#A15 INTEGERS 15 (2015)

A NEW PROOF OF KEMPERMAN'S THEOREM

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Received: 3/15/13, Revised: 2/19/15, Accepted: 4/12/15, Published: 5/8/15

Abstract

Let G be an additively written abelian group, let $A, B \subseteq G$ be finite and nonempty, and consider their sumset $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$. Vosper proved a theorem which characterizes all pairs (A, B) for which $|A + B| < |A| + |B|$ in the special case when $|G|$ is prime. Kemperman extended this characterization to arbitrary abelian groups. Here we give a new proof of Lev's version of Kemperman's Theorem.

1. Introduction

Throughout this paper we shall assume that G is an additively written abelian group. For subsets $A, B \subseteq G$, we define the *sumset* of A and B to be $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$. If $g \in G$ we let $g + A = \{g\} + A$ and $A + g = A + \{g\}$. The *complement* of A is the set $\overline{A} = G \setminus A$, and we let $-A = \{-a \mid a \in A\}$.

The classical direct problem for addition in groups is to find lower bounds on the size of $A + B$. If $G \cong \mathbb{Z}$ (or more generally, G is torsion-free) it is not difficult to argue that $|A + B| \geq |A| + |B| - 1$ holds for every pair of finite nonempty sets (A, B). In 1813 Cauchy proved that this assertion remains true when the order of G is prime and $A + B \neq G$. This result was rediscovered by Davenport in 1935, and it is now known as the Cauchy-Davenport theorem.

[1] Supported in part by an NSERC Discovery Grant (Canada).
[2] Partially supported by the project PAPIIT IA102013.

Theorem 1.1 (Cauchy [1] - Davenport [2]). *If p is prime and $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ are nonempty, then*

$$|A+B| \geq \min\{p, |A|+|B|-1\}.$$

For arbitrary abelian groups we can not expect to have such a lower bound. For instance, if H is a finite proper nontrivial subgroup of G, and $A = B = H$, then we will have $A + B = H$. So any generalization of Theorem 1.1 will have to take subgroup structure into account. Next we introduce an important theorem of Kneser which yields a generalization of Cauchy-Davenport to arbitrary abelian groups.

We define the *stabilizer* of a subset $A \subseteq G$, denoted G_A, to be the subgroup of G defined by $G_A = \{g \in G \mid g + A = A\}$. Note that A is a union of G_A-cosets, and G_A is the maximal subgroup of G with this property. For a subgroup $H \leq G$, we say that a subset A is *H-stable* if $A + H = A$ (equivalently, $H \leq G_A$).

Theorem 1.2 (Kneser [8], version I). *If A and B are finite nonempty subsets of G and $H = G_{A+B}$, then*

$$|A+B| \geq |A+H| + |B+H| - |H|. \tag{1}$$

To further illuminate the bound in Kneser's theorem, let us introduce some further notation. Whenever $H \leq G$ we let $\varphi_{G/H}$ denote the canonical homomorphism from G to the quotient group G/H. Now for $H = G_{A+B}$ let $\tilde{A} = \varphi_{G/H}(A)$ and $\tilde{B} = \varphi_{G/H}(B)$. By definition we have $|A + B| = |\tilde{A} + \tilde{B}||H|$, $|A + H| = |\tilde{A}||H|$ and $|B + H| = |\tilde{B}||H|$. Using these simple equalities, we can express (1) as $|\tilde{A}+\tilde{B}| \geq |\tilde{A}|+|\tilde{B}|-1$, an expression similar to the lower bound from the Cauchy-Davenport Theorem.

Define the *deficiency* of a pair (A, B) to be $\delta(A, B) = |A|+|B|-|A+B|$. We will say that a pair (A, B) is *deficient* if $\delta(A, B) > 0$. The Cauchy-Davenport Theorem implies that every deficient pair (A, B) of nonempty sets with $A+B \neq \mathbb{Z}/p\mathbb{Z}$ satisfies $\delta(A, B) = 1$. Meanwhile, Kneser's theorem asserts that for a deficient pair (A, B) of finite nonempty sets in G, the pair $(\tilde{A}, \tilde{B})$ of G/H as defined above, will be deficient with deficiency $\delta(\tilde{A}, \tilde{B}) = 1$. Indeed, as it is not difficult to see, Kneser's Theorem is equivalent to the assertion (appearing below as Proposition 2.1 (2)) that every deficient pair (A, B) of finite nonempty sets in G satisfies $|A + B| = |A + H| + |B + H| - |H|$ with $H = G_{A+B}$.

Now we shall turn our attention to the structure of deficient pairs. One simple construction for a deficient pair (A, B) is to choose A, B so that $\min\{|A|, |B|\} = 1$. A second, more interesting construction is to choose A and B to be arithmetic progressions with a common difference. In 1956 Vosper proved the following theorem which characterizes deficient pairs in groups of prime order, and these structures feature prominently in his result.

Theorem 1.3 (Vosper [12, 13], version I)**.** *If p is prime and (A, B) is a deficient pair of nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$, then one of the following holds.*

1. $|A| + |B| > p$ *and* $A + B = \mathbb{Z}/p\mathbb{Z}$.
2. $|A| + |B| = p$ *and* $|A + B| = p - 1$.
3. $\min\{|A|, |B|\} = 1$.
4. *A and B are arithmetic progressions with a common difference.*

In 1960 Kemperman proved a structure theorem which characterizes deficient pairs in an arbitrary abelian group. Although this theorem was published few years after Vosper's, it took some time before it achieved the recognition and attention it deserved. This resulted in part from the inherent complexity of deficient pairs, and in part from the difficult nature of Kemperman's paper. Recently, this situation has improved considerably thanks to the work of Grynkiewicz [3, 4], Lev [9], and Hamidoune [5, 6]. Grynkiewicz recasts Kemperman's Theorem and then takes a step further by characterizing those pairs (A, B) with $|A+B| = |A|+|B|$. Lev gives a more convenient "top-down" version of Kemperman's Theorem which we shall adopt here. Finally, Hamidoune showed that all of these results could be achieved using the isoperimetric method.

Here we shall give a new proof of Kemperman's theorem based on some recent work of the second author which generalizes Kemperman's Theorem to arbitrary groups. Although this generalization leans heavily on the isoperimetric method, we shall not adopt these techniques here. Instead we will exploit Kneser's theorem, thus making our proof rather closer in spirit to Kemperman's original than to any of these more recent works. Our paper also differs with the existing literature in our statement of Kemperman's Theorem. The main difference here is that we will work with triples of subsets instead of pairs, and this has the effect of reducing the number of configurations we need to consider.

The remainder of this paper is organized as follows. Over the next two sections, we reduce the original classification problem to a classification problem for certain types of triples of subsets. Section 4 contains our new statement of Kemperman's theorem, and the remaining sections are devoted to its proof.

2. Maximal Pairs

For two pairs of sets (A, B) and (A^*, B^*), we say that (A^*, B^*) is a *superpair* of (A, B) and write $(A, B) \subseteq (A^*, B^*)$ if $A \subseteq A^*$ and $B \subseteq B^*$. If (A, B) is a pair of subsets of an abelian group, then we call (A, B) *maximal* if the only superpair (A^*, B^*) of (A, B) with $A^* + B^* = A + B$ is given by $(A^*, B^*) = (A, B)$. Our main goal in this section is to reduce the original problem to that of classifying maximal deficient pairs.

However, we shall first address some of the uninteresting constructions of deficient pairs. For instance, in the context of a general finite abelian group, consider the behaviour appearing in the first outcome of Theorem 1.3. If $A, B \subseteq G$ satisfy $|A| + |B| > |G|$, then every $g \in G$ satisfies $B \cap (g - A) \neq \emptyset$, and it follows that $A + B = G$. So every such pair (A, B) will be deficient. On the other hand, the deficient pairs (A, B) with $A + B = G$ are precisely those for which $|A| + |B| > |G|$. Another rather uninteresting construction of a deficient pair (A, B) is to take exactly one of A or B to be empty. Accordingly, we will call a pair (A, B) *trivial* if $A = \emptyset$, $B = \emptyset$, or $A + B = G$, and we will generally restrict our attention to nontrivial pairs.

Next we turn our attention to the notion of maximality.

Proposition 2.1. *Let (A, B) be a deficient pair of nonempty sets in G, and let $H = G_{A+B}$. Then, setting $A^* = A + H$ and $B^* = B + H$, we have:*

1. $A^* + B^* = A + B$,
2. $|A^* + B^*| = |A^*| + |B^*| - |H|$ *and,*
3. (A^*, B^*) *is maximal.*

Proof. The first part follows immediately from $A + B + H = A + B$. For the other parts, observe that our assumptions together with Kneser's Theorem imply

$$|A^*| + |B^*| > |A + B| \geq |A^*| + |B^*| - |H|.$$

Since all of the sets in this inequality are unions of H-cosets, the sizes are all multiples of $|H|$, and this yields the second part.

For the third part, suppose that (A', B') is a superpair of (A^*, B^*) with $A^* + B^* = A' + B'$. Then by the second part and Kneser's Theorem we have

$$|A^*| + |B^*| - |H| = |A^* + B^*| = |A' + B'| \geq |A'| + |B'| - |H|.$$

This implies $(A', B') = (A^*, B^*)$, so (A^*, B^*) is maximal. □

Next we show that the problem of classifying deficient pairs reduces to that of classifying maximal deficient pairs. Indeed, every deficient pair is obtained by removing a small number of elements from a maximal deficient pair.

Proposition 2.2. *For every pair of finite nonempty subsets (A, B) of G the following are equivalent.*

1. *The pair (A, B) is deficient.*

2. *There exists a maximal deficient superpair* $(A^*, B^*) \supseteq (A, B)$ *for which* $|A^* \setminus A| + |B^* \setminus B| < |G_{A^*+B^*}|$*, and* $A^* + B^* = A + B$*.*

Proof. First we suppose that (1) holds and define $H = G_{A+B}$, $A^* = A + H$ and $B^* = B + H$. Proposition 2.1 implies that (A^*, B^*) is a maximal deficient superpair of (A, B), and $|A^*| + |B^*| = |A + B| + |H|$. Now the following equation shows that (2) holds

$$\begin{aligned} |A^* \setminus A| + |B^* \setminus B| &= |A^*| + |B^*| - |A| - |B| \\ &= |A + B| + |H| - |A| - |B| \\ &< |H|. \end{aligned}$$

If (2) holds, then set $H = G_{A^*+B^*}$, let $z \in A^* + B^*$ and choose $a \in A^*$ and $b \in B^*$ with $a + b = z$. Now, for every $h \in H$ the elements $a' = a + h$ and $b' = b - h$ satisfy $a' \in A^*$ and $b' \in B^*$ and $a' + b' = z$. So, z has at least $|H|$ distinct representations as a sum of an element in A^* and an element in B^*. It follows from this and $|A^* \setminus A| + |B^* \setminus B| < |H|$ that $A + B = A^* + B^*$. This gives us $|A + B| = |A^* + B^*| = |A^*| + |B^*| - |H| > |A| + |B|$, so (A, B) is deficient and (1) holds. $\square$

3. Trios

In the study of deficient pairs, there is a third set which appears naturally in conjunction with A and B, namely $C = -\overline{A + B}$. Observe that if G is finite and (A, B) is a deficient pair in G, then we have:

- $0 \notin A + B + C$
- $|A| + |B| + |C| > |G|$.

In this case we see that the pair (B, C) is deficient since $B + C$ is disjoint from $-A$ (so $|B + C| \leq |G| - |A| < |B| + |C|$) and similarly (A, C) is deficient. So, in other words, taking the set C as defined above gives us a triple of sets so that every two of them form a deficient pair. Accordingly we now extend our definitions from pairs to triples. To allow for infinite groups we shall permit sets which are infinite but cofinite.

Definition 3.1. If $A, B, C \subseteq G$ satisfy $0 \notin A + B + C$ and each of A, B, C is either finite or cofinite, then we say that (A, B, C) is a *trio*. The trio is *trivial* if at least one of A, B, or C is empty.

Note that a nontrivial trio has at most one infinite set.

Definition 3.2. We define the *deficiency* of the trio (A, B, C) to be the parameter $\delta(A, B, C)$ determined as follows. If G is finite, then $\delta(A, B, C) = |A|+|B|+|C|-|G|$ (equivalently, $\delta(A, B, C)$ is equal to the sum of the sizes of any two of the sets minus the size of the complement of the third). If exactly one of A, B, C is infinite, then we let n be the size of the complement of this set, let ℓ, m be the sizes of the other two sets, and define $\delta(A, B, C) = \ell + m - n$. If two of A, B, C are infinite (in which case the third set is empty) then $\delta(A, B, C) = \infty$, and finally if G is infinite but A, B and C are all finite, then $\delta(A, B, C) = -\infty$. We will say that a trio (A, B, C) is *deficient* if $\delta(A, B, C) > 0$.

Observe that these definitions for trios naturally extend our notions for pairs. More precisely, if $A, B \subseteq G$ are finite, and $C = -\overline{A + B}$ then (A, B, C) is a trio and we have

- $\delta(A, B) = |A| + |B| - |A + B| = |A| + |B| - |\overline{C}| = \delta(A, B, C)$
- (A, B) is deficient if and only if (A, B, C) is deficient.
- (A, B) is trivial if and only if (A, B, C) is trivial.

The following easy observation gives a general connection between the deficiency of a pair of sets and the deficiency of a trio containing these two sets.

Observation 3.3. *If (A, B, C) is a trio and A, B are finite, then $\delta(A, B) \geq \delta(A, B, C)$ with equality precisely when $C = -\overline{A + B}$.*

Proof. By the definition of a trio, C must be disjoint from $-(A + B)$ which implies $C \subseteq -\overline{A + B}$. Thus $\delta(A, B) = |A| + |B| - |A + B| \geq |A| + |B| - |\overline{C}| = \delta(A, B, C)$ with equality precisely when $C = -\overline{A + B}$, as desired. □

Vosper's Theorem has a convenient restatement in terms of trios, as the extra symmetry in a trio eliminates one of the outcomes (and assuming nontriviality eliminates another).

Theorem 3.4 (Vosper, version II)**.** *If (A, B, C) is a nontrivial deficient trio in $\mathbb{Z}/p\mathbb{Z}$ and p is prime, then one of the following holds.*

1. $\min\{|A|, |B|, |C|\} = 1$.
2. *A, B, and C are arithmetic progressions with a common difference.*

Similar to pairs, we define the *supertrio* relation $(A, B, C) \subseteq (A^*, B^*, C^*)$ if $A \subseteq A^*, B \subseteq B^*$, and $C \subseteq C^*$, and call a trio (A, B, C) *maximal* if the only supertrio $(A^*, B^*, C^*) \supseteq (A, B, C)$ is (A, B, C) itself. By definition, every trio (A, B, C) must satisfy $C \subseteq -\overline{A + B}$, $B \subseteq -\overline{A + C}$ and $A \subseteq -\overline{B + C}$. If one of these containments is proper, say $C \subset -\overline{A + B}$, then we may replace C with the set

$-\overline{A+B}$ to obtain a proper supertrio. It follows that the trio (A, B, C) is maximal if and only if $C = -\overline{A+B}$, $B = -\overline{A+C}$ and $A = -\overline{B+C}$. Note that if we begin with an arbitrary trio (A, B, C) and perform this replacement operation on each of the three terms (if possible), the resulting trio will be maximal. In particular, every trio is contained in a maximal one. The next observation, which follows immediately from our definitions, highlights a key relationship between maximal pairs and maximal trios.

Observation 3.5. *Let $A, B, C \subseteq G$ and assume A, B are finite. Then the following are equivalent.*

1. *(A, B) is a maximal deficient pair and $C = -\overline{A+B}$.*
2. *(A, B, C) is a maximal deficient trio.*

The above observation further reduces the general classification problem to that of determining all maximal deficient trios. Next we give a version of Kneser's Theorem for trios which illustrates a key property of maximal deficient trios, and prove the equivalence of the two versions.

Theorem 3.6 (Kneser, version II)**.** *If (A, B, C) is a maximal deficient trio in G, then $G_A = G_B = G_C$ and $\delta(A, B, C) = |G_C|$.*

Proof of Equivalence. To see that version I implies version II, let (A, B, C) be a maximal deficient trio. If (A, B, C) is trivial, then one of A, B, C is empty and by maximality the other two must equal G; the result follows from this. Otherwise we may assume that A, B are both finite. It follows from maximality that $G_A = G_B = G_C$. Since $G_{A+B} = G_C$, the second part of Proposition 2.1 implies $\delta(A, B, C) = |A| + |B| - |\overline{C}| = |A| + |B| - |A + B| = |G_C|$ as desired.

For the other direction, let $A, B \subseteq G$ be finite and nonempty and define $C = -\overline{A+B}$ and $H = G_C = G_{A+B}$. Observe that every supertrio of (A, B, C) has the form (A^*, B^*, C), and choose (A^*, B^*, C) to be a maximal supertrio of (A, B, C). Note that by maximality $G_{A^*} = G_{B^*} = H$. Now version II of Kneser's Theorem implies $\delta(A^*, B^*, C) \leq |H|$ (as either (A^*, B^*, C) is deficient and this holds with equality, or (A^*, B^*, C) is not deficient and $\delta(A^*, B^*, C) \leq 0$). Since $A + H \subseteq A^*$ and $B + H \subseteq B^*$ we have $|A + H| + |B + H| - |A + B| \leq |A^*| + |B^*| - |\overline{C}| = \delta(A^*, B^*, C) \leq |H|$ as desired. □

4. Basic Deficient Trios

Note that if (A, B, C) is a trio, then any permutation of these three sets yields a new trio. In addition, for every $g \in G$ we have that $(A + g, B - g, C)$ is a trio. It is immediately seen that these operations preserve nontriviality, maximality, and

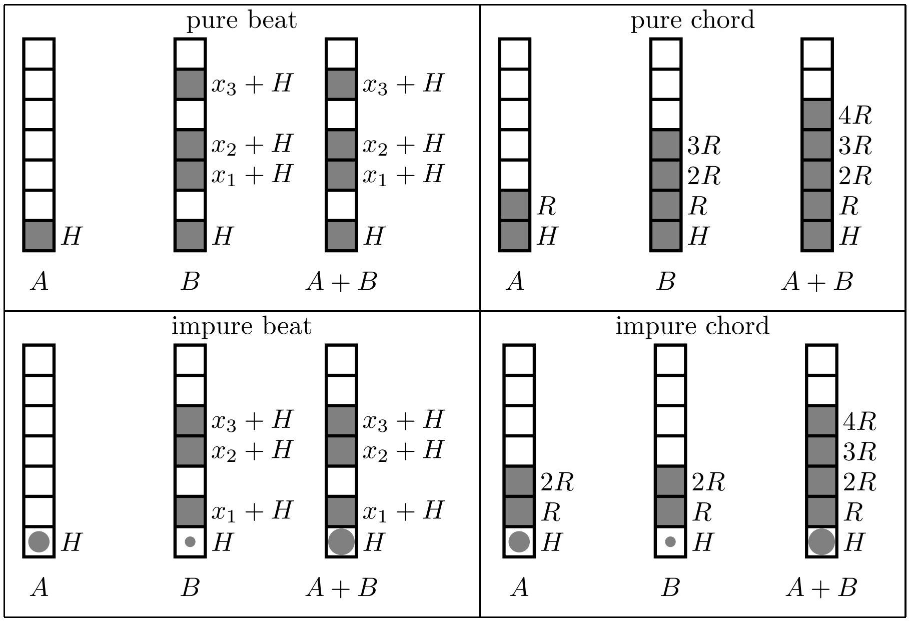

Figure 1: Structure Atlas

deficiency, and we say that two trios are *similar* if one can be turned into the other by a sequence of these operations.

Next we will introduce some terminology to describe the types of behaviour present in the structure of nontrivial maximal deficient trios. We begin with a structure which generalizes those deficient pairs (A, B) with $\min\{|A|, |B|\} = 1$ by allowing for subgroups.

Definition 4.1. Let $H < G$ be finite. A trio Υ is a *pure beat relative to* H if Υ is similar to a trio (A, B, C) which satisfies the following:

1. $A = H$,
2. $G_B = H$, and
3. $C = -\overline{A + B} \neq \emptyset$.

An *arithmetic progression* with *head* a, *difference* g, and *tail* $a + ng$ is a set of the form $\{a, a + g, \ldots, a + ng\}$. We say that the progression is *nontrivial* if it has size at least two.

Definition 4.2. Let $H < G$ be finite with G/H cyclic. A trio Υ is a *pure chord relative to* H, if there exist $R \in G/H$ which generates G/H and a trio (A, B, C) similar to Υ for which the following hold.

1. A and B are H-stable,

2. $\phi_{G/H}(A)$ and $\phi_{G/H}(B)$ are non-trivial arithmetic progressions in G/H with head H and difference R, and

3. $C = -\overline{A+B}$ is not contained in a single H-coset.

It follows immediately from our definitions and Observation 3.5 that every pure beat or pure chord relative to H is a maximal deficient trio with deficiency $|H|$.

For each of these two basic structures, there is a variant which allows for recursive constructions of maximal deficient trios. Before introducing these variants, we require another bit of terminology. For every set $A \subseteq G$ there is a unique minimal subgroup H for which A is contained in an H-coset. We denote this H-coset by $[A]$ and call it the *closure* of A. So, for instance, $[A] \in G/H$ means that A is contained in an H-coset, and is not contained in a coset of a smaller subgroup.

Definition 4.3. A trio Υ is an *impure beat relative to* $H < G$, if there is a trio (A, B, C) similar to Υ for which

1. $[A] = H$,

2. $B \setminus H$ is H-stable,

3. $C \setminus H = -\overline{A+B} \setminus H$, and

4. $A \neq \emptyset$, $B \cap H \neq \emptyset$, and $C \cap H \neq \emptyset$.

In this case $(A, B \cap H, C \cap H)$ is a trio in H which we call a *continuation* of Υ. Note that the subgroup H appearing in this definition is permitted to be infinite in contrast to the previous two definitions and the following one. It follows from our definitions that this continuation will be nontrivial and will be maximal whenever Υ was maximal. Furthermore, we have $|\overline{C \setminus H}| = |B \setminus H|$ from which it follows that the deficiency of the continuation is equal to that of the original trio Υ.

Definition 4.4. Let $H < G$ be finite and assume G/H is cyclic. A trio Υ is an *impure chord relative to* H, if there exist $R \in G/H$ which generates G/H and a trio (A, B, C) similar to Υ satisfying

1. $A \setminus H$ and $B \setminus H$ are H-stable,

2. $\phi_{G/H}(A)$ and $\phi_{G/H}(B)$ are nontrivial arithmetic progressions with head H and difference R,

3. $C \setminus H = -\overline{A+B} \setminus H \neq \emptyset$, and

4. $C \cap H$ is nonempty.

As before, $(A \cap H, B \cap H, C \cap H)$ is a trio in H which we call a *continuation* of Υ. Also as before, this continuation is nontrivial by definition, and will be maximal whenever Υ is maximal. Our definitions imply that $|\overline{C \setminus H}| = |A \setminus H| + |B \setminus H|$ and it follows from this that the deficiency of the continuation is equal to that of Υ.

With this, we can finally present the restatement of Kemperman's structure theorem than we are going to prove in this paper.

Theorem 4.5 (Kemperman)**.** *Let Υ_1 be a maximal nontrivial deficient trio in G_1. Then there exists a sequence of trios $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_m$ in respective subgroups $G_1 > G_2 > \cdots > G_m$ satisfying*

1. *Υ_i is an impure beat or an impure chord with continuation Υ_{i+1} for $1 \leq i \leq m-1$, and*
2. *Υ_m is either a pure beat or a pure chord.*

5. Incomplete Closure

In this section we focus our attention on deficient pairs and trios which contain a set A for which $[A] \neq G$. In particular, we shall prove a stability lemma which shows that every maximal deficient trio containing such a set must be a pure or impure beat. We begin with a lemma which was proved for general groups by Olson [11], but which follows from Kneser's Theorem for abelian groups (as observed by Lev [9]).

Lemma 5.1. *Let A, B be nonempty finite subsets of G and assume that $A + B \neq G$ and $[A] = G$. Then $|A + B| \geq \frac{1}{2}|A| + |B|$.*

Proof. By Theorem 1.2, $H = G_{A+B}$ satisfies $|A+B| \geq |A+H| + |B+H| - |H|$ and $H \neq G$ since $A + B \neq G$. Since A is not contained in any H-coset, $|A+H| - |H| \geq \frac{1}{2}|A+H| \geq \frac{1}{2}|A|$. Combining these two inequalities yields the desired bound. $\square$

For a set $A \subseteq G$ and a subgroup $H \leq G$, we say that A is H-*quasistable* if there exists $R \in G/H$ so that $A \setminus R$ is H-stable. Note that an H-stable set is also H-quasistable. Members of a pure beat or chord relative to $H < G$ are H-stable. The impure versions comprise H-quasistable sets, and their continuations are composed of partial H cosets.

Lemma 5.2. *Let (A, B) be a deficient pair and assume $A \neq \emptyset$, and $[A] \in G/H$ for some $H < G$. Then $A + B$ is H-quasistable. Furthermore, if H is finite, then $\delta(H, B) \geq \delta(A, B)$.*

Proof. By replacing A by $g+A$ for a suitable $g \in G$, we may assume that $[A] = H$. Let $R_1, \ldots, R_k \in G/H$ be the H-cosets which have nonempty intersection with B, and for every $1 \leq i \leq k$ let $B_i = B \cap R_i$. Now we have two inequalities,

1. $|A + B_i| \geq |B_i|$ and
2. if $A + B_i \neq R_i$, then $|A + B_i| \geq \frac{1}{2}|A| + |B_i|$,

the second of which follows from the previous lemma. Since $A + B$ is the disjoint union $\bigcup_{i=1}^{k}(A + B_i)$, it follows that there is at most one $1 \leq i \leq k$ for which $A + B_i \neq R_i$, so $A + B$ is H-quasistable.

For the last part, we may assume that H is finite and that $A + B_i = R_i$ for all $2 \leq i \leq k$. Since $|A + B_1| \geq |A|$ we find

$$\delta(A, B) = |A| + |B| - \sum_{i=1}^{k} |A + B_i| \leq |B| - (k-1)|H| = \delta(H, B)$$

which completes the proof. □

We are now ready to prove our stability lemma for trios which contain a set with closure not equal to G.

Lemma 5.3 (Beat Stability)**.** *If (A, B, C) is a nontrivial, maximal, deficient trio and $[A] \in G/H$ for some $H < G$, then (A, B, C) is either a pure or impure beat relative to H.*

Proof. By possibly moving from (A, B, C) to a similar trio, we may assume that $[A] = H < G$ and that B is finite. Note that A must also be finite, since otherwise H would be infinite and $G \setminus A$ would also be infinite. By Observation 3.5 (A, B) is a deficient pair, so by Lemma 5.2, $A + B$ is H-quasistable. If $A + B$ is H-stable, then H is finite and it follows from maximality that $A = H$ and $H = G_B = G_C$ so (A, B, C) is a pure beat. Otherwise, we may assume (by possibly passing to a similar trio) that $\emptyset \neq (A + B) \cap H \neq H$ and it then follows from maximality that (A, B, C) is an impure beat. □

6. Purification

In this section we will develop a process we call purification which will allow us to make a subtle modification to a deficient trio to obtain a new trio with deficiency no smaller than the original. This will be a key tool in the remainder of the paper.

We have already defined notions of deficiency for pairs of finite sets and for trios. It is also convenient to have a notion of deficiency for a single finite set. If $\emptyset \neq A \subset G$

is finite we define the *deficiency* of A to be

$$\delta(A) = \max_{B \subset G : A+B \neq G} \delta(A,B).$$

Here we only consider finite nonempty sets B. Note that this is indeed well defined since for every $B \subseteq G$ we have $\delta(A,B) = |A|+|B|-|A+B| \leq |A|$ so the maximum in the formula will be obtained. The following theorem of Mann shows that there is always a finite subgroup which achieves this maximum.

Theorem 6.1 (Mann)**.** *If $A \subset G$ is finite and nonempty, there exists a finite subgroup $H < G$ with $\delta(A,H) = \delta(A)$ and $A + H \neq G$.*

Proof. Choose $\emptyset \neq B \subseteq G$ so that $\delta(A,B) = \delta(A)$ and $A + B \neq G$. Now set $H = G_{A+B}$ and apply Kneser's Theorem to obtain

$$\begin{aligned} \delta(A,B) &= |A| + |B| - |A+B| \\ &\leq |A| + |B| - |A+H| - |B+H| + |H| \\ &\leq |A| - |A+H| + |H| \\ &= \delta(A,H). \end{aligned}$$

Finally, $A + H \subseteq A + B + H \neq G$ since $H = G_{A+B}$. □

Next we establish a lemma which is a key part of purification.

Lemma 6.2. *Let $H < G$ and $A \subset G$ be finite and assume (A,H) is deficient. If $B \neq \emptyset$, and $B \subseteq H$, then $\delta(A,B) \leq \delta(A,H)$.*

Proof. We may assume that (A,B) is deficient, as otherwise the result holds immediately. Choose $K \leq H$ so that $[B] \in G/K$ and note that Lemma 5.2 implies $\delta(A,B) \leq \delta(A,K)$. Since $K \leq H$, to complete the proof, it suffices to show $\delta(A,K) \leq \delta(A,H)$ under the assumption $K < H$.

Define $S = (A+H) \setminus A$ and let $S' = \{g \in S \mid g + K \subseteq S\}$ and $S'' = S \setminus S'$. Since (A,H) is deficient $|S'| \leq |S| = |A+H| - |A| < |H|$, and then we must have $|S'| \leq |H| - |K|$ (since $|S'|$, $|H|$, and $|K|$ are all multiples of $|K|$). Thus

$$\delta(A,H) = |H| - |S| \geq |K| - |S''| = |K| - |(A+K) \setminus A| = \delta(A,K)$$

which completes the proof. □

Lemma 6.3 (Trio Purification)**.** *Let (A,B,C) be a deficient trio in G, let $H \leq G$, and assume A and H are finite and (A,H) is deficient. If $R \in G/H$ satisfies $\emptyset \neq R \cap B \neq R$ and $S = -\overline{A+R}$, then $\delta(A, B \cup R, C \cap S) \geq \delta(A,B,C)$.*

Proof. Since (A,B,C) and (A,R,S) are trios, it follows that both $(A,B\cup R,C\cap S)$ and $(A,B\cap R,C\cup S)$ are trios. It is possible for B or C or neither to be infinite, but in all of these cases the following equations hold.

$$\delta(A,B\cup R,C\cap S)=\delta(A,B,C)+|R\setminus B|-|C\setminus S|$$
$$\delta(A,B\cap R,C\cup S)=\delta(A,R,S)-|R\setminus B|+|C\setminus S|.$$

Now summing these yields

$$\delta(A,B\cup R,C\cap S)+\delta(A,B\cap R,C\cup S)=\delta(A,B,C)+\delta(A,R,S).$$

Choose $g\in G$ so that $R=H+g$. Then by the previous lemma we have $\delta(A,B\cap R)=\delta(A,(B\cap R)-g)\leq\delta(A,H)=\delta(A,R)$. Combining this with Observation 3.3 then gives us $\delta(A,R,S)=\delta(A,R)\geq\delta(A,B\cap R)\geq\delta(A,B\cap R,C\cup S)$. Together with the above equation, this yields the desired result. □

The above lemma also applies to pairs to yield the following.

Lemma 6.4 (Pair Purification)**.** *Let $A,B\subseteq G$ and $H<G$ be finite and assume both (A,B) and (A,H) are deficient. Then for every $R\in G/H$ with $B\cap R\neq\emptyset$ we have $\delta(A,B\cup R)\geq\delta(A,B)$.*

Proof. Define $C=-\overline{A+B}$ and set $S=-\overline{A+R}$. Then by Observation 3.3 and the previous lemma,

$$\delta(A,B)=\delta(A,B,C)\leq\delta(A,B\cup R,C\cap S)\leq\delta(A,B\cup R).$$

□

7. Near Sequences

The goal of this section is to establish two important lemmas concerning a type of set called a near sequence. The first is a stability lemma which will show that whenever (A,B,C) is a maximal deficient trio with some additional properties, and A is a near sequence, then (A,B,C) must be a pure or impure chord. The second will show that whenever (A^*,B^*,C^*) is a pure or impure chord, of which (A,B,C) is a deficient subtrio, then every finite set among (A,B,C) must be a near sequence.

We begin by introducing a few important definitions. For this purpose we shall assume that $H<G$ is a finite subgroup and $R\in G/H$ generates the group G/H.

Definition 7.1. We say that $A\subseteq G$ is an *R-sequence* with *head* S and *tail* $S+kR$ if $A=\cup_{i=0}^{k}(S+iR)$.

Definition 7.2. We say that $A \subseteq G$ is a *near R-sequence* if $A+H$ is an R-sequence and $|(A+H) \setminus A| < |H|$.

Definition 7.3. We say that $A \subseteq G$ is a *fringed R-sequence* if

1. $A+H$ is an R-sequence, and
2. if $A+H$ has head S and tail T, then either $A \setminus S$ or $A \setminus T$ is H-stable.

If A is an R-sequence, near R-sequence, or fringed R-sequence, we say that A is *proper* if $|\overline{A}| \geq 2|H|$, and we call it *nontrivial* if $|A| > |H|$. Note that each finite set in a pure chord is a sequence and each finite set in an impure chord is a fringed sequence. Next we prove a technical lemma where fringed sequences emerge.

Lemma 7.4. *Let (A, B) be a nontrivial deficient pair, and assume that A is a nontrivial near R-sequence for $R \in G/H$, and that B is not contained in any H-coset. If there exists an R-sequence B^* with $B \subseteq B^*$ and $A + B^* \neq G$, then $A + B$ is a fringed R-sequence.*

Proof. Suppose (for a contradiction) that the lemma fails, and let A, B be a counterexample for which $|B|$ is minimum. By shifting A (i.e., replacing A by $A+g$ for some $g \in G$) and B we may assume that $A + H = \bigcup_{i=0}^{\ell} iR$ and $B^* = \bigcup_{i=0}^{m} iR$. For convenience let us define $A_i = A \cap iR$ and $B_i = B \cap iR$ for every $i \in \mathbb{Z}$. By replacing B^* with a smaller R-sequence, we may assume that $B_0 \neq \emptyset$ and $B_m \neq \emptyset$. We first prove a series of three claims.

Claim 1. *$B_i \neq \emptyset$ for $0 \leq i \leq m$.*

It follows from repeatedly applying our pair purification lemma to H-cosets $R \in G/H$ for which $\emptyset \neq R \cap B \neq R$ that $(A, B+H)$ is deficient and thus $(A+H, B+H)$ is deficient. It follows from this that the sets $\tilde{A}, \tilde{B} \subseteq \mathbb{Z}$ given by $\tilde{A} = \{0, 1, \ldots, \ell\}$ and $\tilde{B} = \{i \in \mathbb{Z} \mid iR \cap B \neq \emptyset\}$ satisfy $(\tilde{A}, \tilde{B})$ deficient. It follows, e.g., from Lemma 1.3 of Nathanson[10], that $\tilde{B}$ is the interval $\{0, 1, \ldots m\}$ which implies the claim.

Claim 2. *$A + B$ does not contain $\bigcup_{i=1}^{\ell+m-1} iR$.*

Suppose for a contradiction that this claim fails. Let $K_0 = G_{A_0+B_0}$ and $K_1 = G_{A_\ell+B_m}$. We have $K_0, K_1 < H$ since $A+B$ is not a fringed sequence. Now applying Kneser's Theorem to the sumsets $A_0 + B_0$ and $A_\ell + B_m$ we find

$$\begin{aligned}
|A+B| &= (\ell+m-1)|H| + |A_0+B_0| + |A_\ell+B_m| \\
&\geq (\ell+m-1)|H| + |A_0| + |A_\ell| + |B_0| + |B_m| - |K_0| - |K_1| \\
&\geq ((\ell-1)|H| + |A_0| + |A_\ell|) + ((m-1)|H| + |B_0| + |B_m|) \\
&= |A| + |B|
\end{aligned}$$

which gives us the desired contradiction.

Claim 3. $m = 1$

As usual, we suppose for a contradiction that $m > 1$. First consider the set $B' = B \setminus B_m$. It follows easily from the inequality $|A_\ell + B_m| \geq |B_m|$ that $\delta(A, B') \geq \delta(A, B) > 0$ so (A, B') is deficient. By the minimality of our counterexample, it follows that $A+B'$ must contain $\bigcup_{i=1}^{\ell+m-2} iR$. Similarly, $(A, B \setminus B_0)$ is a deficient pair and hence contains $\bigcup_{i=2}^{\ell+m-1} iR$. Putting these together, we find $\bigcup_{i=1}^{\ell+m-1} iR \subseteq A+B$, contradicting Claim 2.

With these claims in place, we are ready to complete the proof. By the pair purification lemma, $(A, B \cup R)$ and $(A, B \cup H)$ are deficient. Thus

$$\begin{aligned} 0 < \delta(A, B \cup R) &= |A| + |B_0| - \ell|H| - |A_0 + B_0| \qquad (2) \\ 0 < \delta(A, B \cup H) &= |A| + |B_1| - \ell|H| - |A_\ell + B_1| \qquad (3) \end{aligned}$$

We also have

$$|A| - |A_0 + B_0| - |A_\ell + B_1| \leq |A| - |A_0| - |A_\ell| \leq (\ell - 1)|H| \qquad (4)$$

Now summing equations (2) and (3) and substituting (4) yields

$$0 < |A| + |B_0| + |B_1| - (\ell + 1)|H|. \qquad (5)$$

It follows from Claim 2 that we may choose a point $z \in \left(\bigcup_{i=1}^{\ell} iR\right) \setminus (A + B)$. Assuming $z \in iR$, we have $B_0 \cap (z - A_i) = \emptyset$ and $B_1 \cap (z - A_{i-1}) = \emptyset$. These together with $|A \setminus (A_{i-1} \cup A_i)| \leq (\ell - 1)|H|$ then imply

$$|B_0| + |B_1| \leq 2|H| - |A_{i-1}| - |A_i| \leq (\ell + 1)|H| - |A| \qquad (6)$$

Inequalities (5) and (6) are contradictory, and this completes the proof. $\square$

Lemma 7.5. *Let (A, B, C) be a nontrivial deficient trio with A, B finite, and assume that every supertrio $(A, B^*, C^*) \supseteq (A, B, C)$ satisfies $(A, B^*, C^*) = (A, B, C)$. Let $H < G$, let $R \in G/H$ and assume A is a nontrivial proper near R-sequence. If neither B nor C is contained in an H-coset, then B and $\overline{C}$ are fringed R-sequences.*

Proof. Suppose (for a contradiction) that there is a counterexample to the lemma using the set A, and then choose B and C so that (A, B, C) is a counterexample for which

1. $\delta(A, B, C)$ is maximum.
2. $|\{S \in G/H \mid \emptyset \neq S \cap B \neq S\}| + |\{S \in G/H \mid \emptyset \neq S \cap C \neq S\}|$ is minimum. (subject to 1).

Observe that if B is a fringed R-sequence, we can automatically conclude that $\overline{C} = -(A + B)$ is a fringed R-sequence by the maximality of C.

Claim 1. *There does not exist an R-sequence D with $A + D \neq G$ so that $B \subseteq D$ or $C \subseteq D$. So, in particular, G must be finite.*

If such a set D exists with $B \subseteq D$, then by applying the previous lemma we deduce that $A + B$ is a fringed R-sequence. But then, the maximality of B implies that B is a fringed R-sequence. Similarly, if such a set D exists with $C \subseteq D$, then G is finite and $A + C$ is a fringed R-sequence. But then $B = -\overline{A + C}$ is also a fringed R-sequence.

Claim 2. *There does not exist $S \in G/H$ with $S \subseteq B$ or with $S \subseteq C$.*

If $S \in G/H$ satisfies $S \subseteq B$ then Claim 1 is violated by $D = -\overline{S + A}$ since $C \subset D$. A similar argument holds if $S \subseteq C$.

Since G is finite by Claim 1, we will show that both B and C (hence $\overline{C}$) are fringed R-sequences. Without loss of generality, $|C| \geq |B|$. In particular, $|C| > |H|$ since $|G \setminus A| \geq 2|H|$ and (A, B, C) is deficient.

Using Claim 2, choose an H-coset $S \in G/H$ so that $\emptyset \neq B \cap S \neq S$. Now setting $B' = B \cup S$ and $C' = C \cap -\overline{A + S}$, Lemma 6.2 implies $\delta(A, B', C') \geq \delta(A, B, C)$ and it follows that $|C \setminus C'| \leq |B' \setminus B| < |H|$. Since $|C| > |H|$ we have that $C' \neq \emptyset$, so (A, B', C') is a nontrivial deficient trio. Choose B'', C'' maximal so that (A, B'', C'') is a supertrio of (A, B', C').

If $B'' \neq B'$ or $C'' \neq C'$ then $\delta(A, B'', C'') > \delta(A, B, C)$ so by the first criteria in our choice of counterexmple, the lemma holds for (A, B'', C''). On the other hand, if $B'' = B'$ and $C'' = C'$ then the quantity in our second optimization criteria improves, so again we find that the lemma holds for (A, B'', C'').

If C'' is not contained in a single H-coset, then B'' is a fringed sequence, but then Claim 1 is violated by $D = B'' + H$ since $B \subset D$. So, we may assume that $C'' \subseteq T$ for some $T \in G/H$. Now let $U \in G/H$ satisfy $U \subseteq -\overline{A + T}$ and suppose for a contradiction that $B \cap U = \emptyset$. In this case $(A, B' \cup U, C')$ is a trio and our pair purification lemma implies

$$\delta(A, B', C') + |H| = \delta(A, B' \cup U, C') \leq \delta(A, C') \leq \delta(A, H) \leq |H|$$

which is a contradiction. It follows that every H-coset contained in $-\overline{A + T}$ must have nonempty intersection with B.

With this knowledge, we now return to our original trio (A, B, C) and modify it to form a new trio by setting $C''' = C \cup T$ and $B''' = B \cap -\overline{A + T}$. It follows from our trio purification lemma that (A, B''', C''') is a trio with $\delta(A, B''', C''') \geq \delta(A, B, C)$. Furthermore, since $-\overline{A + T}$ contains at least two H-cosets, the set B''' cannot be contained in a single H-coset. Now, by repeating the argument from above, we may extend (A, B''', C''') to a trio with the second and third sets maximal, and then the lemma will hold for this new trio. This then implies that C''' is contained in an R-sequence which violates Claim 1. This completes the proof. $\square$

Lemma 7.6 (Sequence Stability)**.** *Let (A, B, C) be a maximal deficient trio with $[A] = [B] = [C] = G$. If A is a proper near sequence, then (A, B, C) is either a pure or an impure chord.*

Proof. Let A be a proper near R-sequence for $R \in G/H$ and assume (without loss) that B is finite. By the previous lemma we deduce that B is a fringed R-sequence. However, then by maximality A is also a fringed R-sequence. Again using maximality, we conclude that (A, B, C) is either a pure or impure chord relative to H. $\square$

Lemma 7.7. *Let (A, B, C) be a deficient trio of which (A^*, B^*, C^*) is a maximal deficient supertrio. If (A^*, B^*, C^*) is a pure or impure chord, and A is finite, then A is a proper near sequence.*

Proof. If (A^*, B^*, C^*) is a pure chord relative to $H \leq G$ then $\delta(A^*, B^*, C^*) = H$ and since (A, B, C) is deficient, we must have $|A^* \setminus A| < |H|$. Since A^* is finite, it is a proper R-sequence for some $R \in G/H$ and it follows immediately that A is a near R-sequence. Next suppose that (A^*, B^*, C^*) is an impure chord relative to $H \leq G$. In this case, there exists a subgroup $K < H$ so that $K = G_{A^*} = G_{B^*} = G_{C^*}$. Now since (A, B, C) is deficient, it follows that $|A^* \setminus A| < |K|$. Since A^* is a proper fringed R-sequence for some $R \in G/H$, we again find that A is a proper near R-sequence. $\square$

8. Proof

In this section we prove Kemperman's Theorem.

Proof of Theorem 4.5. Suppose (for a contradiction) that the theorem fails and let (A, B, C) be a counterexample with $|A| \leq |B| \leq |C|$ so that

1. if there is a finite counterexample, then $|G|$ is minimum;
2. $\overline{C}$ is minimum (subject to 1);
3. the number of terms in $([A], [B], [C])$ equal to G is maximum (subject to 1, 2).

We shall establish properties of our trio with a series of claims.

Claim 1. *The group $H = G_A = G_B = G_C$ is trivial.*

Otherwise we obtain a smaller counterexample by passing to the quotient group G/H and the trio $\big(\varphi_{G/H}(A), \varphi_{G/H}(B), \varphi_{G/H}(C)\big)$.

Claim 2. *None of the sets A, B, or C is contained in a proper coset.*

Suppose for a contradiction that A, B or C is contained in a proper coset, and apply Lemma 5.3. If (A, B, C) is a pure beat, we have an immediate contradiction. Otherwise, we may assume that (A, B, C) is an impure beat relative to H with continuation (A', B', C'). If H is finite, then (A', B', C') contradicts our choice of (A, B, C) for the first criteria. Otherwise H is infinite, and we may assume (without loss) that C' is infinite.

Now, A and B are finite and H-quasiperiodic, so both A and B are both contained in a single H-coset. Hence criteria (1) and (2) agree on (A, B, C) and (A', B', C'). Furthermore, only one term in $([A], [B], [C])$ is equal to G, but by construction, one of A', B' has closure H, and $[C'] = H$ (since C' is cofinite). Therefore (A', B', C') is a counterexample which contradicts our choice.

Claim 3. *A is not a proper near sequence.*

Otherwise it follows from Lemma 7.6 that either (A, B, C) is a pure or impure chord. In the former case we have an immediate contradiction. In the latter a continuation (A', B', C') contradicts our choice of (A, B, C).

Claim 4. *If $D \subseteq G$ satisfies (A, D) deficient and $|D| > \frac{1}{2}|A|$ then $[D] = G$.*

Suppose (for a contradiction) that (A, D) is deficient and $|D| > \frac{1}{2}|A|$ and that $[D] = H + x$ for $H < G$. If A contains points in at least three H-cosets, then $A + D$ is H-quasiperiodic by Lemma 5.2 so $|A + D| \geq 2|H| + |D| \geq 3|D| \geq |D| + |A|$ which contradicts the assumption that (A, D) is deficient. It then follows from Claim 2 that A must contain points in exactly two H-cosets. Now, if $|A| \leq |H|$ then we have $|A + D| \geq |H| + |D| \geq |A| + |D|$ which is contradictory. Otherwise, $|A| > |H|$ and A contains points in exactly two H-cosets, but then A is a near R-sequence for some $R \in G/H$ and this contradicts Claim 3.

Claim 5. *There does not exist a nontrivial finite subgroup $H < G$ so that (A, H) is deficient.*

Suppose for a contradiction that (A, H) is deficient with $\{0\} < H < G$. By Claim 1 we may choose an H-coset $R \in G/H$ so that $\emptyset \neq C \cap R \neq R$. Now setting $C' = C \cup R$ and $B' = B \cap -\overline{A + R}$ our trio purification lemma implies $\delta(A, B', C') \geq \delta(A, B, C)$. It follows from this that $0 \leq |C' \setminus C| - |B \setminus B'| < |H| - |B \setminus B'|$. If $|A + H| = 2|H|$ then A is a near sequence which contradicts Claim 3. Therefore, we have $|A| + |H| > |A + H| \geq 3|H|$. This gives us $|B'| > |B| - |H| \geq |A| - |H| \geq \frac{1}{2}|A|$ so by Claim 4 we have that $[B'] = G$.

Now we let (A^*, B^*, C^*) be a maximal supertrio of (A, B', C'). Since (A^*, B^*, C^*) is maximal with $\delta(A^*, B^*, C^*) \geq \delta(A, B', C') \geq \delta(A, B, C)$ and $|\overline{C^*}| < |\overline{C}|$, the theorem holds for (A^*, B^*, C^*). Therefore, (A^*, B^*, C^*) must either be a pure or impure chord (since $[A] = [B] = [C] = G$). Now Lemma 7.7 implies that A is a proper near sequence, but this contradicts Claim 3.

Claim 6. *Let $D \subseteq G$ be finite and assume that (A, D) is nontrivial and deficient. Then $\delta(A, D) = 1$ and further, either $|D| = 1$ or $[D] = G$.*

It follows immediately from Claim 5 and Mann's theorem that $\delta(A, D) = 1$. Suppose for contradiction that $[D] = H + x$ for $\{0\} < H < G$. Then Lemma 5.2 implies that $A + D$ is H-quasistable. Since $[A] = G$, it follows that H is finite. Again, by Lemma 5.2, $\delta(A, H) \geq \delta(A, D)$, contradicting Claim 5.

Claim 7. *B is not a Sidon set: $|(g + B) \cap B| > 1$ for some $g \in G \setminus \{0\}$.*

Suppose (for a contradiction) that B is a Sidon set. We must have $|A| \geq 3$ as otherwise either $[A] \neq G$ or A is a near sequence. Choose distinct elements $a_1, a_2, a_3 \in A$. Now we have

$$|A + B| \geq |(a_1 + B) \cup (a_2 + B) \cup (a_3 + B)| \geq 3|B| - 3 \geq |A| + |B|$$

which is contradictory.

With this last claim in place, we are now ready to complete the proof. Since B is not a Sidon set, we may choose $g \in G \setminus \{0\}$ so that $B' = B \cap (g + B)$ satisfies $|B'| \geq 2$. Set $C' = C \cup (-g + C)$ and $B'' = B \cup (g + B)$ and $C'' = C \cap (-g + C)$. It now follows from basic principles that (A, B', C') and (A, B'', C'') are trios and

$$\delta(A, B', C') + \delta(A, B'', C'') = 2\delta(A, B, C). \tag{7}$$

If $C'' = \emptyset$ then G must be finite and we have $|C'| = 2|C|$, so

$$\begin{aligned} \delta(A, B', C') &= |A| + |B'| + |C'| - |G| \\ &\geq |A| + 2 + 2|C| - |G| \\ &> |A| + |B| + |C| - |G| \\ &= \delta(A, B, C) \end{aligned}$$

which contradicts Claim 6. Therefore $C'' \neq \emptyset$ and then both (A, B', C') and (A, B'', C'') are nontrivial. Then, (7) and Claim 6 imply that $\delta(A, B', C') = \delta(A, B'', C'') = 1$ and that (A, B', C') and (A, B'', C'') are both maximal. Since $|G \setminus C'| < |G \setminus C|$ the theorem holds for the trio (A, B', C'). Since $|B'| \geq 2$, Claim 6 implies that $[B'] = G$. However, then (A, B', C') must be a pure or impure chord, and then Lemma 7.7 implies that A is a near sequence which contradicts Claim 3. This completes the proof. $\square$

Acknowledgements The authors are extremely grateful to a wonderful anonymous referee who not only caught an embarrassing number of mistakes, but also made numerous insightful suggestions which dramatically improved the clarity and quality of this article.

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