$

This follows from the fact that if $\alpha \in \frac{1}{k}\mathcal{N}_k$ then $k\alpha = \nu(p)$ for some $p \in \mathbb{C}[V]_{\leq k}$. Consequently, $p^r \in \mathbb{C}[V]_{\leq rk}$ with $\nu(p^r) = rk\alpha$, so $\alpha \in \frac{1}{rk}\mathcal{N}_{rk}$. Hence $\frac{1}{k}\mathcal{N}_k \subset \frac{1}{rk}\mathcal{N}_{rk}$, and (19) follows upon taking convex hulls.

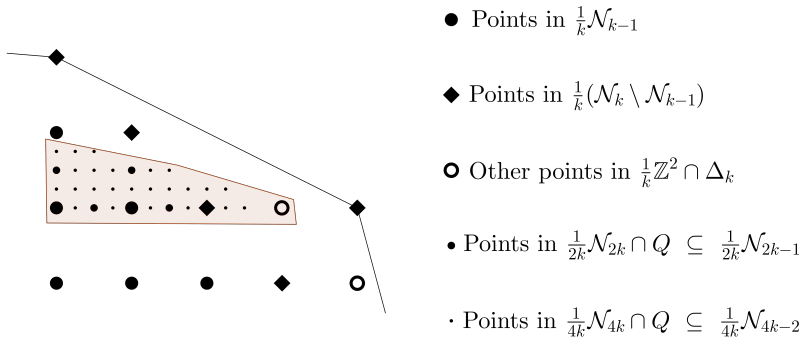The following result will be used in the next section.

Proposition 3.9. *Let $\Delta \subset \mathbb{R}^m$ be the Okounkov body of some variety, Δ° its interior, and $Q \subset \Delta^\circ$ a compact convex subset with nonempty interior. For a sufficiently large positive integer k_0,*

$$\frac{1}{k_0}\mathcal{N}_{k_0} \cap Q = \frac{1}{k_0}\mathbb{Z}^m \cap Q. \tag{20}$$

If $|\mathcal{N}_k \cap kQ|$ denotes the number of points in $\mathcal{N}_k \cap kQ$, then

$$\frac{1}{k^m}|\mathcal{N}_k \cap kQ| \longrightarrow vol(Q) \quad \text{as } k \to \infty. \tag{21}$$

---

[8] Precisely, it consists of the k points along the hypotenuse and the $k-1$ points on the neighbouring parallel line.

● Points in $\frac{1}{k}\mathcal{N}_{k-1}$

◆ Points in $\frac{1}{k}(\mathcal{N}_k \setminus \mathcal{N}_{k-1})$

○ Other points in $\frac{1}{k}\mathbb{Z}^2 \cap \Delta_k$

• Points in $\frac{1}{2k}\mathcal{N}_{2k} \cap Q \;\subseteq\; \frac{1}{2k}\mathcal{N}_{2k-1}$

· Points in $\frac{1}{4k}\mathcal{N}_{4k} \cap Q \;\subseteq\; \frac{1}{4k}\mathcal{N}_{4k-2}$

Figure 2: Suppose $\frac{1}{k}\nu(p) = \alpha$ where $p \in \mathbb{C}[V]_{\le k-1}$ (one of the large black circles). Then

$$p^2 \in \mathbb{C}[V]_{\le 2k-2} \;\; \text{and} \;\; z_1 p^2, z_2 p^2 \in \mathbb{C}[V]_{\le 2k-1},$$

giving $\frac{1}{2k}\nu(p^2) = \alpha$, $\frac{1}{2k}\nu(z_1 p^2) = \alpha + (\frac{1}{2k}, 0)$, and $\frac{1}{2k}\nu(z_2 p^2) = \alpha + (0, \frac{1}{2k})$ as points of $\mathcal{N}_{2k}$. We can do the same sort of thing to get points of $\mathcal{N}_{4k}$. In the above example, the 'missing point' of Q in $\frac{1}{k}\mathbb{Z}^m \setminus \frac{1}{k}\mathcal{N}_k$ will be covered with a point of $\frac{1}{8k}\mathcal{N}_{8k}$; hence (20) holds with $k_0 = 8k$.

Although the proof is elementary, it is rather technical, so we omit it. In Lemma 2.3 of [19], it is shown (using a theorem of Khovanskii in [13]) that there is some constant $C > 0$ such that (20) holds for all k_0 for which the distance from Q to $\partial \Delta$ is greater than C/k_0.

We can give an idea for why (20) holds. For Q compact in Δ°, the number of points in $kQ \cap \mathcal{N}_k$ is almost the same as the number of points in $kQ \cap \mathcal{N}_{k-1}$. By taking certain products and powers of polynomials (as in the verification of (19)), one can fill in the gaps by 'translation from existing nodes'. See Figure 2 for an illustration.

Equation (21) follows easily from (20). Up to a boundary correction of order k^{m-1}, the number of points in $\frac{1}{k_0}\mathbb{Z}^m \cap Q$ is equal to the number of m-dimensional cubes in the corresponding grid that cover Q, and the volume of each cube is $(\frac{1}{k})^m$. The total volume of these cubes converges to the total volume of Q.

Note that Δ is compact: for sufficiently large k, the number of points in $\mathcal{N}_k = \dim \mathbb{C}[V]_{\le k}$ is given by the Hilbert polynomial of V, which is a polynomial in k of degree m. Hence $\frac{1}{k^m}|\mathcal{N}_k|$ is uniformly bounded. The same argument as in the previous paragraph shows that $\frac{1}{k^m}|\mathcal{N}_k| \to \text{vol}(\Delta)$, which must be finite. Since Δ is a closed convex set with finite nonzero volume, it must be compact.

4 Chebyshev transform and transfinite diameter

Let $V \subset \mathbb{C}^n$ be an algebraic variety. Throughout this section we will assume that we are working in local coordinates on V constructed from a Noether normalization, with the associated Okounkov body as defined in the previous section. Let us denote these local coordinates by z. We will define directional Chebyshev constants associated to polynomials on V, and the Chebyshev transform. Then we will define transfinite diameter in terms of a special basis of $\mathbb{C}[V]$. Finally, we will prove the main theorem that gives transfinite diameter as an integral of the Chebyshev transform over the Okounkov body.

Given $k \in \mathbb{N}$ and a multi-index $\alpha \in \mathcal{N}_k$, we define the normalized class of polynomials

$$\mathcal{M}(k, \alpha) := \big\{ p \in \mathbb{C}[V]_{\le k} : \text{TT}(p) = z^\alpha \; (\text{i.e., } \text{TC}(p) = 1) \big\}.$$

Let $K \subset V$ be a set and w an admissible weight on K.

Definition 4.1. For k a positive integer and $\alpha \in \mathcal{N}_k$. We define the *Chebyshev constant* $T_k^w(K, \alpha)$ by

$$T_k^w(K, \alpha) := \inf\big\{ \|p\|_{K,w,k} : \; p \in \mathcal{M}(k, \alpha) \big\}^{1/k}.$$

A polynomial t for which $T_k^w(K, \alpha) = \|t\|_K^{1/k}$ will be called a *Chebyshev polynomial*.

Lemma 4.1. *We have*

$$T_{k_1+k_2}^w(K, \alpha_1 + \alpha_2)^{k_1+k_2} \le T_{k_1}^w(K, \alpha_1)^{k_1} T_{k_2}^w(K, \alpha_2)^{k_2}.$$

Proof. Let t_{k_1}, t_{k_2} satisfy $\|t_{k_1}\|_{K,w,k_1} = T_{k_1}^w(K, \alpha_1)^{k_1}$ and $\|t_{k_2}\|_{K,w,k_2} = T_{k_2}^w(K, \alpha_2)^{k_2}$. Then $t_{k_1} t_{k_2} \in \mathcal{M}(k_1 + k_2, \alpha_1 + \alpha_2)$, and so

$$T_{k_1+k_2}^w(K, \alpha_1 + \alpha_2) \le \|t_{k_1} t_{k_2}\|_{K,w,k_1+k_2} \le \|t_{k_1}\|_{K,w,k_1} \|t_2\|_{K,w,k_2}.$$

$\square$

The lemma says that $(k, \alpha) \mapsto T_k^w(K, \alpha)^k$ is a *submultiplicative function*. This property allows us to define directional Chebyshev constants via a limiting process.

Lemma 4.2. *The limit*

$$T^w(K,\theta) := \lim_{\substack{k\to\infty \\ \alpha/k\to\theta}} T_k^w(K,\alpha)$$

exists for each $\theta \in \Delta^\circ$.

Proof. Let $\{\alpha_j\}, \{\beta_j\}, \{s_j\}, \{t_j\}$ be sequences of multi-indices such that as $j \to \infty$, we have

$$s_j, t_j \to \infty, \quad \frac{\alpha_j}{s_j}, \frac{\beta_j}{t_j} \to \theta, \tag{22}$$

and

$$T_{s_j}^w(K,\alpha_j) \to L_1 := \liminf_{\substack{k\to\infty \\ \alpha/k\to\theta}} T_k^w(K,\alpha), \quad \text{and} \quad T_{t_j}^w(K,\beta_j) \to L_2 := \limsup_{\substack{k\to\infty \\ \alpha/k\to\theta}} T_k^w(K,\alpha).$$

The existence of such an approximating sequence follows from Proposition 3.9.[5]

It is sufficient to show that $L_2 \le L_1$. To this end, let $\epsilon > 0$, and choose an index j_0 large enough that $T_{s_{j_0}}(K,\alpha_{j_0})^{s_{j_0}} < L_1 - \epsilon$. Choose a polynomial $p \in \mathcal{M}(s_{j_0}, \alpha_{j_0})$ such that $\|p\|_{K,w,s_{j_0}}^{1/s_{j_0}} < L_1 - \epsilon$. Now, for each j let q_j be the largest non-negative integer for which

$$\beta_j = q_j \alpha_{j_0} + \ell_j$$

and all entries of ℓ_j are non-negative. Using the fact that all entries of α_{j_0} are positive, and each entry of β_j goes to infinity, it is straightforward to verify that $q_j \to \infty$ and the set of ℓ_j's is a finite subset of $\mathbb{Z}^m$. Using (22), we also have $\frac{q_j s_{j_0}}{t_j} \to 1$ as $j \to \infty$.

For every j, consider the polynomial $P_j := p^{q_j} r_j$, where r_j is a polynomial for which $\nu(r_j) = \ell_j$. Then $P_j \in \mathcal{M}(t_j, \alpha_j)$, so that

$$\begin{aligned} T_{t_j}^w(K,\alpha_j)^{t_j} \le \|P_j\|_{K,w,t_j} \le \|P_j\|_{K,w,t_j} &\le (\|p\|_{K,w,s_{j_0}})^{q_j} \|r_j\|_{K,w,t_j - q_j s_{j_0}} \\ &\le (L_1 - \epsilon)^{q_j s_{j_0}} \|r_j\|_K \|w\|_K^{t_j - q_j s_{j_0}}. \end{aligned}$$

Since $\{\ell_j\}$ is finite, we need only select the r_j's from a fixed finite collection when constructing the P_j's; hence we may assume the quantity $\|r_j\|_K$ is uniformly bounded in j. Taking t_j-th roots in the above inequality, and the limit as $j \to \infty$, yields

$$L_2 = \lim_{j\to\infty} T_{t_j}^w(K,\alpha_j)^{t_j} \le L_1 - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we are done. $\qquad\square$

Definition 4.2. For each $\theta \in \Delta^\circ$ we call $T^w(K,\theta)$ the *directional Chebyshev constant of K* (with weight w) associated to θ. The function $\theta \mapsto \log T^w(K,\theta)$ will be called the *Chebyshev transform of (K,w)*.

Remark 5. The Chebyshev transform takes a set K and weight w, and outputs a real-valued function on Δ°. Translated to the complex geometric setting, this corresponds to taking a Hermitian metric on the line bundle and transforming it to a function on the (interior of the) Okounkov body. Apart from this change in terminology, Witt Nyström's definition in [19] is the same as above.

Lemma 4.3.   *1. The Chebyshev transform $\theta \mapsto \log T^w(K,\theta)$ is convex on Δ°. Hence it is either continuous or identically $-\infty$ on Δ°.*

   2. If the Chebyshev transform is continuous, then for any compact $Q \subset \Delta^\circ$, the quantity

$$\sup\{|\log T_k^w(K,\alpha(j)) - \log T^w(K,\theta_j)| : \alpha(j)/k =: \theta_j \in Q\}$$

   goes to zero as $k \to \infty$.

Proof. To prove the first statement of part (1), it is sufficient to show that for each $\theta, \phi \in \Delta^\circ$ and $t \in [0,1]$,

$$T^w(K, t\theta + (1-t)\phi) \le T^w(K,\theta)^t T^w(K,\phi)^{1-t}. \tag{23}$$

Fix $\theta, \phi \in \Delta^\circ$ and $t \in [0,1]$. Choose sequences $\{\alpha_j\}, \{\beta_j\}$ in $\mathbb{N}^m$, and $\{s_j\}, \{t_j\}$ in $\mathbb{N}$, such that $|\alpha_j|, |\beta_j|, s_j, t_j \to \infty$ as $j \to \infty$ and

$$\frac{\alpha_j}{|\alpha_j|} \to \theta, \ \frac{\beta_j}{|\beta_j|} \to \phi, \ \frac{s_j}{s_j + t_j} \to t.$$

If we put $\gamma_j := t_j \alpha_j |\beta_j| + s_j \beta_j |\alpha_j| =: \tilde{\alpha}_j + \tilde{\beta}_j$, then by Lemma 4.1,

$$T^w(K,\gamma_j) = T^w(K, \tilde{\alpha}_j + \tilde{\beta}_j) \le T^w(K, \tilde{\alpha}_j)^{|\tilde{\alpha}_j|/|\gamma_j|} T^w(K, \tilde{\beta}_j)^{|\tilde{\beta}_j|/|\gamma_j|}. \tag{24}$$

We have $\frac{\tilde{\alpha}_j}{|\tilde{\alpha}_j|} = \frac{\alpha_j}{|\alpha_j|}, \frac{\tilde{\beta}_j}{|\tilde{\beta}_j|} = \frac{\beta_j}{|\beta_j|}$, and further calculation gives

$$\frac{\gamma_j}{|\gamma_j|} \to t\theta + (1-t)\phi, \quad \frac{|\tilde{\alpha}_j|}{|\gamma_j|} \to t, \quad \text{and} \ \frac{|\tilde{\beta}_j|}{|\gamma_j|} \to 1 - t.$$

Taking the limit as $j \to \infty$ in (24) yields (23), proving the first statement of (1). The second statement then follows immediately.

We leave the proof of part (2) as an exercise in the triangle inequality. (Use Lemma 4.2 and the fact that the Chebyshev transform is uniformly continuous on Q.) $\qquad\square$

---

[5]Apply the proposition with Q being a small closed m-dimensional cube containing θ.

Notation 4.4. For a positive integer k, let

M_k denote the dimension of $\mathbb{C}[V]_{\leq k}$ (or number of points in $\mathcal{N}_k$)

h_k denote the number of points in $\mathcal{N}_k \setminus \mathcal{N}_{k-1}$ (so $h_k = M_k - M_{k-1}$).

For each $k \in \mathbb{N}$, let $\mathcal{E}_k = \{\mathbf{e}_j\}_{j=1}^{M_k}$ be a basis of polynomials for $\mathbb{C}[V]_{\leq k}$, with distinct leading terms with respect to their local power series expansions, i.e.,

$$v(\mathbf{e}_j) \neq v(\mathbf{e}_k) \text{ whenever } j \neq k. \tag{25}$$

Recall that the bases used in the algorithm to construct the Okounkov body have this property. We will also assume that $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots$, and put $\mathcal{E} := \bigcup_k \mathcal{E}_k$, which is a basis for $\mathbb{C}[V]$ with the above property.

Definition 4.3. A *monic basis of* $\mathbb{C}[V]$ is a basis $\mathcal{E}$ that satisfies (25) as well as the normalization condition $\text{TC}(\mathbf{e}_j) = 1$ for each $\mathbf{e}_j \in \mathcal{E}$.

We now define transfinite diameter.

Definition 4.4. Let k and s be positive integers with $s \leq M_k$. Given a finite collection of points $\{\zeta_1, \ldots, \zeta_s\}$, define the *Vandermonde matrix*

$$VM_{\mathcal{E},k}^w(\zeta_1, \ldots, \zeta_s) := \begin{bmatrix} \mathbf{e}_1(\zeta_1)_{w,k} & \cdots & \mathbf{e}_1(\zeta_m)_{w,k} \\ \vdots & \ddots & \vdots \\ \mathbf{e}_s(\zeta_1)_{w,k} & \cdots & \mathbf{e}_s(\zeta_s)_{w,k} \end{bmatrix} = [\mathbf{e}_j(\zeta_l)_w]_{j,l=1}^s,$$

as well as the *Vandermonde determinant*

$$VDM_{\mathcal{E},k}^w(\zeta_1, \ldots, \zeta_s) := \det\bigl(VM_{\mathcal{E},k}^w(\zeta_1, \ldots, \zeta_s)\bigr).$$

For a set $K \subset V$, define

$$V_{\mathcal{E},k}^w(K,s) := \sup\{|VDM_{\mathcal{E},k}^w(\zeta_1, \ldots, \zeta_m)| : \{\zeta_1, \ldots, \zeta_s\} \subset K\}. \tag{26}$$

The quantity

$$d_{\mathcal{E},k}^w(K) := \left(V_{\mathcal{E},k}^w(K, M_k)\right)^{1/(kM_k)} \text{ is the } k\text{-th order diameter of } K,$$

and

$$d^w(K) := \limsup_{k \to \infty} d_{\mathcal{E},k}^w(K) \text{ is the } transfinite \text{ } diameter \text{ of } K.$$

Remark 6. Since we take the absolute value in (26), rearranging the order of the rows in the Vandermonde matrix does not make any difference to the definition of k-th order diameters. We can therefore assume, at each stage k, that the rows of $VM_{\mathcal{E},k}^w$ have been ordered (and relabelled) so that $v(\mathbf{e}_j) \prec_l v(\mathbf{e}_k)$ whenever $j < k$.

Theorem 4.5. *Suppose the Chebyshev transform is an integrable function. Then the transfinite diameter computed with respect to any monic basis $\mathcal{E}$ is given by $d^w(K) = \lim\limits_{k \to \infty} d_{\mathcal{E},k}^w(K)$, i.e., the limit exists, and*

$$d^w(K) = \exp\left(\frac{1}{vol(\Delta)} \int_{\Delta^\circ} \log T^w(K, \theta)d\theta\right), \tag{27}$$

where $d\theta$ denotes the usual volume in $\mathbb{R}^m$.

In what follows, $\mathcal{E} = \{\mathbf{e}_j\}_{j=1}^\infty$ will be assumed to be a monic basis. It will also be convenient to introduce some more notation.

Notation 4.6. 1. The exponent of $\text{TT}(\mathbf{e}_j)$ will be denoted by $\alpha(j)$.

2. Given a polynomial p, write $o(p)$ to denote an (arbitrary) polynomial q for which $v(p) \prec v(q)$.

By definition, $\{\alpha(j) : j = 1, \ldots, M_k\} = \mathcal{N}_k$.

Lemma 4.7. *Fix $k \in \mathbb{N}$. The inequality*

$$V_{\mathcal{E},k}^w(K, M_k) \leq M_k! \prod_{j=1}^{M_k} T_k^w(K, \alpha(j))^k$$

holds.

Proof. For each $j \in \{1, \ldots, M_k\}$, let

$$\mathbf{t}_{j,k}(z) = \mathbf{e}_j(z) + o(\mathbf{e}_j)$$

denote the Chebyshev polynomial for $(k, \alpha(j))$, i.e., $\|\mathbf{t}_{j,k}\|_K = T_k^w(K, \alpha(j))^k$. Now choose a set of M_k points such that $V_{\mathcal{E},k}^w(K, M_k) = |VDM_{\mathcal{E},k}^w(\zeta_1, \ldots, \zeta_{M_k})|$; then

$$|VDM_{\mathcal{E},k}^w(\zeta_1, \ldots, \zeta_{M_k})| = \left|\det \begin{bmatrix} \mathbf{t}_{1,k}(\zeta_1)_{w,k} & \cdots & \mathbf{t}_{1,k}(\zeta_{M_k})_{w,k} \\ \vdots & \ddots & \vdots \\ \mathbf{t}_{M_k-1,k}(\zeta_1)_{w,k} & \cdots & \mathbf{t}_{M_k-1,k}(\zeta_{M_k})_{w,k} \\ \mathbf{t}_{M_k,k}(\zeta_1)_{w,k} & \cdots & \mathbf{t}_{M_k,k}(\zeta_{M_k})_{w,k} \end{bmatrix}\right|$$

by the invariance of the determinant under row operations. Expanding the determinant and using the triangle inequality, we obtain

$$
\begin{aligned}
V_{\mathcal{E},k}^{w}(K, M_k) &\leq \sum_{s} |\mathbf{t}_{1,k}(\zeta_{s(1)})| \cdots |\mathbf{t}_{M_k,k}(\zeta_{s(M_k)})| \\
&\leq \sum_{s} T_k^{w}(K, \alpha(1))^k \cdots T_k^{w}(K, \alpha(M_k))^k
\end{aligned}
$$

where the sum is over all permutations s of $\{1, \dots, M_k\}$, and there are $M_k!$ of these in all. $\square$

Lemma 4.8. *Fix $k \in \mathbb{N}$. Then*

$$
V_{\mathcal{E},k}^{w}(K, M_k) \geq \prod_{j=1}^{M_k} T_k^{w}(K, \alpha(j))^k.
$$

Proof. For convenience we will drop the subscripts in polynomial evaluation in what follows, writing e.g. $\mathbf{e}_j(\zeta) = \mathbf{e}_j(\zeta)_{w,k}$.

Let us also introduce the following notation: for $j = 0, \dots, M_k - 1$, set

$$
W_j(\zeta_1, \dots, \zeta_{M_k-j}) := \det \begin{bmatrix} \mathbf{e}_{j+1}(\zeta_1) & \cdots & \mathbf{e}_{j+1}(\zeta_{M_k-j}) \\ \vdots & \ddots & \vdots \\ \mathbf{e}_{M_k}(\zeta_1) & \cdots & \mathbf{e}_{M_k}(\zeta_{M_k-j}) \end{bmatrix}.
$$

In particular $W_0(\zeta_1, \dots, \zeta_{M_k}) = VDM_{\mathcal{E},k}^{w}(\zeta_1, \dots, \zeta_{M_k})$ and $W_{M_k-1}(\zeta_1) = \mathbf{e}_{M_k}(\zeta_1)$.

We start by observing that for any collection of points $\{\zeta_1, \dots, \zeta_{M_k}\} \subseteq K$,

$$
V_{\mathcal{E},k}^{w}(K, M_k) \geq W_0(\zeta_1, \dots, \zeta_{M_k}).
$$

Using this, we can derive an inequality involving the W_1-determinant. To this end, fix $\{\zeta_1, \dots, \zeta_{M_k-1}\} \subseteq K$. Then $W_0(\zeta_1, \dots, \zeta_{M_k-1}, z)$ (where $z \in K$ is arbitrary) is equal to

$$
\det \begin{bmatrix} \mathbf{p}_1(\zeta_1) & \cdots & \mathbf{p}_1(\zeta_{M_k-1}) & \mathbf{p}_1(z) \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{e}_{M_k}(\zeta_1) & \cdots & \mathbf{e}_{M_k}(\zeta_{M_k-1}) & \mathbf{e}_{M_k}(z) \end{bmatrix},
$$

where $\mathbf{p}_1 = \mathbf{e}_1 + o(\mathbf{e}_1)$ and the $o(\mathbf{e}_1)$ expression is a linear combination of the form $\sum_{j=2}^{M_k} \alpha_j \mathbf{e}_j$ obtained via row operations on $VM_{\mathcal{E},k}^{w}(\zeta_1, \dots, \zeta_{M_k-1}, z)$, the Vandermonde matrix. If the matrix is nonsingular there exist row operations that give a polynomial satisfying $\mathbf{p}(\zeta_1) = \cdots = \mathbf{p}(\zeta_{M_k-1}) = 0$. Then

$$
\begin{aligned}
V_{\mathcal{E},k}^{w}(K, M_k) &\geq \left| \det \begin{bmatrix} 0 & \cdots & 0 & \mathbf{p}_1(z) \\ \mathbf{e}_2(\zeta_1) & \cdots & \mathbf{e}_2(\zeta_{M_k-1}) & \mathbf{e}_2(z) \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{e}_{M_k}(\zeta_1) & \cdots & \mathbf{e}_{M_k}(\zeta_{M_k-1}) & \mathbf{e}_{M_k}(z) \end{bmatrix} \right| \\
&= |\mathbf{p}_1(z)| \cdot |W_1(\zeta_1, \dots, \zeta_{M_k-1})|.
\end{aligned}
$$

Choosing $\eta \in K$ such that $|\mathbf{p}_1(\eta)| = \|\mathbf{p}_1\|_K \geq T_k^{w}(K, \alpha(1))^k$, we have

$$
V_{\mathcal{E},k}^{w}(K, M_k) \geq |W_0(\zeta_1, \dots, \zeta_{M_k-1}, \eta)| = T_k^{w}(K, \alpha(1))^k |W_1(\zeta_1, \dots, \zeta_{M_k-1})|. \tag{28}
$$

Now consider fixing points $\zeta_1, \dots, \zeta_{M_k-2}$ and carrying out a similar argument as above: construct a polynomial $\mathbf{p}_2 = \mathbf{e}_2 + o(\mathbf{e}_2)$ with $\mathbf{p}_2(\zeta_1) = \cdots = \mathbf{p}_2(\zeta_{M_k-2}) = 0$ using row operations on the matrix for W_1, then choose $\eta \in K$ for which $|\mathbf{p}_2(\eta)| = \|\mathbf{p}_2\|_K$. This gives the inequality

$$
|W_1(\zeta_1, \dots, \zeta_{M_k-2}, \eta)| \geq T_k^{w}(K, \alpha(2))^k |W_2(\zeta_1, \dots, \zeta_{M_k-2})|. \tag{29}
$$

We use (28) to estimate the left-hand side of (29), observing that the upper bound on the left-hand side of (28) is valid for an arbitrary collection of M_{k-1} points of K. Hence

$$
V_{\mathcal{E},k}^{w}(K, M_k) \geq T_k^{w}(\alpha(1))^k T_k^{w}(K, \alpha(2))^k W_2(\zeta_1, \dots, \zeta_{M_k-2}).
$$

Now it is easy to see that the argument can be iterated to obtain the estimate

$$
V_{\mathcal{E},k}^{w}(K, M_k) \geq \left(\prod_{j=1}^{s} T_k^{w}(K, \alpha(j))^k \right) W_s(\zeta_1, \dots, \zeta_{M_k-s})
$$

for successive values $s = 3, 4, \dots$. The lemma is proved when $s = M_k - 1$. $\square$

Theorem 4.5 is proved by putting together the above lemmas.

Proof of Theorem 4.5. We have by Lemmas 4.7 and 4.8,

$$\prod_{j=1}^{M_k} T_k^w(K, \alpha(j))^k \le V_{\mathcal{E},k}^w(K, M_k) \le M_k! \prod_{j=1}^{M_k} T_k^w(K, \alpha(j))^k.$$

Let us take kM_k-th roots in this inequality and let $k \to \infty$. Since $(M_k!)^{1/(kM_k)} \to 1$ as $k \to \infty$,

$$d^w(K) = \lim_{k \to \infty} V_{\mathcal{E},k}^w(K, M_k)^{1/(kM_k)} = \lim_{k \to \infty} \left(\prod_{j=1}^{M_k} T_k^w(K, \alpha(j))^k \right)^{1/(kM_k)}.$$

We need to show that the limit on the right-hand side of the above converges to the right-hand side of (27). To see this, observe that the limit may be rewritten as

$$\lim_{k \to \infty} \exp\left(\frac{1}{M_k} \sum_{j=1}^{M_k} \log T_k^w(K, \alpha(j)) \right). \tag{30}$$

We look at the limit of the expression inside the parentheses. If we fix a compact convex body $Q \subset \Delta^\circ$, then by Lemma 3.10(2), we have

$$\lim_{k \to \infty} \frac{1}{M_k} \sum_{\alpha(j) \in kQ}^{M_k} \log T_k^w(K, \alpha(j)) = \lim_{k \to \infty} \frac{1}{M_k} \sum_{\alpha(j) \in kQ} \log T^w(K, \theta_j),$$

where $\theta_j = \alpha(j)/k$. Now (2.12) in Proposition 2.15 implies that the discrete measure $\frac{1}{M_k} \sum_{j=1}^{M_k} \delta_{\theta_j}$ converges weak-* on Δ° to the uniform measure $\frac{1}{\text{vol}(\Delta)} d\theta$ as $k \to \infty$.[6] Hence

$$\lim_{k \to \infty} \frac{1}{M_k} \sum_{\alpha(j) \in kQ} \log T^w(K, \theta_j) = \frac{1}{\text{vol}(\Delta)} \int_Q \log T^w(K, \theta) d\theta.$$

Since Q was arbitrary, one can consider a sequence of compact convex sets increasing to Δ° and obtain

$$\lim_{k \to \infty} \frac{1}{M_k} \sum_{j=1}^{M_k} \log T^w(K, \theta_j) = \frac{1}{\text{vol}(\Delta)} \int_{\Delta^\circ} \log T^w(K, \theta) d\theta,$$

using a standard convergence argument. □

Remark 7. If $T^w(K, \theta)$ is nonzero and uniformly bounded from above then the Chebyshev transform is integrable, since the Okounkov body Δ is compact. It is straightforward to get an upper bound when K is compact, or when w is admissible.

Witt Nyström's formula in [19] is actually stated in terms of a ratio. The analogous result in our setting is to take another set L and admissible weight u, and compute the associated quantities. It is easy to see that

$$\frac{d^w(K)}{d^u(L)} = \exp\left(\frac{1}{\text{vol}(\Delta)} \int_{\Delta^\circ} (\log T^w(K, \theta) - \log T^u(L, \theta)) d\theta \right). \tag{31}$$

In place of the ratio of transfinite diameters on the left-hand side, the formula in [19] has a *mixed Monge-Ampère energy*, which for the above case is

$$\mathcal{E}(P_K(\varphi), P_L(\psi)) := \int_V (P_K(\varphi) - P_L(\psi)) \sum_{j=0}^m (dd^c P_K(\varphi))^j \wedge (dd^c P_L(\psi))^{m-j}.$$

Here, $P_K(\varphi), P_L(\psi)$ are plurisubharmonic extremal functions defined in terms of the sets K, L and functions $\varphi = -\log w, \psi = -\log u$.[7]

The Monge-Ampère energy appears in Witt Nyström's formula because his proof is based on pluripotential theory and the L^2 theory associated to Bernstein-Markov measures. It is equivalent to (31) because of the (highly nontrivial) equality

$$\frac{d^w(K)}{d^u(L)} = \mathcal{E}(P_K(w), P_L(u)),$$

which is a version of *Rumely's formula* proved in [1] (see also [16] for a self-contained exposition). The original statement and proof of Rumely's formula is in [18].

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[6](2.12) holds for compact convex bodies, and these generate all Borel sets. The claim then follows by standard measure theory.

[7]From the point of view of pluripotential theory, it is more natural to consider the logs of the weights.

5 Comparison with classical results

The arguments in the previous section are almost the same as those of Zaharjuta in [20]. He derived an integral formula for the classical transfinite diameter of a compact set $K \subset \mathbb{C}^n$, denoted $d(K)$. Let $\mathcal{E} = \{z^{\alpha(j)}\}_{j=1}^{\infty}$ be the enumeration of the monomials in $z_1, \ldots, z_n$ according to the *grevlex order* on multi-indices $\alpha(j) = (\alpha(j)_1, \ldots, \alpha(j)_n)$, which is defined as follows: if $j < k$ then

- either $|\alpha(j)| < |\alpha(k)|$; or
- $|\alpha(j)| = |\alpha(k)|$, and there exists $\nu \in \{1, \ldots, n\}$ such that

$$\alpha(j)_1 = \alpha(k)_1, \ldots, \alpha(j)_{\nu-1} = \alpha(k)_{\nu-1}, \quad \text{and } \alpha(j)_\nu < \alpha(k)_\nu.$$

Let

$$\Sigma = \{\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n : \theta_k \geq 0 \ \forall k, \ \sum_k \theta_k = 1\} \tag{32}$$

be the standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$, and $\Sigma^\circ := \{\theta \in \Sigma : \theta_k > 0 \ \forall k\}$ its interior. Zaharjuta showed that Chebyshev constants parametrized by $\theta \in \Sigma^\circ$ can be defined as follows: let

$$\tau_j(K) := \inf\left\{\|p\|_K^{1/|\alpha(j)|} : p(z) = z^{\alpha(j)} + \sum_{\nu=1}^{j-1} c_\nu z^{\alpha(\nu)}\right\}, \text{ and } \tau(K, \theta) := \lim_{\substack{j \to \infty \\ \frac{\alpha(j)}{|\alpha(j)|} \to \theta}} \tau_j(K). \tag{33}$$

Remark 8. We will also use the notation $\tau_\alpha(K) = \tau_{\alpha(j)}(K) = \tau_j(K)$ when $\alpha = \alpha(j)$ for some positive integer j.

Next consider $V_{\mathcal{E},k}(K, M_k)$ as in Definition 4.4 (we suppress the dependence on the trivial weight $w \equiv 1$), and let

$$L_k := \sum_{s=1}^{k} sh_s. \tag{34}$$

Zaharjuta showed that the limit

$$d(K) := \lim_{k \to \infty} \left(V_{\mathcal{E},k}(K, M_k)\right)^{1/L_k},$$

the *transfinite diameter of K*, exists and satisfies the following formula.

Theorem 5.1. *We have* $d(K) = \exp\left(\dfrac{1}{vol(\Sigma)} \displaystyle\int_{\Sigma^\circ} \log \tau(K, \theta) d\theta\right)$.

Later, Jedrzejowski [11] showed a similar formula for the *homogeneous* transfinite diameter in $\mathbb{C}^n$; let us denote it by $\widehat{d}(K)$. The homogeneous transfinite diameter for a compact set K is defined by

$$\widehat{d}(K) := \lim_{k \to \infty} \left(\sup\{|VDMH(\zeta_1, \ldots, \zeta_{h_k})|^{1/(kh_k)} : \{\zeta_1, \ldots, \zeta_{h_k}\} \subset K\}\right),$$

where, with $\mathbf{e}_j := z^{\alpha(j)}$,

$$VDMH(\zeta_1, \ldots, \zeta_{h_k}) = \det \begin{bmatrix} \mathbf{e}_{M_{k-1}+1}(\zeta_1) & \cdots & \mathbf{e}_{M_{k-1}+1}(\zeta_{h_k}) \\ \vdots & \ddots & \vdots \\ \mathbf{e}_{M_k}(\zeta_1) & \cdots & \mathbf{e}_{M_k}(\zeta_{h_k}) \end{bmatrix}.$$

One can construct homogeneous Chebyshev constants $\widehat{\tau}(K, \theta)$ as limits of constants $\widehat{\tau}_j(K)$, where the latter are defined as in (33), but with the inf restricted to homogeneous polynomials (i.e. $c_\nu = 0$ if $|\alpha(\nu)| < |\alpha(j)|$). The homogeneous formula is the following.

Theorem 5.2. *We have* $\widehat{d}(K) = \exp\left(\dfrac{1}{vol(\Sigma)} \displaystyle\int_{\Sigma^\circ} \log \widehat{\tau}(K, \theta) d\theta\right)$.

Remark 9. Bloom and Levenberg have studied weighted versions of directional Chebyshev constants [4]. For a fixed admissible weight w, $\tau_j^w(K)$ is defined as in (33) by a sup over quantities of the form $\|w^{|\alpha(j)|}p\|_K$. A weighted generalization of Theorem 5.1 was proved later in [5].

In what follows, we expand a bit on the relationship between the above theorems and Theorem 4.5.

5.1 Homogeneous transfinite diameter

Consider $\mathbb{C}^n$ with coordinates given by $(z_0, z_1, \ldots, z_{n-1})$, and consider a compact subset of the form $\{1\} \times K$ (i.e. $z_0 = 1$), where K is compact in $\mathbb{C}^{n-1}$. We will also assume for what follows that K avoids the hyperplane $z_{n-1} = 0$. We describe the relation between Theorems 4.5 and 5.2.

Consider the monomials in $\mathbb{C}[z_0, \ldots, z_{n-1}]$ with the ordering $\prec$ defined as the lexicographic order for which $z_{n-1} \prec \cdots \prec z_1 \prec z_0$. The homogeneous polynomials $\{\mathbf{e}_j\}_{j=M_{k-1}+1}^{M_k}$ of degree k are given by

$$\mathbf{e}_{M_{k-1}+1} = z_{n-1}^k, \ \mathbf{e}_{M_{k-1}+2} = z_{n-2}z_{n-1}^{k-1}, \ \ldots, \ \mathbf{e}_{M_k} = z_0^k.$$

On the variety $V := \{z_0 = 1\}$, polynomials are given by $\mathbb{C}[z_1, \ldots, z_{n-1}]$ and for $z = (1, z_1, \ldots, z_{n-1}) \in \{1\} \times K \subset V$ we can identify $\mathbb{C}[z_0, \ldots, z_{n-1}]_k$ (the homogeneous polynomials of degree k in n variables) with $\mathbb{C}[V]_{\leq k} \simeq \mathbb{C}[z_1, \ldots, z_{n-1}]_{\leq k}$, via

$$z_0^{\alpha_0} z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} = z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \quad (\alpha_0 + \cdots + \alpha_{n-1} = k).$$

We also have the relation to $\mathbb{C}[v_0, v_1, \ldots, v_{n-2}]$ given by

$$|z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}}| = |w(v)^k v_0^{\alpha_0} v_1^{\alpha_1} \cdots v_{n-2}^{\alpha_{n-2}}|$$

where v-coordinates are given by

$$v_0 = 1/z_{n-1}, \quad v_j = z_j/z_{n-1} \text{ for } j = 1, \ldots, n-2, \tag{35}$$

and weight w defined by the formula $w(v) = 1/z_{n-1}$. Identifying $z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} \in \mathbb{C}[V]_{\leq k}$ with $v_0^{\alpha_{n-1}} v_1^{\alpha_2} \cdots v_{n-2}^{\alpha_{n-2}} \in \mathbb{C}[v_0, v_2, \ldots, v_{n-1}]_{\leq k}$, polynomial evaluation is related by $|p(z)| = |w(v)^k p(v)|$.

Now observe that

(i) h_k (computed with respect to $\mathbb{C}[z_0, z_1, \ldots, z_n]$) is the same as M_k for the spaces $\mathbb{C}[V]_{\leq k}$ and $\mathbb{C}[v_0, v_1, \ldots, v_{n-2}]_{\leq k}$.

(ii) Under the identifications described above, the lex order $\prec_l$ on $\mathbb{C}[z_0, \ldots, z_n]_k$ translates to lex order on the corresponding monomials in $\mathbb{C}[v_0, v_1, v_{n-2}]_{\leq k}$, and to the *reverse* of grevlex order (which we will denote by $\prec_g$) on the corresponding monomials in affine coordinates on $V = \{z_0 = 1\}$. An example of a pair of monomials when $n = 4$ and $k = 5$, together with the corresponding pairs in the other spaces, is

$$z_0^3 z_1 z_3 \succ_l z_0^2 z_1^2 z_4 \text{ in } \mathbb{C}[z_0, z_1, z_2, z_3, z_4] \quad \sim \quad v_0^3 v_1 v_3 \succ_l v_0^2 v_1^3 \text{ in } \mathbb{C}[v_0, v_1, v_2, v_3]$$
$$\sim \quad z_1 z_3 \prec_g z_1^2 z_4 \text{ in } \mathbb{C}[z_1, z_2, z_3, z_4].$$

These identifications yield, for any collection of points $\{\zeta_1, \ldots, \zeta_{h_k}\} \subset V$,

$$VDMH(\zeta_1, \ldots, \zeta_{h_k}) = VDM_{\mathcal{E}, k}^w(\zeta_1, \ldots, \zeta_{h_k})$$

where the right-hand side may be interpreted either in z-coordinates with $w \equiv 1$ (unweighted) and $\mathcal{E}$ being monomials in z, or in v-coordinates with monomials in v and the weight $w = w(v)$ described above. The equivalence of the above determinants, and the fact that the same root is taken at each stage (kh_k-th or kM_k-th, see (i) above), yields the equality $\widehat{d}(\{1\} \times K) = d^w(\{1\} \times K)$.

Chebyshev constants are also related. (For convenience of notation, let us simply write K for $\{1\} \times K$ in what follows.) Given $\theta \in \Sigma$, i.e., $\theta = (\theta_0, \ldots, \theta_{n-1})$ with $\theta_j > 0$ and $\sum \theta_j = 1$, we have

$$\widehat{\tau}(K, \theta) = T^w(K, \theta') \tag{36}$$

where we interpret the right-hand side in v coordinates, in which $w = w(v)$ and $\theta' = (\theta_0, \ldots, \theta_{n-2})$.[8] As θ varies over Σ°, θ' varies over the projection of these points to the first $n-1$ coordinates, which fills the interior of the region S in $\mathbb{R}^{n-1}$ given by $\theta_0 + \cdots + \theta_{n-2} \leq 1$, $\theta_j \geq 0$. Now $d\theta'$, the volume in $\mathbb{R}^{n-1}$, is the push forward of $d\theta$ (the $n-1$-dimensional volume in the plane containing Σ), scaled by a factor of $\text{vol}(\Sigma)/\text{vol}(S)$. Hence

$$\frac{1}{\text{vol}(\Sigma)} \int_{\Sigma^\circ} \widehat{\tau}(K, \theta) d\theta = \frac{1}{\text{vol}(S)} \int_{S^\circ} T^w(K, \theta') d\theta'.$$

Since S is the Okounkov body of $\mathbb{C}^{n-1}$, Theorem 5.2 is just Theorem 4.5 under a change of variable.

5.2 Transfinite diameter

Theorem 5.1 is slightly different, but closely related. We describe its precise relationship to Theorem 4.5 in what follows. We will work with a compact set $K \subset \mathbb{C}^{n-1}$ (with variables $z_1, \ldots, z_{n-1}$) and reuse the material above, relating K to $\{1\} \times K \subset \mathbb{C}^n$. In particular, z_0 denotes the additional variable, and v-coordinates are defined as in (35). It is easy to see that $d(K)$ is related to $\widehat{d}(\{1\} \times K)$: we have

$$VDMH(\zeta_1, \ldots, \zeta_{M_k}) = VDM_{\mathcal{E}, k}(\zeta_1', \ldots, \zeta_{M_k}')$$

where M_k is counted with respect to $\mathbb{C}[z_1, \ldots, z_{n-1}]$ (and therefore coincides with 'h_k for $\mathbb{C}[z_0, \ldots, z_{n-1}]$'), and we use the notation $\mathbb{C}^n \ni \zeta_j = (1, \zeta_j') \in \mathbb{C} \times \mathbb{C}^{n-1}$.

We saw previously that $\widehat{d}(\{1\} \times K) = d^w(K)$, but $d(K)$ in Theorem 5.1 takes a slightly different root at each step k, which affects the normalization. We omit the calculation of the relevant limit.

Proposition 5.3. *We have* $\lim_{k \to \infty} \dfrac{L_k}{kM_k} = \dfrac{n-1}{n}$, *where* L_k *is as in (34). Hence*

$$d^w(K) = \widehat{d}(\{1\} \times K) = d(K)^{(n-1)/n},$$

where the above quantities are from Theorems 4.5, 5.2 and 5.1 respectively. □

---

[8] Since the Okounkov body is constructed with reference to local coordinates, Theorem 4.5 only applies directly to the v-coordinate setting.

We now turn to the integral formulas. Suppose $\theta \in S^\circ$, and consider the constants $T^w(K, \theta)$ defined as on the right-hand side of (36). These constants (associated to v-coordinates) transform to constants $T(K, \tilde{\theta})$ associated to z-coordinates, with parameters related by

$$S^\circ \ni \theta = (\theta_0, \ldots, \theta_{n-2}) \iff (\theta_1, \ldots, \theta_{n-1}) = \tilde{\theta} \in \tilde{S}^\circ, \tag{37}$$

whenever $\theta_0 + \cdots + \theta_{n-1} = 1$.[9] The constants $T(K, \tilde{\theta})$ satisfy the following homogeneity property.

Lemma 5.4. *Let $\theta \in \tilde{S}^\circ$. Then for all $t \in [0, 1]$,*

$$\log T(K, t\theta) = t \log T(K, \theta).$$

Moreover, if $r\theta = \varphi \in \Sigma^\circ$ for some $r > 1$ then

$$r \log T(K, \theta) = \log \tau(K, \varphi)$$

where $\tau(K, \varphi)$ is the directional Chebyshev constant given by (33).

Proof. Fix a positive integer k, and a positive integer $j < M_k$. Define the Okounkov body with respect to v-coordinates as above, and let w be the corresponding weight. Let p be a Chebyshev polynomial such that $T_k^w(K, \alpha(j))^k = \|w^k p\|_K$. In z-coordinates, the weight becomes trivial and the sup translates to a sup over quantities of the form $\|p\|_K$, which we will denote by $T_k(K, \tilde{\alpha}(j))^k$. (The exponent $\tilde{\alpha}(j)$ is such that $z^{\tilde{\alpha}(j)}$ corresponds to $v^{\alpha(j)}$.) It is easy to see that p must be a Chebyshev polynomial of degree $s = |\tilde{\alpha}(j)|$ (i.e., one that attains the inf in (33) for $\tau_j(K) = \tau_{\tilde{\alpha}(j)}(K)$), and so

$$\tau_j(K)^s = T_k(K, \tilde{\alpha}(j))^k. \tag{38}$$

Suppose, for some rational number $\frac{a_j}{b_j} \in (0, 1)$, that the $(n-1)$-tuple $\frac{a_j}{b_j}\tilde{\alpha}(j)$ has integer entries (i.e. b_j divides all components). Then $\frac{a_j}{b_j}\tilde{\alpha}(j) = \tilde{\alpha}(j')$ for some $j' < j$. It is easy to see that $|\tilde{\alpha}(j)| = s$ implies $|\tilde{\alpha}(j')| = \frac{a_j}{b_j}s$. Similar to (38), one also has

$$\tau_{j'}(K)^{\frac{a_j}{b_j}s} = T_k(K, \frac{a_j}{b_j}\tilde{\alpha}(j))^k. \tag{39}$$

Equations (38) and (39) yield the lemma after some analysis. Precisely, take sequences of positive integers a_j, b_j such that $b_j \to \infty$ and $a_j/b_j \to t \in (0, 1)$. Then take a sequence of exponents α_j such that b_j divides each component of α_j, and a sequence of integers $k_j > |\alpha_j| =: s_j$ such that $\alpha_j/k_j \to \theta \in S^\circ$. Let $\beta_j := \frac{a_j}{b_j}\alpha_j$; then $\beta_j/k_j \to t\theta$ where $\beta_j = \frac{a_j}{b_j}\alpha_j$.

Define $\varphi := \lim_{j \to \infty} \alpha_j/|\alpha_j|$. Then $\varphi = r\theta \in \Sigma$, where $r = \lim_{j \to \infty} k_j/s_j$. For the same reason, $\varphi = \lim_{j \to \infty} \beta_j/|\beta_j|$ also holds, and as $j \to \infty$,

$$T_{k_j}(K, \beta(j))^{(b_j k_j)/(a_j s_j)} = \tau_{\beta_j}(K) \to \tau(K, \varphi), \text{ and} \tag{40}$$

$$T_{k_j}(K, \alpha(j))^{k_j/s_j} = \tau_{\alpha_j}(K) \to \tau(K, \varphi). \tag{41}$$

Now note that as $j \to \infty$ we also have $T_{k_j}^w(K, \beta(j))^{(b_j k_j)/(a_j s_j)} \to T^w(K, t\theta)^{r/t}$ and $T_{k_j}^w(K, \alpha(j))^{k_j/s_j} \to T(K, \theta)^r$ on the left-hand sides. By (40) and (41) we may equate all of the asymptotic quantities and take logs. Both statements of the lemma follow immediately. $\square$

Using the above lemma, we can directly relate the integrals of Theorems 4.5 and 5.1. First, we transform the integral in Theorem 4.5 to z-coordinates:

$$\frac{1}{\text{vol}(S)} \int_{S^\circ} \log T^w(K, \theta) \, d\theta = \frac{1}{\text{vol}(\tilde{S})} \int_{\tilde{S}^\circ} \log T(K, \tilde{\theta}) \, d\tilde{\theta}.$$

Now observe that $\tilde{S}$ can be expressed as the union $\tilde{S} = \bigcup_{t \in [0,1]} t\Sigma$ where $\Sigma \in \mathbb{R}^{n-1}$ is the $(n-2)$-dimensional simplex defined as in (32). Using the map

$$(0, 1) \times \Sigma^\circ \ni (t, \theta') \mapsto t\theta' = \tilde{\theta} \in \tilde{S}^\circ,$$

we see that the volume element $d\tilde{\theta}$ on $\tilde{S}^\circ$ may be decomposed as $t^{n-2} d\theta' dt$. Continuing, we have

$$\frac{1}{\text{vol}(\tilde{S})} \int_{\tilde{S}^\circ} \log T(K, \tilde{\theta}) \, d\tilde{\theta} = \frac{1}{\text{vol}(\tilde{S})} \int_0^1 \int_{\Sigma^\circ} \log T(K, t\theta') t^{n-2} d\theta' dt$$

$$= \frac{1}{\text{vol}(\tilde{S})} \int_0^1 \int_{\Sigma^\circ} \log \tau(K, \theta') t^{n-1} d\theta'$$

$$= \frac{1}{\text{vol}(\tilde{S})} \int_0^1 t^{n-1} dt \int_{\Sigma^\circ} \log \tau(K, \theta') d\theta'$$

$$= \frac{1}{n\text{vol}(\tilde{S})} \int_{\Sigma^\circ} \log \tau(K, \theta') d\theta', \tag{42}$$

---

[9] We will write $\tilde{S}$ for the standard triangle associated to z-coordinates (and write $\tilde{S}^\circ$ for its interior) but use the same labels K, p for sets and polynomials.

where we use the previous lemma. We compute $\text{vol}(\tilde{S})$ using the same decomposition:

$$\text{vol}(\tilde{S}) = \int_0^1 \text{vol}(t\Sigma)dt = \int_0^1 t^{n-2}\text{vol}(\Sigma)dt = \frac{1}{n-1}\text{vol}(\Sigma).$$

Finally, substitute the above expression for $\text{vol}(\tilde{S})$ into (42). Altogether, we have

$$\frac{1}{\text{vol}(S)} \int_{S^\circ} T^w(K,\theta)d\theta = \left(\frac{n-1}{n}\right)\frac{1}{\text{vol}(\Sigma)} \int_{\Sigma^\circ} \tau(K,\theta')d\theta'.$$

Observe that the normalization agrees with Proposition 5.3.

6 Further properties

In this section, we study further properties of Chebyshev constants and associated notions. Specific results will be given on the complexified sphere. More general results will be the subject of future research.

6.1 General collections of polynomials.

We want to reuse the methods of Theorem 4.5 in a more general context, so let us extract the essential ingredients required for the proof. The Vandermonde determinant used in the limiting process is defined in terms of a collection of polynomials $\mathcal{E} = \{\mathbf{e}_j\}_{j=1}^\infty$ with some additional structure related to a grading with respect to multiplication: $\mathcal{E} = \bigcup_{k=0}^\infty \mathcal{E}_k$ with $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots$, and $\mathcal{E}_k\mathcal{E}_l = \{pq : p \in \mathcal{E}_k, q \in \mathcal{E}_l\}$ is contained in the span of $\mathcal{E}_{k+l}$ for each pair of non-negative integers k, l. The structure associated to this grading allows us to compute the Okounkov body and associated Chebyshev constants, and consists of the following things.

1. There is a function $\nu : \text{span}(\mathcal{E}) \to \mathbb{N}_0^m$ which is one-to-one on $\mathcal{E}$, and an associated convex body given as follows. Let $\mathcal{N}_k = \{\nu(\mathbf{e}_j)\}_{j=1}^{M_k}$ where $\mathcal{E}_k = \{\mathbf{e}_1, \ldots, \mathbf{e}_{M_k}\}$. Then let $\Delta_{\mathcal{E},k}$ be the convex hull of $\frac{1}{k}\mathcal{N}_k$ and let $\Delta_{\mathcal{E}}$ be the convex hull of $\bigcup_k \Delta_{\mathcal{E},k}$.

2. For each $k \in \mathbb{N}$ and $\alpha \in \mathcal{N}_k$ there is a class of polynomials $\mathcal{M}(k,\alpha) \subset \text{span}(\mathcal{E}_k)$ such that $\mathbf{e}_j \in \mathcal{M}(k,\alpha)$ if $\nu(\mathbf{e}_j) = \alpha$. These classes satisfy the properties

$$\mathcal{M}(k,\alpha) + p \subseteq \mathcal{M}(k,\alpha) \quad \text{if } p \in \text{span}(\mathcal{E}_k) \text{ and } \nu(p) \prec \alpha,$$
$$\mathcal{M}(k_1,\alpha_1)\mathcal{M}(k_2,\alpha_2) \subseteq \mathcal{M}(k_1+k_2, \alpha_1+\alpha_2).$$

3. The discrete measure $\frac{1}{M_k}\sum_{\alpha \in \mathcal{N}_k} \delta_{\alpha/k}$ converges weak-* to $\frac{1}{\text{vol}(\Delta_{\mathcal{E}})}d\theta$, where $d\theta$ is the usual volume measure on $\mathbb{R}^m$ restricted to $\Delta_{\mathcal{E}}$.

In addition, there is a weighted polynomial evaluation with respect to some admissible weight $w : K \to \mathbb{C}$, with

$$p(z)_{w,k}q(z)_{w,l} = (pq)(z)_{w,k+l}.$$

With these properties, one can then construct, for $\alpha \in \mathcal{N}_k$,

$$T_k(K,\alpha) := \inf\{\|p\|_{K,w,k} : p \in \mathcal{M}(\alpha,k)\}^{1/k}$$

(we suppress the dependence on w here and in what follows), as well as the function on the interior of the convex body, $T : \Delta_{\mathcal{E}}^\circ \to [0,\infty)$ given by

$$T(K,\theta) := \lim_{\substack{k\to\infty \\ \alpha/k\to\theta}} T_k(K,\alpha).$$

Then defining

$$d_{\mathcal{E},k}(K) := \sup\{|VDM_{\mathcal{E},k}(\zeta_1,\ldots,\zeta_{M_k})|^{1/(kM_k)} : \{\zeta_1,\ldots,\zeta_{M_k}\} \subset K\}$$

as in Definition 4.4, we have

$$d_{\mathcal{E}}(K) = \exp\left(\frac{1}{\text{vol}(\Delta_{\mathcal{E}})} \int_{\Delta_{\mathcal{E}}^\circ} \log T(K,\theta)d\theta\right). \tag{43}$$

Remark 10. The main point is that such a formula arises for *any* collection $\mathcal{E}$ of polynomials with the type of structure given above; for example, the basis of a graded subalgebra of $\mathbb{C}[V]$. (In the complex geometric setting, Okounkov bodies associated to subalgebras of global sections have been studied e.g., in [10].)

6.2 Monomials on the sphere

Consider again the complexified sphere $V \subset \mathbb{C}^3$ given by (3.1)which is spanned by monomials of the form $z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}$ with $\alpha_1, \alpha_2 \in \mathbb{N}_0$ and $\alpha_3 \in \{0, 1\}$. Let $\mathcal{E}$ denotes the subcollection of monomials of the form $z_1^{\alpha_1} z_2^{\alpha_2}$ that span $\mathbb{C}[z_1, z_2] \subseteq \mathbb{C}[V]$; as usual, for $\alpha = (\alpha_1, \alpha_2)$, write $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2}$, and order the monomials by grevlex. We treat them as in the classical theory:

1. let M_k denote the number of monomials in $\mathcal{E}$ of degree $\leq k$;

2. let $\nu : \mathbb{C}[z_1, z_2] \to \mathbb{N}_0^2$ return the leading exponent of a polynomial (with respect to grevlex);

3. let $\mathcal{M}(\alpha, k) \subset \mathbb{C}[z_1, z_2]_{\leq k}$ denote the class of monic polynomials of the form

$$z^\alpha + (\text{lower terms wrt grevlex});$$

4. and let $VDM(\zeta_1, \ldots, \zeta_{M_k})$ denote the Vandermonde determinant associated to $\mathcal{E}$ with the standard polynomial evaluation at affine points.

Define Chebyshev constants and transfinite diameter:

$$\begin{aligned} T_{\mathcal{E},k}(K, \alpha) &:= \inf\{\|p\|_K : p \in \mathcal{M}(k, \alpha)\}^{1/k}, \text{ and} \\ d_{\mathcal{E},k}(K) &:= \sup\{|VDM(\zeta_1, \ldots, \zeta_{M_k})|^{1/(kM_k)} : \zeta_j \in K \; \forall j\}. \end{aligned}$$

The Okounkov body $\Delta_{\mathcal{E}}$ associated to $\mathcal{E}$ is the standard triangle,

$$\Delta_{\mathcal{E}} = S = \{\theta = (\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1, \theta_2 \geq 0, \theta_1 + \theta_2 \leq 1\}.$$

The limits

$$d_{\mathcal{E}}(K) := \lim_{k \to \infty} d_{\mathcal{E},k}(K) \quad \text{and} \quad T_{\mathcal{E}}(K, \theta) := \lim_{\substack{k \to \infty \\ \alpha/k \to \theta}} T_{\mathcal{E},k}(K, \alpha)$$

exist (for all $\theta \in S^\circ$ in the latter), and

$$d_{\mathcal{E}}(K) = \exp\left(\frac{1}{\text{vol}(S)} \int_{S^\circ} \log T_{\mathcal{E}}(K, \theta) \, d\theta\right). \tag{44}$$

Observe that $\mathcal{E}$ (and hence $\mathbb{C}[z_1, z_2]$) cannot distinguish between points with different z_3-coordinates: if $\pi : V \to \mathbb{C}^2$ is the projection $(z_1, z_2, z_3) \mapsto (z_1, z_2)$, then for all $\mathbf{e}_j \in \mathcal{E}$, we have $\mathbf{e}_j(\zeta) = \mathbf{e}_j(\eta)$ whenever $\pi(\zeta) = \pi(\eta)$. As an immediate consequence, for any collections $\{\zeta_1, \ldots, \zeta_{M_k}\}$ and $\{\eta_1, \ldots, \eta_{M_k}\}$ of points of V,

$$VDM(\zeta_1, \ldots, \zeta_{M_k}) = VDM(\eta_1, \ldots, \eta_{M_k}) \text{ whenever } \eta_j \in \pi^{-1}(\{\pi(\zeta_j)\}). \tag{45}$$

Proposition 6.1. *Let $K \subset V$ be compact. Then*

$$d_{\mathcal{E}}(K) = d_{\mathcal{E}}(\pi^{-1}(\pi(K))) = d(\pi(K))^2,$$

where $d(\cdot)$ denotes the classical transfinite diameter in $\mathbb{C}^2$.

Proof. The first equality follows immediately by applying a standard limiting argument to (45). For the second, we view $\mathcal{E}$ as the standard monomial basis for $\mathbb{C}[z_1, z_2]$, and hence $VDM(\cdot)$ is the same Vandermonde determinant that gives the classical transfinite diameter. The exponent of 2 comes from taking a (kM_k)-th (rather than an L_k-th) root in the limiting process (see Proposition 5.3 with $n = 2$). $\qquad \square$

Remark 11. When $K = V \cap \mathbb{R}^3$ is the real sphere, then $\pi(K) \subset \mathbb{R}^2$ is the real disk and the explicit value $d(\pi(K)) = 1/\sqrt{2e}$ has been calculated in [3]. Hence in this case $d_{\mathcal{E}}(K) = 1/2e$.

6.3 Relations between Chebyshev constants.

Consider the complexified sphere $\mathbf{V}(z_1^2 + z_2^2 + z_3^2 - 1)$ given projectively by $Z_1^2 + Z_2^2 + Z_3^2 = Z_0^2$. Consider local coordinates $v = (v_0, v_1)$ about the point

$$[Z_0 : Z_1 : Z_2 : Z_3] = [v_0 : v_1 : 1 : v_3] = [0 : 0 : 1 : i] \in \mathbb{P}^3,$$

and let Δ be the Okounkov body calculated in these coordinates. In these coordinates, $V = \mathbf{V}(v_1^2 + 1 + v_3^2 - v_0^2)$, and one can calculate that Δ is again the triangle given by Proposition 3.8. As a result of Theorem 4.5, the integral formula

$$d^w(K) = \exp\left(\frac{1}{\text{vol}(\Delta)} \int_{\Delta^\circ} \log T^w(K, \theta) d\theta\right) \tag{46}$$

holds, where w is a weight on a compact subset K of $\{Z_0, Z_2 \neq 0\}$. We will show how the integrands in (44) and (46) are related in a particular case.

First, we will transform (44) to v-coordinates. Similar to the previous section, we use $v_1 = z_1/z_2$, $v_0 = 1/z_2$, $v_3 = z_3/z_2$, and polynomial evaluation is related by $v_0^k p(v_0, v_1) = p(z_1, z_2)$.
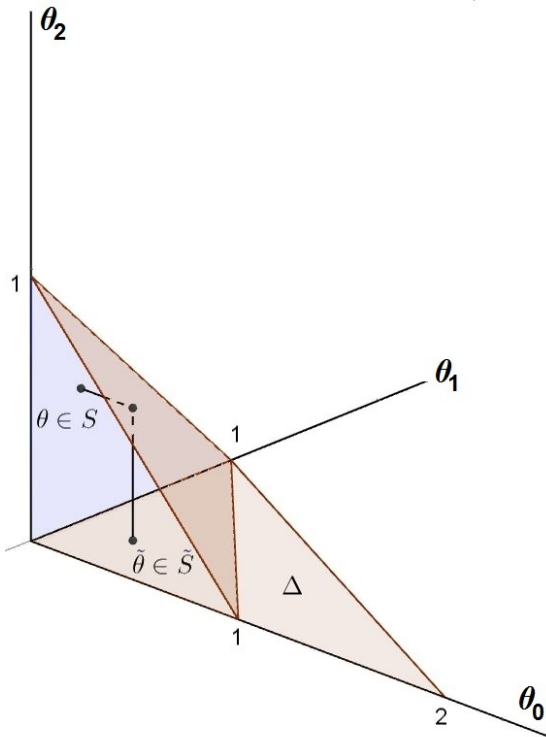
Figure 3: Relations between the convex bodies. S is constructed from monomials in affine coordinates, and $\tilde{S} \subset \Delta$ from monomials in v-coordinates. The parameters are related by projection: we have $\theta = (\theta_1, \theta_2)$ and $\tilde{\theta} = (\theta_0, \theta_1)$ where $(\theta_0, \theta_1, \theta_2)$ is a point in the simplex $\theta_0 + \theta_1 + \theta_2 = 1$, $\theta_j > 0 \;\forall j$.

Let us use tilded quantities ($\tilde{\mathcal{E}}$, $\tilde{\theta}$, etc.) to denote the v-coordinate versions of the quantities in (44), and use the same $\tilde{\theta}$ variable in equation (46); then $\tilde{S} \subset \Delta$. With the weight $w(v) = v_0$, (44) and (46) become

$$d_{\tilde{\mathcal{E}}}^w(K) \;=\; \exp\left(\frac{1}{\mathrm{vol}(\tilde{S})} \int_{\tilde{S}^\circ} \log T_{\tilde{\mathcal{E}}}^w(K, \tilde{\theta}) \, d\tilde{\theta} \right), \tag{47}$$

$$d^w(K) \;=\; \exp\left(\frac{1}{\mathrm{vol}(\Delta)} \int_{\Delta^\circ} \log T^w(K, \tilde{\theta}) \, d\tilde{\theta} \right). \tag{48}$$

Figure 3 shows the relation between the different convex bodies and parameters.

Definition 6.1. Let $A(r_1, r_2)$ be the annulus $\{\zeta \in \mathbb{C} : r_1 < |\zeta| < r_2\}$, with $r_1 < 1 < r_2$, and let U be an open subset of V. Suppose there exists a holomorphic map

$$A(r_1, r_2) \times U \ni (\zeta, v) \mapsto \zeta * v \in V$$

such that $\zeta * (\eta * v) = (\zeta \eta) * v$ whenever $v, \eta * v \in U$ and $\zeta, \eta, \zeta\eta \in A(r_1, r_2)$. The map $*$ is called a *local circle action on U*, and $K \subset U$ is *locally circled under* $*$ if it is invariant under the restriction of the action to the unit circle:

$$e^{i\theta} * v \in K \text{ if and only if } v \in K, \text{ for all } \theta \in \mathbb{R} \quad (\text{i.e. } e^{i\theta} * K = K).$$

A local circle action on a hypersurface $V \subset \mathbb{C}^3$ arises naturally as follows. Locally (say in an open set U), V is a graph over the v_0, v_1 variables; let us write $v_3 = v_3(v_0, v_1)$ for some holomorphic function v_3. Scalar multiplication $\zeta(v_0, v_1) = (\zeta v_0, \zeta v_1)$ lifts to a map

$$\zeta * (v_1, v_2, v_3) := (\zeta v_1, \zeta v_2, v_3(\zeta v_1, \zeta v_2)) \tag{49}$$

which satisfies the above definition, as long as v_3 extends to a well-defined holomorphic function on some neighbourhood of a set of the form $\{e^{i\theta} w : \theta \in \mathbb{R}\}$, for some $w \in \pi(U)$ (here π denotes projection onto the (v_0, v_1) variables). This holds, for example, if v_3 has a Laurent series expansion at w, $v_3(w_0, w_1) = \sum_{i,j=-\infty}^{\infty} c_{ij} w_0^i w_1^j$.

In particular, when V is the sphere, let $r < 1$ and consider

$$W_r = \{(v_0, v_1) \in \mathbb{C}^2 : |v_0|, |v_1| \le r\}.$$

In v-coordinates, we have $v_3(v_0, v_1) = (v_0^2 - v_1^2 - 1)^{1/2}$. Then $\zeta * v$ given by (49) is a local circle action defined on

$$\tilde{W}_r = \{(v_0, v_1, v_3(v_0, v_1)) : (v_0, v_1) \in W_r\}.$$

Let us see how a locally circled set can be generated in affine coordinates.

Lemma 6.2. *Let $K \subset \tilde{W}_r \subset V$ be a compact, locally circled set under the action $*$ on the sphere. Using $*$, one can generate K from a smaller set $K' \subset \mathbb{C} \times [0, \infty)$ as follows:*

$$K = \{e^{i\theta} * (z_1, z_2, z_3) \in \tilde{W}_r : z_1 = w, z_2 = r, (w, r) \in K'\}.$$

Proof. Fix $v \in K$, which we will write as $z \in K$ in affine coordinates. We compute the holomorphic function $z(\zeta) = (z_1(\zeta), z_2(\zeta), z_3(\zeta))$ given by the transformation $\zeta \mapsto \zeta * v$:

$$z_1(\zeta) = \frac{\zeta v_1}{\zeta v_0} = \frac{v_1}{v_0} = z_1, \quad z_2(\zeta) = \frac{1}{\zeta v_0} = \frac{1}{\zeta} z_2,$$

and this determines $z_3(\zeta)$ by lifting to $\tilde{W}_r$. Since $e^{i\theta} * v \in K$ for all θ, we can choose θ_v such that $z_2(e^{i\theta_v}) = e^{-i\theta_v} z_2 \in [0, \infty)$.

We now vary v and put $I = \{r = z_2(e^{i\theta_v}) : v \in K\} \subset [0, \infty)$. Then

$$K' := \{(w, r) \in \mathbb{C} \times I : (w, r, z_3) \in K \text{ for some } z_3 \in \mathbb{C}\}$$

is the desired set. $\qquad \square$

This is a rotation about the origin in the z_2-plane, lifted to the variety.

We return back to v-coordinates to relate Chebyshev constants. The notation in the following proposition is as in (47) and (48) above, with the weight given by $w(v_0, v_1) = v_0$.

Proposition 6.3. *Let $r \in (0, 1)$, and suppose $K \subset \tilde{W}_r \subset V$ is locally circled under $*$. Then*

$$T_{\tilde{\mathcal{E}}}^w(K, \tilde{\theta}) = T^w(K, \tilde{\theta})$$

for all $\tilde{\theta} \in \tilde{S} \subset \Delta$.

Proof. By definition, it is sufficient to show that for any $\alpha \in \mathbb{N}$ with $|\alpha| \leq k$,

$$\inf\{\|p\|_{K,w} : p \in \mathbb{C}[v_0, v_1], \mathrm{TT}(p) = v^\alpha\} = \inf\{\|p\|_{K,w} : p \in \mathbb{C}[V], \mathrm{TT}(p) = v^\alpha\},$$

or more compactly, $T_{\mathcal{E},k}^w(K, \alpha) = T_k^w(K, \alpha)$. Note that any $p \in \mathbb{C}[v_0, v_1]_{\leq k}$ is also a polynomial in $\mathbb{C}[V]_{\leq k}$ with no term involving v_3, so the inf on the right-hand side is over a larger collection. Hence $T_{\mathcal{E},k}^w(K, \alpha) \geq T_k^w(K, \alpha)$.

To prove the reverse inequality, fix $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \leq k$ (i.e., $\alpha/k \in \tilde{S}$). Let $p \in \mathbb{C}[V]_{\leq k}$ with $\mathrm{TT}(p) = v^\alpha$, and for the moment, let $v = (v_0, v_1)$ be fixed in K. Define $\psi(\zeta) := p(\zeta * v)_{w,k}$; then ψ is holomorphic on a neighborhood of the unit circle, as can be seen by writing it out in a series expansion:

$$\begin{aligned} \psi(\zeta) = p(\zeta * v)_{w,k} &= \zeta^k v_0^k \sum_{j=0}^{\infty} \sum_{|\beta|=j} c_\beta (\zeta v_0)^{\beta_0} (\zeta v_1)^{\beta_1} \\ &= \sum_{j=0}^{\infty} \left(\sum_{|\beta|=j} c_\beta v_0^{\beta_0+k} v_1^{\beta_1} \right) \zeta^{j+k}. \end{aligned}$$

Since K is locally circled, $\zeta * v \in K$ for all $|\zeta| = 1$, so that $|p(\zeta * v)| \leq \|p\|_K$. Plugging this into the Cauchy integral formula for the coefficient of ζ^{j+k}, we have

$$\left| \sum_{|\beta|=j} c_\beta v_0^{\beta_0+k} v_1^{\beta_1} \right| = \left| \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{p(\zeta * v)}{\zeta^{j+k+1}} d\zeta \right| \leq \|p\|_K, \quad \forall j \in \mathbb{N}. \tag{50}$$

In particular, this is true when $j = |\alpha| \leq k$. Since v was an arbitrary point of K, let us now treat it as a variable and define the polynomial

$$\tilde{p}(v) := \sum_{|\beta|=|\alpha|} c_\beta v_0^{\beta_0} v_1^{\beta_1} \in \mathbb{C}[v_1, v_2]_{\leq k}.$$

Clearly, by construction $\mathrm{TT}(\tilde{p}) = v^\alpha$, and by (50), $\|\tilde{p}\|_{K,w,k} \leq \|p\|_{K,w,k}$. Hence

$$T_{\mathcal{E},k}^w(K, \alpha) \leq \|\tilde{p}\|_{K,w,k} \leq \|p\|_{K,w,k}$$

and since $p \in \mathbb{C}[V]_{\leq k}$ was an arbitrary polynomial with $\mathrm{TT}(p) = v^\alpha$, we can take the inf over all such polynomials to obtain $T_{\mathcal{E},k}^w(K, \alpha) \leq T_k^w(K, \alpha)$. $\qquad \square$

Remark 12. The notion of a locally circled set is adapted from the notion of a *circled set in $\mathbb{C}^n$*. Recall that a compact set $K \subset \mathbb{C}^n$ is circled if $e^{i\theta} z \in K$ whenever $z \in K$. For such sets, the Chebyshev constants and homogeneous Chebyshev constants (of Theorems 5.1 and 5.2 respectively) are equal. The proof is essentially the same as that of the above proposition.

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