Buhmann Covariance Functions, their Compact Supports, and their Smoothness

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Abstract

We consider the Buhmann class of radially symmetric and compactly supported covariance functions, that includes the most prominent classes used in both numerical analysis and spatial statistics literature. We discuss a very simple difference operator and report the conditions for which the application of it to Buhmann functions preserves positive definiteness on m-dimensional Euclidean spaces. We also show a relation between the Zastavnyi-Porcu problem [39] for Buhmann functions and a class of completely monotone functions.
Keywords: Compact Support; Completely Monotonic; Fourier transforms; Laplace transforms.

1 Introduction

Interpolation of data has a notable importance in both numerical analysis and geostatistical communities: in geostatistics, the underlying structure of the data is assumed to be a stochastic process. Best linear unbiased prediction is known under the name of kriging in the geostatistical context [23], and it is equivalent to interpolation through radial basis functions in numerical analysis [5].

Several radial basis functions of compact support that give rise to nonsingular interpolation problems have been proposed, and we cite [19, 12, 5, 30, 20, 32, 33] and the reviews in [8] and [21], amongst others. In particular, the motivation behind Buhmann's [5] tour de force is to propose radial functions of compact support that also give positive definite matrices and have genuinely banded interpolation matrices (similarly to the multiquadric and Gaussian kernels). Of such nature are those ones we will discuss in the present paper. Early examples of radial functions with compact support that have a simple piecewise polynomial structure are due to [32]. Then Schaback and Wu [22], Wendland [30] and finally Schaback [20] established several of their special properties, such as certain optimality facts about their degree and smoothness. The functions proposed by Buhmann are closely related to the so-called multiply monotone radial basis functions as discussed in [15].

Radial basis functions are known under the name of covariance (correlation) functions in the geostatistical community: the use of compactly supported covariance functions has been advocated in a number of papers, and we refer the reader to [13], with the references therein, and to [7] for a recent effort under the framework on multivariate Gaussian fields. Covariance functions with compact support represent the building block for the construction of methods allowing to overcome the big data problem ([11]). The recent work of [4] brought even more attention on the role of some classes of compactly supported covariances for asymptotically optimal prediction on a bounded set of $\mathbb{R}^d$. From the cited works it has become apparent that the smoothness at the origin (intended as even extension) of a compactly supported and isotropic covariance function plays a crucial role for both estimation and prediction. Wendland functions [30] have been especially popular, being compactly supported over balls of $\mathbb{R}^d$ with arbitrary radii, and additionally allowing for parameterization of differentiability at the origin, in a similar way to the Matérn family ([25]).

The works of [30], [13] and [40] put emphasis on linear operators, called *Montée*, that allow to increase the smoothness of a given radial function, being positive definite on $\mathbb{R}^d$. This is done at the expense of losing positive definiteness, which is only achieved on $\mathbb{R}^{d-2}$, for $d \geq 3$.

Figure 1 depicts the following situation: Dashed lines report Wendland functions [30] with unit compact support, for $k = 0, 1, 2$ (from left to right) being the parameter that allows to govern differentiability at the origin. Dashed-dotted lines report the respective

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Wendland functions for a compact support equal to 0.75. The functions depicted with continuous lines report their weighted differences (see Equation (17) and Equation (3)). We can clearly appreciate that the level of differentiability changes for these last.

The level of the differentiability at the origin has important consequences on the geometrical properties of a Gaussian random field with a given covariance function [25]. For instance, Figure 2 show two realizations from a zero mean Gaussian random field with covariance a weighted differences of Wendlands functions with $k = 0$ (on the left) and a Wendland function with $k = 0$ (on the right). See Equation (17) and Equation (3).

Zastavnyi [36] has shown that Buhmann functions include as special case many other classes of compactly supported covariance functions, such as Askey [1] Wendland [30] and Missing Wendland [20] functions, as well as the Zastavnyi [34, 35, 36] and Trigub [26, 27] classes. Finally, also Wu functions [32] and the celebrated spherical model [29] are included as special cases. This fact motivated [39] to explore functions obtained as weighted difference of Buhmann functions, with the purpose of

a. Determining the parameter range such that positive definiteness in some m-dimensional Euclidean space is preserved, and **b.** inspecting how the difference of two Buhmann functions with a given level of differentiability at the origin implies a change of smoothness.

This paper revisits Buhmann functions and their weighted differences as studied in [39]. We expose the state of the art, the Zastavnyi-Porcu problem and the structure of the solution. We then provide some results that highlight a relationship between the solution of the Zastavnyi-Porcu problem and some classes of completely monotone functions.

The plan of the paper is the following. Section 2 introduces the Buhmann class and offers a formal statement of the problem. Section 3 exposes the structure of the solution and the new results. Section 4 illustrates connections with previous literature.

2   Buhmann functions. Statement of the problem

2.1   Buhmann's class and relations with previous literature

A real valued function $f : \mathbb{R}^m \to \mathbb{R}$ is positive definite if, for any finite dimensional collection $\{x_k\}_{k=1}^N \subset \mathbb{R}^m$ and constants $\{c_k\}_{k=1}^N \subset \mathbb{R}$, we have $\sum_{k=1}^N \sum_{h=1}^N c_k f(x_k - x_h) c_h \geq 0$.

We focus throughout on the class Φ_m of continuous functions $\varphi : [0, \infty) \to \mathbb{R}$ such that $\varphi(\|x\|)$ is positive definite on $\mathbb{R}^m$. Thus, $\varphi(\| \cdot \|)$ with $\varphi(0) = 1$ is the correlation function of some Gaussian field. Apparently, the functions $C(\cdot) := \varphi(\| \cdot \|)$ are radially symmetric and Schoenberg's theorem (1938, see [6] for a more recent discussion) uniquely identifies them as scale mixtures of the type

$$\varphi(t) = \int_{[0,\infty)} \Omega_m(rt) F(\mathrm{d}r), \qquad t \geq 0, \tag{1}$$

with F a uniquely determined probability measure, and $\Omega_m(\cdot)$ being the characteristic function of a random vector that is uniformly distributed on the spherical shell of $\mathbb{R}^m$. Daryl and Porcu [6] put emphasis on the measure F, termed Schoenberg measure there. It is well known [13] that any random vector X of $\mathbb{R}^m$ with characteristic function φ can be written as $X = \eta R$, with η having Ω_m as characteristic function, R a positive random variable distributed according to F, and η and R are stochastically independent. The identity above is intended as equality in distribution.

The class Φ_m is nested, with the following inclusion relation

$$\Phi_1 \supset \Phi_2 \supset \ldots \supset \Phi_\infty := \bigcap_{m \geq 1} \Phi_m, \qquad m \in \mathbb{N},$$

being strict (see, for example, [6, 13]). A function $f : (0, \infty) \to \mathbb{R}$ is called completely monotone if it is infinitely often differentiable and $(-1)^n f^{(n)}(x) \geq 0$, for all $n \in \mathbb{Z}_+$ and for all $x > 0$. The set of completely monotone functions on $(0, \infty)$ is denoted $\mathcal{CM}$. By Schoenberg's theorem, $\varphi(t) \in \Phi_\infty$ if and only if $\varphi(\sqrt{t}) \in \mathcal{CM}$ with $\varphi(0) = \varphi(+0) < \infty$.

We denote $C(\mathbb{R}^m)$ the set of continuous functions on $\mathbb{R}^m$, for $m = 1, 2, \ldots$. Let $\delta, \mu, \nu \in \mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$ and $\alpha \in \mathbb{C}$. Zastavnyi [36] (2006) proposed the following even functions given on $\mathbb{R}$:

$$\varphi_{\delta,\mu,\nu,\alpha}(x) := \begin{cases} \int_{|x|}^1 (s^2 - x^2)^{\nu-1} (1 - s^\delta)^{\mu-1} s^{\alpha-2\nu+1} \, \mathrm{d}s, & |x| < 1 \\ \\ 0, & |x| \geq 1. \end{cases} \tag{2}$$

If $\delta, \mu, \nu \in \mathbb{C}_+$, then arguments in Proposition 1 and Theorem 1 in [36] show, respectively, that $\varphi_{\delta,\mu,\nu,\alpha} \in C(-1, 1)$ if and only if $\alpha \in \mathbb{C}_+$ and that $\varphi_{\delta,\mu,\nu,\alpha} \in C(\mathbb{R})$ if and only if $\alpha, \mu + \nu - 1 \in \mathbb{C}_+$. If $\delta, \mu, \nu, \alpha \in \mathbb{C}_+$, then $\varphi_{\delta,\mu,\nu,\alpha}(0) = B(\alpha/\delta, \mu)/\delta$, with B denoting the Beta function.

The functions $\varphi_{\delta,\mu,\nu,\alpha}$ coincide (modulo some positive factors) with the functions

$$\phi_{\delta,\varrho,\lambda,\alpha}(x) \equiv 2\varphi_{2\delta,\varrho+1,\lambda+1,2\alpha+2}(x), \qquad x \in \mathbb{R},$$

introduced by Martin Buhmann [5]. We thus term them Buhmann functions throughout. The class $\varphi_{\delta,\mu,\nu,\alpha}$ includes a wealth of interesting special cases. For instance, $\mu\delta\varphi_{\delta,\mu,1,\delta}(x) = (1-|x|^{\delta})_{+}^{\mu}$, which implies that $\mu\varphi_{1,\mu,1,1}$ coincides with the Askey functions [1]. Also, we have that

$$\varphi_{1,\mu,\nu,2\nu-1}(x) \equiv h_{\mu,\nu}(x) \equiv \frac{2^{\nu-1}\Gamma(\nu)}{\mu}\psi_{\mu,\nu-1}(x), \qquad x \in \mathbb{R}, \tag{3}$$

with the functions $h_{\mu,\nu}$ being introduced by Zastavnyi (2002) [35, 34] and defined as follows: $h_{\mu,\nu}(x) := 0$ for $|x| \geq 1$ and

$$h_{\mu,\nu}(x) := \int_{|x|}^{1}(2u-|x|)g_{\mu,\nu}(u)g_{\mu,\nu}(u-|x|)\,du, \ |x| < 1,$$

$$\text{where } g_{\mu,\nu}(u) := u^{\mu-1}(1-u^2)^{\nu-1}, u \in (0,1), \ \mu, \nu \in \mathbb{C}_{+}. \tag{4}$$

Functions of the form (4) arise in the study of exponential type entire functions without zeros in the lower half-plane [37, Proposition 5.1]. The functions $\psi_{\mu,\nu-1}$, with $\mu > 0$, $\nu \in \mathbb{N}$, have been introduced in 1995 by Wendland [30], and they have been termed Wendland functions in both numerical analysis and geostatistical literatures: for $\mu > 0, k \in \mathbb{Z}_{+}$, we have

$$\psi_{\mu,0}(x) := \psi_{\mu}(x) := (1-|x|)_{+}^{\mu}, \ \ \psi_{\mu,k} := I^{k}\psi_{\mu} \ (k \in \mathbb{N}),$$

where $I(f)(x) := \int_{|x|}^{+\infty} sf(s)\,\mathrm{d}s$ is the Matheron's [14] Montée operator (provided the integral is well defined), and where I^{k} is the k-fold application of the operator I. Arguments in [30] and subsequently [13] show that $I\varphi$ belongs to the class Φ_{m-2} whenever $\varphi \in \Phi_{m}$, for $m \geq 3$. For $k < 2m$, the k-fold application of the Montée operators shows that $I^{k}\varphi \in \Phi_{m-2k}, k \in \mathbb{N}$.

Gneiting [13, Equation (17)] has proposed a generalization of Wendland functions on the basis of the fractional Montée operator, which coincides with the normalized Buhmann functions $\varphi_{1,\mu+1,\nu,2\nu}(x)/\varphi_{1,\mu+1,\nu,2\nu}(0)$, $\mu, \nu > 0$, as well as with the functions $h_{\mu,\nu+1}(x)/h_{\mu,\nu+1}(0) \equiv \psi_{\mu,\nu}(x)/\psi_{\mu,\nu}(0)$ (see Equation (6)). Arguments in [5] show that Wu functions [32] and consequently the spherical model are special cases of the Buhmann class.

For $r \in \mathbb{Z}_{+}$ and $k \in \mathbb{N}$, we have

$$h_{r+k,r+1}(x) \equiv B(r+k, 2r+1)A_{r,2k-1}(x),$$

with the splines $A_{r,2k-1}$ introduced by Trigub (1987), and we refer to [26], [27, § 6.2.13, 6.2.16, 6.3.12] for their analytical expression which is not reported here. Equation above in turn highlights the explicit connection between Trigub splines and Wendland functions: $A_{r,2k-1}(x) \equiv \psi_{r+k,r}(x)/\psi_{r+k,r}(0)$, for $r \in \mathbb{Z}_{+}$ and $k \in \mathbb{N}$.

For a proof of the identities above, the reader is referred to Zastavnyi and Trigub [35, Remarks 10 and 11], to [34, Theorems 12 and 13], [36] and [38, § 4.7].

Arguments in Proposition 4 of [36] show that, for $\delta, \mu, \nu \in \mathbb{C}_{+}$ and $x \in \mathbb{R}$,

$$\varphi_{2,\frac{\mu}{2},\frac{\mu}{2}+\nu,2\nu-1}(x) \equiv \frac{2^{\mu-1}\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\mu}{2}+\nu\right)}{\Gamma(\mu)}\varphi_{1,\mu,\nu,2\nu-1}(x),$$

$$2\nu\varphi_{\delta,\mu+1,\nu,2\nu}(x) \equiv \delta\mu\varphi_{\delta,\mu,\nu+1,2\nu+\delta}(x), \tag{5}$$

and, for $\mu, \nu \in \mathbb{C}_{+}$ and $x \in \mathbb{R}$, we also have the obvious identities:

$$\varphi_{1,\mu+1,\nu,2\nu}(x) \equiv \frac{\mu}{2\nu}\varphi_{1,\mu,\nu+1,2\nu+1}(x) \equiv \frac{\mu}{2\nu}h_{\mu,\nu+1}(x) \equiv 2^{\nu-1}\Gamma(\nu)\psi_{\mu,\nu}(x). \tag{6}$$

2.2 Buhmann class and its Fourier and Laplace Transforms

After the illustration of the relation between Buhmann and other celebrated classes of radial basis function, we need some preliminary material in order to provide a better description of the results coming subsequently. For a function h defined on $(0,\infty)$ and $m \in \mathbb{C}$, we define the Hankel transform $\mathfrak{F}_{m}$ as follows:

$$\mathfrak{F}_{m}(h)(t) := t^{1-\frac{m}{2}}\int_{0}^{\infty}h(u)u^{\frac{m}{2}}J_{\frac{m}{2}-1}(tu)\,\mathrm{d}u = \int_{0}^{\infty}h(u)u^{m-1}j_{\frac{m}{2}-1}(tu)\,\mathrm{d}u, \qquad t > 0, \tag{7}$$

where J_{λ} is the Bessel function of the first kind (see [28, Sec. 3.1]) and

$$j_{\lambda}(x) := \frac{J_{\lambda}(x)}{x^{\lambda}} = \frac{1}{2^{\lambda}}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+\lambda+1)} \cdot \frac{\left(-\frac{x^2}{4}\right)^{k}}{k!}, \ x \in \mathbb{C}, \ \lambda \in \mathbb{C}. \tag{8}$$

Remark 1. For $m \in \mathbb{N}$ the transform $\mathfrak{F}_{m}$ is connected with the Fourier transforms F_{m} of radial functions through the identity

$$F_{m}(h(\|\cdot\|))(\boldsymbol{x}) = (2\pi)^{\frac{m}{2}}\mathfrak{F}_{m}(h)(\|\boldsymbol{x}\|), \qquad \boldsymbol{x} \in \mathbb{R}^{m}.$$

These facts and Bochner-Khintchine theorem (see, for example [18, 24, 27]) imply that, if h is a continuous functions on $[0,\infty)$ and $\int_{0}^{\infty}t^{m-1}|h(t)|\,\mathrm{d}t < \infty$, then $h \in \Phi_{m}$ if and only if $\mathfrak{F}_{m}(h)$ is nonnegative on the positive real line.

For $\delta, \mu, \alpha + 1 \in \mathbb{C}_+$ and $\nu \in \mathbb{C}$, we define the function $I_{\delta,\mu,\nu,\alpha} : \mathbb{R}_+ \to \mathbb{C}$ through

$$
\begin{aligned}
I_{\delta,\mu,\nu,\alpha}(t) \quad &:= \quad t^{-\alpha-1-\delta(\mu-1)} \int_0^t (t^\delta - u^\delta)^{\mu-1} u^{\alpha-\nu+\frac{1}{2}} J_{\nu-\frac{1}{2}}(u)\,\mathrm{d}u \\
&= \quad \int_0^1 (1-x^\delta)^{\mu-1} x^\alpha j_{\nu-\frac{1}{2}}(tx)\mathrm{d}x\,, \qquad\qquad t > 0\,.
\end{aligned}
\tag{9}
$$

The following result reports succinctly a collection of useful results from [36] (Theorems 2, 3 and Proposition 4 (Assertions 1,3)).

Theorem 2.1 (Zastavnyi [36]). *Let the functions I and $\mathfrak{F}_m$ as being defined through Equations (9) and (7), respectively. Denote with I' the first derivative of I. Then, the following assertions are true:*

1. *Let $\delta, \mu, \nu, m, \alpha + m \in \mathbb{C}_+$. Then $\mathfrak{F}_m(\varphi_{\delta,\mu,\nu,\alpha})(t) = 2^{\nu-1}\Gamma(\nu)I_{\delta,\mu,\frac{m-1}{2}+\nu,m-1+\alpha}(t)$. Moreover, if $n, m-n+2\nu \in \mathbb{C}_+$, then*

$$
\mathfrak{F}_m(\varphi_{\delta,\mu,\nu,\alpha})(t) = \frac{2^{\frac{n-m}{2}}\Gamma(\nu)}{\Gamma(\frac{m-n}{2}+\nu)}\mathfrak{F}_n(\varphi_{\delta,\mu,\frac{m-n}{2}+\nu,m-n+\alpha})(t), \qquad t \geq 0.
$$

2. *Let $\delta, \mu, \alpha + 1 \in \mathbb{C}_+$ and $\nu \in \mathbb{C}$. Then*

$$
I'_{\delta,\mu,\nu,\alpha}(t) = -t I_{\delta,\mu,\nu+1,\alpha+2}(t), \qquad t > 0.
\tag{10}
$$

3. *If $\mu, \nu \in \mathbb{C}_+$, $t > 0$, then*

$$
I_{1,\mu,\nu,2\nu-1}(t) = \frac{2^{\frac{1}{2}-\nu}\Gamma(\mu)\Gamma(2\nu)}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(\mu+2\nu)}\,{}_1F_2\left(\nu; \frac{\mu+2\nu}{2}, \frac{\mu+2\nu+1}{2}; -\frac{t^2}{4}\right).
\tag{11}
$$

4. *Let $\delta, \mu, \nu \in \mathbb{C}_+$ and $\alpha \in \mathbb{C}$. Then*

$$
\varphi_{\delta,\mu,\nu+1,\alpha+2}(t) = 2\nu \int_{|t|}^\infty u\,\varphi_{\delta,\mu,\nu,\alpha}(u)\,\mathrm{d}u\,, \ t \neq 0.
\tag{12}
$$

A relevant remark is that Equation (11) describes the spectral density of the Wendland functions. Another remarkable consequence of Theorem 2.1 is that, for $\mu, \nu, m, 2\nu - 1 + m \in \mathbb{C}_+$,

$$
\mathfrak{F}_m(h_{\mu,\nu})(t) = \mathfrak{F}_m(\varphi_{1,\mu,\nu,2\nu-1})(t) = \frac{2^{\frac{1-m}{2}}\Gamma(\nu)}{\Gamma(\frac{m-1}{2}+\nu)}\mathfrak{F}_1(h_{\mu,\frac{m-1}{2}+\nu})(t), \qquad t \geq 0,
\tag{13}
$$

which in turn shows, in concert with [35, Lemma 12], that in some cases the Hankel transforms above can be written in closed form. Specifically, we have

$$
\begin{aligned}
&\mathfrak{F}_m(h_{\mu,\nu})(t) = \mathfrak{F}_m(\varphi_{1,\mu,\nu,2\nu-1})(t) = 2^{\nu-1}\Gamma(\nu)I_{1,\mu,\frac{m-1}{2}+\nu,m-1+2\nu-1}(t) = \\
&D(m,\mu,\nu) \cdot {}_1F_2\left(\frac{m-1}{2}+\nu; \frac{m-1}{2}+\nu+\frac{\mu}{2}, \frac{m-1}{2}+\nu+\frac{\mu+1}{2}; -\frac{t^2}{4}\right), \\
&\text{with } D(m,\mu,\nu) := \frac{2^{-\frac{m}{2}}\Gamma(\nu)\Gamma(\mu)\Gamma(m-1+2\nu)}{\Gamma\left(\frac{m}{2}+\nu\right)\Gamma(\mu+m-1+2\nu)}, \ \mu, \nu, m, 2\nu-1+m \in \mathbb{C}_+\,.
\end{aligned}
\tag{14}
$$

Let us use the abuse of notation $\widehat{h}_{\mu,\nu}$ for the one dimensional Fourier transform of the function $h_{\mu,\nu}$. We also denote with L the Laplace transform operator. For $\mu, \nu \in \mathbb{C}_+$, arguments in Zastavnyi and Trigub [35, Equation (44)] show that

$$
L\left(t^{2\nu+\mu-1}\widehat{h}_{\mu,\nu}(t)\right)(x) := \int_0^\infty e^{-tx}\,t^{2\nu+\mu-1}\widehat{h}_{\mu,\nu}(t)\,\mathrm{d}t = \frac{\Gamma^2(\nu)\Gamma(\mu)2^{2\nu-1}}{x^\mu(1+x^2)^\nu}, \ x > 0.
\tag{15}
$$

Thus, for $\mu, \nu, m, 2\nu - 1 + m \in \mathbb{C}_+$ and $x > 0$, we have

$$
\begin{aligned}
&L\left(t^{m-1+2\nu+\mu-1}\mathfrak{F}_m(h_{\mu,\nu})(t)\right)(x) \\
&= \frac{2^{\frac{1-m}{2}}\Gamma(\nu)}{\Gamma(\frac{m-1}{2}+\nu)}L\left(t^{m-1+2\nu+\mu-1}\mathfrak{F}_1(h_{\mu,\frac{m-1}{2}+\nu})(t)\right)(x) \\
&= \frac{2^{\frac{1-m}{2}}\Gamma(\nu)}{\Gamma(\frac{m-1}{2}+\nu)} \cdot \frac{\Gamma^2(\frac{m-1}{2}+\nu)\Gamma(\mu)2^{m-1+2\nu-1}}{(2\pi)^{\frac{1}{2}}} \cdot \frac{1}{x^\mu(1+x^2)^{\frac{m-1}{2}+\nu}} \\
&= \frac{C(m,\mu,\nu)}{x^\mu(1+x^2)^{\frac{m-1}{2}+\nu}}, \\
&\text{with } C(m,\mu,\nu) := \frac{2^{-\frac{m}{2}}\Gamma(\nu)\Gamma(\mu)\Gamma(m-1+2\nu)}{\Gamma\left(\frac{m}{2}+\nu\right)} = D(m,\mu,\nu)\Gamma(\mu+m-1+2\nu).
\end{aligned}
\tag{16}
$$

Remark 2. Equation ([16]) is the crux of the proof of the main part of Theorem 11 in [34]:

(i) *If* $\nu > \frac{1}{2}$ *and* $\mu \geq \max\{\nu, 1\}$, *then* $h_{\mu,\nu} \in \Phi_1$. *If, additionally,* $(\mu; \nu) \neq (1; 1)$, *then there exist constants* $c_i > 0$, $i = 1, 2$, *depending on* μ *and* ν *only, such that*

$$c_1 \leq (1 + t^2)^\nu \cdot \widehat{h}_{\mu,\nu}(t) \leq c_2, \qquad t \in \mathbb{R}.$$

(ii) *If* $\nu \geq 1$, *then* $h_{\mu,\nu} \in \Phi_1 \iff \mu \geq \nu$.

(iii) *If* $m \geq 2$, *then* $h_{\mu,\nu} \in \Phi_m \iff \nu > \frac{1}{2}$ *and* $\mu \geq \frac{m-1}{2} + \nu$. *In this case, there exist two constants* $c_i > 0$, $i = 1, 2$, *depending on* μ, ν *and* m, *and such that*

$$c_1 \leq (1 + t^2)^{\frac{m-1}{2} + \nu} \cdot \mathfrak{F}_m(h_{\mu,\nu})(t) \leq c_2, \qquad t \geq 0.$$

This theorem is related to the positiveness of the function $I_{1,\mu,\nu,2\nu-1}(t)$ for all $t > 0$. Theorems on positiveness of the functions $I_{\delta,\mu,\nu,\alpha}(t)$ are obtained in [36, Theorems 4,5,6] (the well-known cases given before Theorem 4 from [36]).

2.3 The Zastavnyi-Porcu Problem

We are now able to state formally the main problem of this paper.

Problem 1. [39] Let $\mu > 0$, $\nu > \frac{1}{2}$, $\mu + \nu > 1$. Then $h_{\mu,\nu} \in C(\mathbb{R})$ (see [35, 34]). Let $\varepsilon > 0$ and

$$f_{\mu,\nu,\varepsilon,\beta_1,\beta_2}(x) := \beta_2^\varepsilon h_{\mu,\nu}\left(\frac{x}{\beta_2}\right) - \beta_1^\varepsilon h_{\mu,\nu}\left(\frac{x}{\beta_1}\right), \ x \in \mathbb{R}. \tag{17}$$

Let $m \in \mathbb{N}$. *Show the conditions on* (ε, μ, ν) *such that, for any* $\beta_2 > \beta_1 > 0$, *we have*

$$f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \in \Phi_m. \tag{18}$$

Next section details the structure of the solution and provides the exact smoothness of the new covariance resulting from the differences of Buhmann functions. Original results related to completely monotone functions are also included.

3 Structure of the Solution

Let us start with a general assertion regarding the structure of Problem [1].

Proposition 1. [39] The following conditions are equivalent:

1. Condition ([18]) is satisfied.

2. For any $\beta_2 > \beta_1 > 0$, the function $t \mapsto \beta_2^{\varepsilon+m}\mathfrak{F}_m(h_{\mu,\nu})(\beta_2 t) - \beta_1^{\varepsilon+m}\mathfrak{F}_m(h_{\mu,\nu})(\beta_1 t)$ is nonnegative in interval $(0, \infty)$.

3. The function $t^{\varepsilon+m}\mathfrak{F}_m(h_{\mu,\nu})(t) = 2^{\nu-1}\Gamma(\nu)t^{\varepsilon+m}I_{1,\mu,\frac{m-1}{2}+\nu,m-1+2\nu-1}(t)$ increases in the interval $(0, \infty)$.

4. The following inequality is true:

$$(\varepsilon + m)I_{1,\mu,\frac{m-1}{2}+\nu,m-1+2\nu-1}(t) + tI'_{1,\mu,\frac{m-1}{2}+\nu,m-1+2\nu-1}(t) =$$
$$(\varepsilon + m)I_{1,\mu,\frac{m-1}{2}+\nu,m-1+2\nu-1}(t) - t^2 I_{1,\mu,\frac{m-1}{2}+\nu+1,m-1+2\nu+1}(t) \geq 0, \ \forall t > 0$$

5. Let $n := \frac{m-1}{2} + \nu$. Then,

$$(\varepsilon + m)\mathfrak{F}_1(h_{\mu,n})(t) - \frac{t^2}{2n}\mathfrak{F}_1(h_{\mu,n+1})(t) \geq 0, \qquad \forall t > 0.$$

6. We have

$$L\left(t^{2n+\mu-1}\left((\varepsilon + m)\widehat{h}_{\mu,n}(t) - \frac{t^2}{2n}\widehat{h}_{\mu,n+1}(t)\right)\right)(x) =$$
$$\Gamma^2(n)\Gamma(\mu)2^{2n-1}\left(\frac{\varepsilon + m}{x^\mu(1 + x^2)^n} - \frac{2n}{x^\mu(1 + x^2)^{n+1}}\right) \in \mathcal{CM}.$$

7.

$$\frac{\varepsilon - 2\nu + 1 + (\varepsilon + m)x^2}{x^\mu(1 + x^2)^{\frac{m-1}{2} + \nu + 1}} \in \mathcal{CM} \tag{19}$$

Remark 3. It follows from Hausdorff-Bernstein-Widder theorem (see, for example, [9, 18, 24, 31]) that if $g \in C[0, +\infty)$ and its Laplace transform

$$Lg(x) := \int_0^{+\infty} e^{-xs}g(s)\, ds$$

converges for all $x > 0$, then $g(s) \geq 0$ for $s \geq 0$ if and only if $Lg \in \mathcal{CM}$.

The proof Proposition 1 is an easy consequence of Remark 1 in concert with the Hausdorff-Bernstein-Widder theorem (see Remark 3), Theorem 2.1 (statements 1 and 2), and equalities (13) and (15). Note that, if $\mu, \nu > 0$, then $x^{-\mu}(1 + x^2)^{-\nu} \in \mathcal{CM}$ if and only if $\widehat{h}_{\mu,\nu}(t) \geq 0$ for all $t > 0$, if and only if $I_{1,\mu,\nu,2\nu-1}(t) \geq 0$ for all $t > 0$ (see Equations (15) and (14)).

Proposition 1 has been combined by [39] with the following facts:

1. If $\nu \geq 1$, then $x^{-\mu}(1 + x^2)^{-\nu} \in \mathcal{CM}$ if and only if $\mu \geq \nu$. The sufficiency of this result can be found in [10], and the necessity has been proved in [16], [33, Lemma 8]).

2. If $0 < \nu < 1$, $\mu \geq 1$, then $x^{-\mu}(1 + x^2)^{-\nu} \in \mathcal{CM}$ [16] and [37, Example 5.4], [38, § 4.7, Example 4.7.7].

3. If $\nu > 0$, $\mu \geq 2\nu$, then $x^{-\mu}(1 + x^2)^{-\nu} \in \mathcal{CM}$ [2].

4. If $n = 1, 2, 3$, then $(a + x^2)/(x^n(1 + x^2)^n) \in \mathcal{CM}$ if and only if $a \geq 1/(2^{n-1} + 1)$ [35, § 2].

The first three sufficient conditions above provide the following assertion: if $\nu > 0$ and $\mu \geq \min\{2\nu; \max\{1, \nu\}\}$, then $x^{-\mu}(1 + x^2)^{-\nu} \in \mathcal{CM}$.

The combination of these facts with Proposition 1 has just offered the proof of the following

Theorem 3.1. *[39] The following assertions are true:*

1. *If* (18) *is true, then* $\varepsilon \geq 2\nu - 1$.

2. *If* $m \in \mathbb{N}$, $\nu > \frac{1}{2}$, $\varepsilon \geq 2\nu - 1$ *and* $\mu \geq (m-1)/2 + \nu + 3$, *then condition* (18) *is true. If, in addition,* $\varepsilon = 2\nu - 1$, *then* (18) *is true if and only if* $\mu \geq (m-1)/2 + \nu + 3$.

3. *Suppose that for some* $n = 1, 2, 3$, *we have* $\varepsilon \geq 2^{1-n}(m + (2\nu - 1)(2^{n-1} + 1))$, $(m-1)/2 + \nu + 1 - n > 0$ *and*

$$\mu - n \geq \min\left\{m - 1 + 2\nu + 2 - 2n; \max\{1; \frac{m-1}{2} + \nu + 1 - n\}\right\}.$$

   Then, condition (18) *is true.*

We now provide a characterization result for the following problem: whether the condition (18) is satisfied for fixed $\beta_2, \beta_1 > 0$ (and not for any $\beta_2 > \beta_1 > 0$ as in Problem 1).

Theorem 3.2. *Let* $\mu > 0$, $\nu > \frac{1}{2}$, $\mu + \nu > 1$ *and* $m \in \mathbb{N}$. *Let* $\varepsilon \in \mathbb{R}$, $\beta_2, \beta_1 > 0$ *and* $a := \beta_2/\beta_1$. *Then* $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \in \Phi_m$ *if and only if*

$$\frac{1}{x^\mu(1 + x^2)^{\frac{m-1}{2} + \nu}} - \frac{a^{2\nu - 1 - \varepsilon}}{x^\mu(1 + a^2 x^2)^{\frac{m-1}{2} + \nu}} \in \mathcal{CM}. \tag{20}$$

If $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \in \Phi_m$ *and* $\beta_2 > \beta_1 > 0$, *then* $\varepsilon \geq 2\nu - 1$.

Proof. From Remarks 1 and 3, it follows that the following conditions are equivalent:

1. $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \in \Phi_m$.

2. $\beta_2^{\varepsilon + m} \mathfrak{F}_m(h_{\mu,\nu})(\beta_2 t) - \beta_1^{\varepsilon + m} \mathfrak{F}_m(h_{\mu,\nu})(\beta_1 t) \geq 0$ for all $t \in (0, \infty)$.

3. $L\left(t^{m-1+2\nu+\mu-1}\left(\beta_2^{\varepsilon + m} \mathfrak{F}_m(h_{\mu,\nu})(\beta_2 t) - \beta_1^{\varepsilon + m} \mathfrak{F}_m(h_{\mu,\nu})(\beta_1 t)\right)\right)(x) \in \mathcal{CM}$.

Furthermore, we have (see (16)):

$$\beta^{\varepsilon + m} L\left(t^{m-1+2\nu+\mu-1} \mathfrak{F}_m(h_{\mu,\nu})(\beta t)\right)(x) = \frac{\beta^{\varepsilon - 2\nu + 1}}{x^\mu(1 + x^2/\beta^2)^{\frac{m-1}{2} + \nu}}, \ \beta > 0, x > 0.$$

Suppose that condition (20) is satisfied. Since completely monotone functions are non-negative on $(0, +\infty)$, we have that $1 - a^{2\nu - 1 - \varepsilon} \geq 0$. If, additionally, $a = \beta_2/\beta_1 > 1$, then $\varepsilon \geq 2\nu - 1$. The proof is completed. ∎

Direct inspection of the proof of Proposition 1 as well as the proof of Theorem 3.2 shows that the following proposition is true.

Proposition 2. Let $\mu, \nu, m, 2\nu - 1 + m > 0$ and $\varepsilon \in \mathbb{R}$. Then following conditions are equivalent:

1. $\dfrac{\varepsilon - 2\nu + 1 + (\varepsilon + m)x^2}{x^\mu(1 + x^2)^{\frac{m-1}{2} + \nu + 1}} \in \mathcal{CM}$.

2. $\dfrac{1}{x^\mu(1 + x^2)^{\frac{m-1}{2} + \nu}} - \dfrac{a^{2\nu - 1 - \varepsilon}}{x^\mu(1 + a^2 x^2)^{\frac{m-1}{2} + \nu}} \in \mathcal{CM}, \forall a > 1$.

3. $\dfrac{1}{x^\mu(1 + x^2)^{\frac{m-1}{2} + \nu}} - \dfrac{a_n^{2\nu - 1 - \varepsilon}}{x^\mu(1 + a_n^2 x^2)^{\frac{m-1}{2} + \nu}} \in \mathcal{CM}$ for some sequence $a_n > 1, a_n \to 1$.

If the first condition is satisfied, or if the second condition is satisfied for some $a > 1$, then $\varepsilon \geq 2\nu - 1$.

Proof. The crux of the proof is in the following equality, which holds for $a > 1$:

$$\int_1^a \frac{(\varepsilon - 2\nu + 1 + (\varepsilon + m)x^2 t^2)\, t^{2\nu - \varepsilon - 2}}{x^\mu (1 + x^2 t^2)^{\frac{m-1}{2} + \nu + 1}}\, dt = \frac{1}{x^\mu (1 + x^2)^{\frac{m-1}{2} + \nu}} - \frac{a^{2\nu - 1 - \varepsilon}}{x^\mu (1 + a^2 x^2)^{\frac{m-1}{2} + \nu}}.$$

Since completely monotone functions are closed under scale mixtures, the equality above provides the implication $1 \Rightarrow 2$. Assertion $2 \Rightarrow 3$ is obvious. To prove assertion $3 \Rightarrow 1$, it is necessary to take in the last equation $a = a_n$, divide both sides by $a_n - 1$ and take the limit as $n \to \infty$. Finally, we make use of the well-known fact: if a sequence of completely monotone functions converges pointwise on $(0, +\infty)$, then the limit function is also completely monotone. ∎

We conclude this section detailing the exact smoothness of the differences of Buhmann functions.

Theorem 3.3. *[39] Let $\nu \in \mathbb{N}$, $\mu > 0$, $\varepsilon \in \mathbb{R}$, $\beta_2, \beta_1 > 0$, and $\beta_2 \neq \beta_1$. Let $q := \min(\beta_1, \beta_2)$. Then,*

1. *If $\varepsilon \neq 2\nu - 1$, then $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \in C^{2\nu-2}(-q, q)$, and $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \notin C^{2\nu-1}(-q, q)$.*

2. *If $\varepsilon = 2\nu - 1$, $\mu \notin \{1, 2\}$, then $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \in C^{2\nu}(-q, q)$, and $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \notin C^{2\nu+1}(-q, q)$.*

3. *If $\varepsilon = 2\nu - 1$, $\mu = 1$ or $\mu = 2$, then $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2}$ is an even polynomial of degree at most $\mu + 2\nu - 2$ on $[-q, q]$, and therefore $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2} \in C^\infty(-q, q)$.*

4   Connections with previous literature

The k-fold application of the Montée operator to the Askey function $\mathcal{A}_\mu(t) = (1 - t)_+^\mu$, for $\mu \geq (m + 1)/2$, results in Wendland functions $\psi_{\mu,k}$ as described in previous section. Table 1 depicts the role of the mapping $f_{\mu,\nu,\varepsilon,\beta_1,\beta_2}$ as defined through Equation (17). In particular, we consider $f_{\mu,k+1,2k+1,\beta_1,\beta_2}$, for $k = 0, 1, 2$. Expressions of the corresponding Wendland functions are reported in the second column of the same table. According to Theorem 3.1, $f_{\mu,k+1,2k+1,\beta_1,\beta_2} \in \Phi_m$ if and only if $\mu \geq (m + 7)/2 + k$. The third and fourth column allow to describe the action of the mapping $f_{\mu,k+1,2k+11,\beta_1,\beta_2}$. When $\mu \notin \{1, 2\}$, one can clearly appreciate the increase in terms of differentiability at the origin.

The Wendland radial basis functions are piecewise polynomial compactly supported reproducing kernels in Hilbert spaces which are norm-equivalent to Sobolev spaces. But they only cover the Sobolev spaces $H^{d/2+k+1/2}(\mathbb{R}^d)$, when $k \in \mathbb{N}$. Motivated by this fact, Robert Schaback [20] covered the case of the integer order spaces in even dimensions. Namely, he derived the missing Wendland functions working for half-integer k and even dimensions, reproducing integer-order Sobolev spaces in even dimensions, and showing that they turn out to have two additional non-polynomial terms: a logarithm and a square root.

Other walks through dimensions have been recently proposed by [17] through the Generalized Askey functions $\varphi_{n,k,m} : [0, \infty) \to \mathbb{R}$ defined through

$$\varphi_{n,k,m}(t) = t^{k-n}(1-t)_+^{n+m+1}\, {}_1F_2(n - k, n + 1, n + m + 2, 1 - 1/t), \qquad t \geq 0.$$

The parameters (n, m, k) are then shown to be crucial in order to determine when $\varphi_{n,k,m} \in \Phi_m$ (see their Proposition 2.3).

To our knowledge, the only case of compactly supported correlation functions which is not covered by this work is the case of the Euclid''s hat ([19, 12]), which is the self-convolution of an indicator function supported on the unit ball in $\mathbb{R}^d$. As noted by [20], while Euclid's hat is not differentiable and Wu functions have zeros in their Fourier transform, Wendland's functions have no such drawbacks. They are polynomials on $[0, 1]$ and yield positive definite $2k$-times differentiable radial basis functions on $\mathbb{R}^d$. Given these properties, their polynomial degree $\lfloor d/2 \rfloor + 3k + 1$ is minimal.

k	$\psi_{\mu,k}$	$D_{\texttt{before}}$	$D_{\texttt{after}}$
0	$(1 - x)_+^\mu$	$C(\{0\})$	$C^2(\{0\})$
1	$(1 - x)_+^{\mu+1}(1 + (\mu + 1)x)$	$C^2(\{0\})$	$C^4(\{0\})$
2	$(1 - x)_+^{\mu+2}\left(1 + (\mu + 2)x + \frac{1}{3}\left((\mu + 2)^2 - 1\right)\right)$	$C^4(\{0\})$	$C^6(\{0\})$

Table 1: Examples of the Wendland functions $\psi_{\mu,k}$ for $k = 0, 1, 2$ and $\mu \notin \{1, 2\}$. For all cases, $x \geq 0$. The first column reports the values of k, and the corresponding expression in the second column reports the analytic expression of $\psi_{\mu,k}$. In the third column, $D_{\texttt{before}}$ depicts the differentiability of $\psi_{\mu,k}$. In the last column, $D_{\texttt{after}}$ stays for the differentiability at the origin of $f_{\mu,k+1,2k+1,\beta_1,\beta_2}$. Observe that when $\mu = 1$ or $\mu = 2$, then $D_{\texttt{after}} = \infty$.
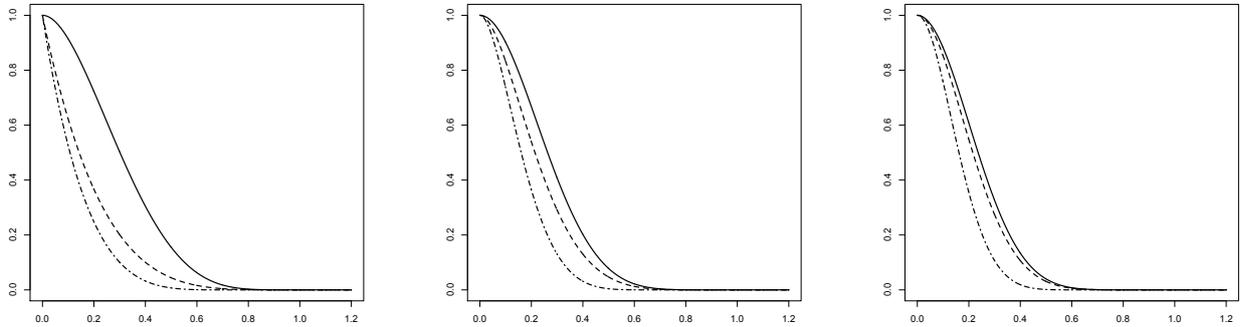
Figure 1: Continuous Lines: $f_{4.5+k,k+1,2k+1,0.75,1}(x)$, $x \in [1, 1.2]$, defined according to Equation (17) for $k = 0, 1, 2$ (from left to right). Dashed and Dashed-dot Lines report $h_{4.5+k,k+1}(x/0.75)$ and $h_{4.5+k,k+1}(x/1)$ respectively for $k = 0, 1, 2$. All the functions are normalized with their value at the origin.
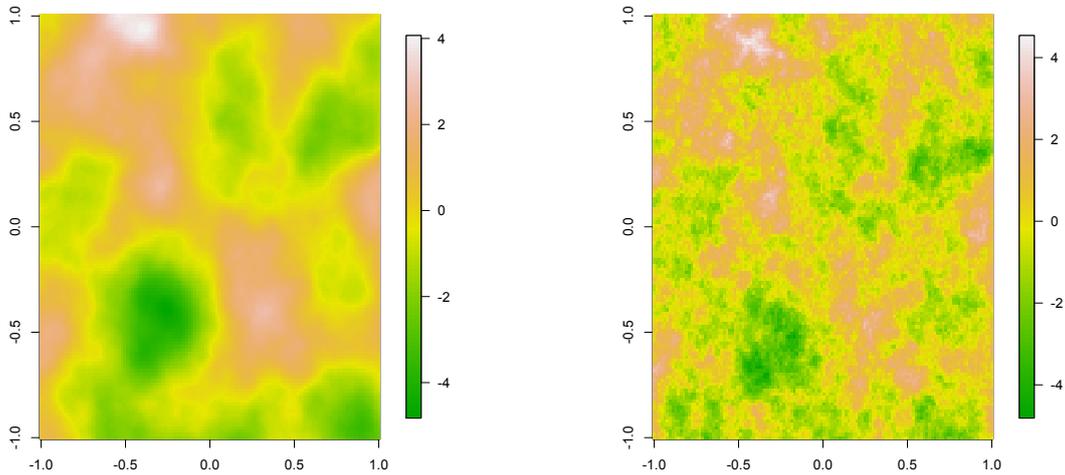

Figure 2: Two realizations from a zero mean Gaussian random field with covariance $f_{4.5,1,1,0.75,1}(x)$ (left) and $h_{4.5,1}(x/1)$ (right). Both functions are normalized with their value at the origin.

The proof of Theorem 3.1 highlights explicit connections with previous literature devoted to (sub) classes of completely monotone functions. A function $f : [0, \infty) \to \mathbb{R}$ is called Logarithmically completely monotonic on $(0, \infty)$, and denoted $f \in \mathcal{L}(0, \infty)$ if and only if it is infinitely often differentiable on $(0, \infty)$ and

$$(-1)^n \left[\log f(x) \right]^{(n)} \geq 0, \qquad x \geq 0.$$

By well known results (see [3], with the references therein) $f \in \mathcal{L} \Leftrightarrow f^\alpha \in \mathcal{CM} \Leftrightarrow (f)^{1/n} \in \mathcal{CM}$, for all $\alpha > 0$ and $n \in \mathbb{N}$.

Berg, Porcu and Mateu [3] introduced the so called *Auxiliary* family

$$f_{\alpha, \beta}(x) = \frac{1}{x^\alpha (1 + x^\beta)}, \qquad x > 0,$$

with α, β positive parameters. They show that $f_{\alpha, \beta} \in \mathcal{L}$ for $\alpha \geq 0$ and $0 \leq \beta \leq 1$. Additionally, $f_{\alpha, 2} \in \mathcal{L}$ if and only if $\alpha \geq 2$. This resut is one of the crux for describing the complete monotonicity of the Dagum family. Known cases, when $x^{-\mu}(1 + x^2)^{-\nu} \notin \mathcal{CM}$ (if $\mu, \nu > 0$, then this is equivalent to the inequality $I_{1,\mu,\nu,2\nu-1} \geq 0$ is not true for all $t > 0$): 1) $\mu < 0$ or $\mu = 0$, $\nu \neq 0$ (it is obvious); 2) $\mu < \nu$; 3) $0 < \mu = \nu < 1$. Proof of the latter two cases, see, for example, [16], [33, Lemma 8] and [36, Theorem 5].

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