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FACTOR - UNION REPRESENTATION OF PHENOTYPE SYSTEMS

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Abstract: An algorithm is provided in order to decide whether a given phenotype system is a factor-union system and to construct a corresponding factor-union representation (if such a representation exists).

Factor-union phenotype systems were introduced by Cotterman (cf. [1]). A characterization of such systems was given in [2]. In [4] Markowsky provided an algorithm in order to decide whether a given phenotype system is a factor-union system and to construct a corresponding factor-union representation (if such a representation exists). The aim of this paper is to present another such algorithm which is very simple.

In the following let G denote a fixed non-empty finite set (of genes) and put $M := \{A \in 2^G | 1 \leq |A| \leq 2\}$. The following two definitions are essentailly a "more algebraic" reformulation of the corresponding definitions originally given by Cotterman (cf. [1]).

Definition 1. By a *phenotype system* (with respect to G) one means an equivalence relation on M.

In the following let α denote a fixed phenotype system.

Definition 2. By a *factor-union representation* (FU-*representation*) of α on means an ordered pair (F, f) where F is some set (of so-called "factors") and f is a mapping from G to 2^F such that $\{(A, B) \in \in M^2 | \bigcup_{x\in A} f(x) = \bigcup_{x\in B} f(x)\} = \alpha\}$. α is called a *factor-union system* (FU-*system*) if there exists an FU-representation of α.

In the following, for every $\beta \subseteq (2^G \setminus \{\emptyset\})^2$ let $<\beta>$ denote the congruence on $(2^G \setminus \{\emptyset\}, \cup)$ generated by β.

Remark 1. $<\alpha>$ is the transitive hull of $\{(A \cup C, B \cup C) | (A, B) \in \in \alpha; C \subseteq G\}$.

Remark 2. Let (F, f) be an FU-representation of α. Since $(2^G \setminus \{\emptyset\}, \cup)$ is the free semilattice with free generating set G (every $g \in G$ is here identified with the one-element set $\{g\}$), f can be extended to a homomorhism g from $(2^G \setminus \{\emptyset\}, \cup)$ to $(2^F, \cup)$. Now $\ker g$ is a congruence on $(2^G \setminus \{\emptyset\}, \cup)$ and $< (\ker g) \cap M^2 > \subseteq \ker g$. Here, in general, equality does not hold as can be seen from the following example: Put $G := \{a, b, c, d\}$ (a, b, c, d mutually distinct), $F := \{1, 2, 3, 4, 5\}$ and $f := \{(a, \{1, 2\}), (b, \{2, 3\}), (c, \{3, 4\}), (d, \{1, 2, 3, 5\})\}$. Then (F, f) is an FU-representation of $\{\{a\}\}^2 \cup \{\{b\}\}^2 \cup \{\{c\}\}^2 \cup \{\{d\}, \{a, d\}, \{b, d\}\}^2 \cup \{\{a, b\}\}^2 \cup \{\{a, c\}\}^2 \cup \{\{b, c\}\}^2 \cup \{\{c, d\}\}^2$, $< (\ker g) \cap M^2 >= = \{\{a\}\}^2 \cup \{\{b\}\}^2 \cup \{\{c\}\}^2 \cup \{\{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}^2 \cup \{\{a, b\}\}^2 \cup \cup \{\{a, c\}\}^2 \cup \{\{b, c\}\}^2 \cup \{\{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}^2 \cup \{\{a, b, c\}\}^2$ and $\ker g =< (\ker g) \cap M^2 > \cup \{\{a, c\}, \{a, b, c\}\}^2$.

Definition 3. By a *(join-)semilattice* on means a commutative idempotent semigroup, i.e. an algebra $(S, \vee)$ of type 2 in which the laws $x \vee y = y \vee x$, $x \vee x = x$ and $(x \vee y) \vee z = x \vee (y \vee z)$ hold. The corresponding partial order $\leq$ is defined by $x \leq y$ iff $x \vee y = y$ $(x, y \in S)$. $a \in S$ is called *meet-irreducible* if for all $x, y \in S$ with $x \wedge y = a$ it holds $a \in \{x, y\}$ ($\wedge$ denotes the infimum with respect to $\leq$).

The aim of this paper is to prove the following

Theorem.

(i) *α is an* FU-*system iff* $<\alpha> \cap M^2 = \alpha$.

(ii) *If* $<\alpha> \cap M^2 = \alpha$ *and if L denotes the set of all meet-irreducible elements of the (join-)semilattice* $(2^G \backslash \{\emptyset\}, \cup)/ <\alpha>$ *then* $(L, \{(x, \{y \in L | y \not\geq [\{x\}] <\alpha>\}) | x \in G\})$ *is an* FU-*representation of* α.

The proof of the theorem makes use of the following

Proposition (cf. [3]). *Let $(S, \vee)$ be a finite join-semilattice and let L denote the set of all meet-irreducible elements of $(S, \vee)$. Then* $\{(x, \{y \in \in L | y \not\geq x\}) | x \in S\}$ *is an injective homomorphism from $(S, \vee)$ to* $(2^L, \cup)$.

Proof. Put $f := \{(x, \{y \in L | y \not\geq x\}) | x \in S\}$. Since S is finite, every element a of S is the meet of elements of L and hence the meet of all elements of L which are $\geq a$. This shows injectivity of f. A straightforward calculation yields $f(x \vee y) = f(x) \cup f(y)$ for all $x, y \in S$.

Proof of the Theorem. First assume α to be an FU-system. Then there exists an FU-representation (F, f) of α. Let g denote the homomorphism from $(2^G \backslash \{\emptyset\}, \cup)$ to $(2^F, \cup)$ extending f. (This homomorphism exists according to Remark 2 following Definition 2.) Then $\ker g$ is a congruence on $(2^G \backslash \{\emptyset\}, \cup)$ and since (F, f) is an FU-representation of α, $\alpha = (\ker g) \cap M^2$. Because of $\alpha \subseteq \ker g$ we have $<\alpha> \subseteq \ker g$ and hence $\alpha \subseteq <\alpha> \cap M^2 \subseteq (\ker g) \cap M^2 = \alpha$ which shows $<\alpha> \cap M^2 = \alpha$. Conversely, assume $<\alpha> \cap M^2 = \alpha$. Let L denote the set of all meet-irreducible elements of $(2^G \backslash \{\emptyset\}, \cup)/ <\alpha>$ and put $h := \{(x, \{y \in L | y \not\geq x\}) | x \in (2^G \backslash \{\emptyset\})/ <\alpha>\}$. Then, according to the above proposition, h is an injective homomorphism from $((2^G \backslash \{\emptyset\})/ <\alpha>, \cup)$ to $(2^L, \cup)$. Now for all $A, B \in M$ the following are equivalent: $\bigcup_{x \in A} h([\{x\}] <\alpha>) = \bigcup_{x \in B} h([\{x\}] <\alpha>)$, $h([A] <\alpha>) = h([B] <\alpha>)$, $[A] <\alpha> = [B] <\alpha>$, $(A, B) \in <\alpha>$, $(A, B) \in \alpha$. This completes the proof of the theorem.

Remark. The FU-systems (with respect to G) are exactly the restrictions of the congruences of $(2^G \backslash \{\emptyset\}, \cup)$ (or of $(2^G, \cup)$) to M; for let β be some congruence of $(2^G \backslash \{\emptyset\}, \cup)$ and suppose $\alpha = \beta \cap M^2$. Then $\alpha \subseteq \beta$ and hence $<\alpha> \subseteq \beta$. From this we conclude $\alpha \subseteq <\alpha> \cap M^2 \subseteq \subseteq \beta \cap M^2 = \alpha$ whence $\alpha = <\alpha> \cap M^2$, i.e. α is an FU-system.

The above theorem gives rise to the following

Algorithm. Construct the undirected graph (with vertex-set $2^G \setminus \{\emptyset\}$) corresponding to α. Next construct in an obvious graph-theoretical manner $<\alpha>$ as the transitive hull of $\{(A \cup C, B \cup C) | (A, B) \in \alpha; C \subseteq \subseteq G\}$. Then check if $<\alpha> \cap M^2 = \alpha$. If this is the case then construct the Hasse-diagram of $(2^G \setminus \{\emptyset\}, \cup) / <\alpha>$ in order to obtain the FU-representation of α described within the above theorem.

Example (human ABO blood group system). Put $G := \{A, B, O\}$ and $\alpha := \{\{A\}, \{A, O\}\}^2 \cup \{\{B\}, \{B, O\}\}^2 \cup \{\{O\}\}^2 \cup \{\{A, B\}\}^2$. Then $<\alpha> = \alpha \cup \{\{A, B\}, \{A, B, O\}\}^2$ and hence $<\alpha> \cap M^2 = \alpha$. Therefore α is an FU-system. The above theorem yields the FU-representation $(\{a, b, c\}, \{(A, \{b\}), (B, \{a\}), (O, \emptyset)\})$ of α where $a := [\{A\}] <\alpha>$, $b := := [\{B\}] <\alpha>$ and $c := [\{A, B\}] <\alpha>$.

Remark. From the above theorem it follows that if α is an FU-system (with respect to G) then there exists an FU-representation (F, f) of α with $|F| \leq 2^{|G|} - 1$. Knowing this, the problem whether α is an FU-system or not can be decided in a finite number of steps. (Take an arbitrary fixed set of cardinality $2^{|G|} - 1$ as the set of possible factors.)

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