Mathematica Pannonica
2/2 (1991), 25 – 36

ON REFLEXIVE SHEAVES WITH LOW SECTIONAL GENERA ON THREEFOLDS

Edoardo **Ballico**

Department of Mathematics, University of Trento, I-38050 Povo (Trento), Italy.

Received January 1990

AMS Subject Classification: 14 F 05; 14 J 30, 14 M 99

Keywords: reflexive sheaf, vector bundle, ampleness, curvilinear sheaf, sectional genus, adjoint map, projective 3-fold, projective variety.

Abstract: Here we define the sectional genus $g(F)$ of a reflexive sheaf F over a projective variety V. Here we classify (char 0) all such pairs (V,F) with $\dim(V)=3$, V smooth, F not locally free but curvilinear, F ample and spanned, and $2g(F)-2\leq c_3F$; we have $V\cong\mathbf{P}^3$, all such F are described explicitely and the set of such F is parametrized by $\mathbf{P}^3$.

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K})=0$. Fix a complete variety V; set $n=\dim(V)$; here we need only the case $n=3$. Recall that a coherent sheaf F on V is called reflexive if the natural map from F to its double dual F^{**} is an isomorphism; this is the case for instance if F is locally free, but it happens in several other interesting cases: see [6] for the background, motivation (i.e. their link with space curves) and the general theory of reflexive sheaves. We fix a rank$-n(n-1)$ reflexive sheaf E on V. We say (as usual) that E is ample if the tautological line bundle $O_{\mathbf{P}}(E)(1)$ on $\mathbf{P}(E)$ is ample. From now on we will assume that E is spanned by its global sections and that the set S of points of V at which E is not locally free is finite (if $n=3$ the last condition is automatically statisfied ([6], 1.4)). Then (since char($\mathbf{K}$)= 0 and S is finite) a standard form of Bertini theorem (same proof as for $n = 3$ ([6]), which in turn is essentially a standard form of Bertini for spanned vector bundles on $V\backslash S$) gives that for a general $s \in H^0(V, E)$, its zero-locus $C := (s)_0$ is a pure-dimensional curve which is smooth outside S (and C must contain S, as shown in [6], Th. 4.1); if V is smooth around S, C is locally Cohen-Macaulay; since C is generically reduced by the finiteness of S, C is reduced; set $g := p_a(C)$. We will call g the sectional genus of E (see the introduction of [1] for a discussionn of why (in the locally free case) among many other competing ones, this is a very natural and useful definition). But we do not claim or use that if E is ample the general such C is integral of at least connected; in particular we will consider also the case "$g < 0$" (which often will be easily shown to lead to a contradiction). From now on in this paper we assume V smooth. In particular $K_0(V) \cong K^0(V)$ and the Chern classes are defined for all coherent sheaves on V. Set $L := c_1(E)$; L is a line bundle; note also that $C = (s)_0$ represents $c_{n-1}(E)$ (exactly as in the locally free case, to which, when S is finite, it could be easily reduced). From now on we assume $n = 3$. The aim of this paper is the proof of Theorem 0.1 below. To clarify its statement we need the following "adjunction formula" ([6], th. 4.1) valid for any section s with $C := (s)_0$ of dimension 1 and arithmetic genus g:

$$2g - 2 = (K_V + L)C + c_3(E) \tag{1}$$

It is known (see [6], Prop. 2.6) that $c_3(E) \geq 0$, $c_3(E) > 0$ if and only if $S \neq \emptyset$, and that indeed $c_3(E)$ is a very good measure of the "number" of singularities of E. A reflexive sheaf E on V is called curvilinear if for each $P \in S$ there are formal parameters x, y, z of the completion A of $O_{V,P}$ such that the completion of the stalk of E at P is isomorphic to Coker(j), where $j : A \to 3A$ is defined by: $j(u) = (xu, yu, zy)$; by [6], 4.1.1, if there is $s \in H^0(E)$ with as $(s)_0$ a smooth curve, then E is curvilinear; by [3], Prop. 4, if E is curvilinear and spanned, a general $s \in H^0(E)$ has $(s)_0$ smooth.

Theorem 0.1. *Let V be a smooth complete variety with* $\dim(V) = 3$, *and E an ample spanned* rank $- 2$ *reflexive sheaf on V with sectional genus g; set $c_i = c_i(E)$. Assume $2g - 2 \leq c_3$ and E not locally free. Then there are exactly* 4 *families of (V, E):*

(i) $V \cong \mathbf{P}^3$, $L \cong O(3)$, $g = 0$, $c_3 = 1$ *and E is described in the following way.* Fix$P \in \mathbf{P}^3$ *and consider homogeneous coordinates $x_0, \dots, x_3$, such* *that* $P = (1;0;0;0)$. *Let* E *be the cokernel of the map* $j : O_V \to 3O_V(1)$ *given by* $j(c) = (cx_1, cx_2, cx_3)$; *then* E *is a solution; any solution with these invariants differs from* E *by the action of an element of* $\mathrm{Aut}(\mathbf{P}^3)$; *any two solutions are isomorphic if and only if they have the same singular set,* P.

(ii) $V \cong \mathbf{P}^3$, $L \cong O(4)$, $c_2 = 7$, $c_3 = 2g - 2 = 8$; *furthermore* $h^0(E(-1)) \neq 0$, *and, for general* E, *a general* $m \in H^0(E(-1))$ *has as* $(m)_0$ *the complete intersection of 2 quadrics.*

(iii) $V \cong \mathbf{P}^3$, $L \cong O(4)$, $c_2 = 6$, $c_3 = 2g - 2 = 4$; *furthermore* $h^0(E(-1)) \neq 0$ *and, for general* E, *a general* $m \in H^0(E(-1))$, *has as* $(m)_0$ *a rational normal curve.*

(iv) V *is a smooth quadric in* $\mathbf{P}^4$, $L \cong O(3)$, $c_2 = 6$, $c_3 = 2g - 2 = 2$; *furthermore* $h^0(E(-1)) \neq 0$ *and a general* $m \in H^0(E(-1))$ *has as* $(m)_0$ *a plane conic.*

In particular the space of solutions in case (i) is parametrized by $\mathbf{P}^3$. We will give some more informations on the possible sheaves E's in cases (ii), (iii) and (iv) in §2, repsectively in (β) case 5) (β), case 5), and (γ); here suffice to say that by the general recipe in [6], Th. 4.1, the datum $(m)_0$ (plus a suitable divisor on $(m)_0$) is sufficient to reconstruct E.

At the end of the paper we discuss briefly the case E not "curvilinear".

The proof of 0.1 depends heavily on [1] (which in turn depends on [11], hence on Mori's theory and its applications ([8], plus the classifications of Fano threefolds due to Iskovskih and Mori-Mukai); both [1] and the present paper we inspired by [11].

The paper is dedicated to the memory of Giorgio Gamberni.

1. Fix a smooth, irreducible, complete variety V and a rank-2 ample reflexive sheaf E on V; assume that E is spanned by its global sections. Let $S \subset V$ be the set of points of V at which E is not locally free. We will always assume $S \neq \emptyset$ (i.e. $c_3(E) > 0$). Set $L := c_1(E)$ and $c_i = c_i(E)$. All these notations will be always assumed, even if not explicitly stated. We write O and K instead of O_V and K_V; for a closed subscheme A of V, I_A will denote its ideal sheaf. For any sheaf G on V, we write $H^i(A)$ and $h^i(A)$ instead of $H^i(V, A)$ and $h^i(V, A)$. Fix a general $s \in H^0(E)$; set $C := (s)_0$. By the assumptions (since $\mathrm{char}(\mathbf{K}) = 0$) C is a reduced pure dimensional curve which is smooth outside $S \cap C$. Furthermore (e.g. [6], proof of Th. 4.1) $S \subset C$. The choice of s induces an exact sequence:

$$0 \to O \to E \to L \otimes I_C \to 0. \tag{2}$$

C will always denote $(s)_0$ with $s \in H^0(E)$, s general enough; thus s will give (2) and C will satisfy (2).

Remark 1.1. *Let T be a closed subvariety of V and $\pi : T' \to T$ a finite morphism. Then every quotient sheaf of $\pi^*((E|T))$ is ample.* Proof. By definition of ampleness and Grothendieck's definition of P we are reduced to the known case in which E is a line bundle. $\diamondsuit$

Lemma 1.2. *Fix an integral curve $T \subset V$ and let $\pi : T' \to T$ be its normalization. Then $\deg(\pi^*(E/T) \geq 2$ and we have equality if and only if $T \cong P^1$, $T \cap S = \emptyset$ and E/T is the direct sum of two line bundles of degree 1.*

Proof. The "if" part is obvious. Set $F := \pi^*(E|T)$. Since T' is smooth, F is the direct sum of a rank-2 locally free sheaf F' and a torsion sheaf F'', with $F'' = 0$ if $T \cap S = \emptyset$. By 1.1 and the fact that π is finite, we get that F' is ample. By construction F' is spanned. Thus if $p_a(T') > 0$, $\deg(F) \geq \deg(F') \geq 3$. Assume $p_a(T') = 0$. By [11], 3.2.1, $\deg(F') \geq 2$ and $\deg(F') = 2$ if and only if F' is the direct sum of two line bundles of degree one. Note that if $T \cap S \neq \emptyset$, we have $F'' \neq \emptyset$ (hence $\deg(F) > \deg(F') \geq 2$) because a coherent sheaf on a reduced variety is locally free if its fibers have constant dimension. Thus we may assume also $T \cap S = \emptyset$ i.e. E locally free near T. Then the proof of [11], 3.2.1, gives that T is smooth. $\diamondsuit$

Remark 1.3. *Under the assumption of* 1.2 *and with the notations* (2), *assume* $\dim(T \cap C) = 0$; *then LT* (i.e. $\deg(\pi^*(L))$) *is at least* $1+$ $+\mathrm{card}(C \cap T)$. *Furthermore if T is smooth around $S \cap T$, $LT \geq 1+$ $+\mathrm{length}(C \cap T)$.* Proof. Look at (2) and note that $S \subset C$. Restrict (2) to T and pull it back by π^*; the first map in the corresponding sequence (2)' is again injective because $O_{T'}$ has no torsion. There is a map $j : \pi^*(I_{C,V} \otimes L \otimes O_T) \to O_{T'}$ whose image defines the ideal sheaf of a non-negative divisor $\mathbf{a}$, $\mathrm{supp}(\mathbf{a})$ must contain every point of $\pi^{-1}(C \cap T)$. Since by (2)' and 1.1 $\pi^*(L|T)(-\mathbf{a})$ is ample, we get the first part of first inequality. The last inequality follows from the fact that $\mathbf{a} = T \cap C$ as schemes, 1.2 and (2)'. $\diamondsuit$

Remark 1.4. *If $T \subset V$ is a curve, then $LT \geq 2$; indeed if $T \cap S = \emptyset$, this is* [1], 1.1; *if $T \cap S \neq \emptyset$, then $T \cap C \neq \emptyset$ and* 1.4 *follows from* 1.3.

Remark 1.5. *By* 1.4 *(V,L) is its own reduction in the sense of* [8], 0.11.

To prove the amplenesss of the last 3 families of sheaves in the statement of 0.1, we need a lemma.

Lemma 1.6. *Assume only that E is spanned. E is ample if and only if for every integral curve $T \subset V$, E/T is ample; the last condition is equivalent (if $\pi : T' \to T$ is the normalization) to the ampleness of the locally free part of $\pi^*(E/T)$.*

Proof. A similar crieterion is true for every spanned line bundle on every variety. Since the restriction of the tautological line bundle of $\mathbf{P}(E)$ to every fiber of $P(E) \to V$ is ample, we get the first part. To get second one, note that we may check the ampleness of the tautological line bundle of $\mathbf{P}(E|T)$ after the base-change by π (e.g. see [4], prop. 2.1). $\diamondsuit$

2. We use the notations introduced in §1; we will use heavily the proofs in [1]; thus the reader need a copy of [1] nearby. We assume that E is ample and spanned (unless otherwise stated).

First assume that $K + L$ is semi-ample. Then there is an integer $m > 0$ such that $m(K + L)$ is spanned. Since $(K + L)C \leq 0$ for every C as in (2) and we may find such a C through a general point of V (e.g. count the dimensions and use the spannedness of E and that $c_2(E) \neq \emptyset$) we get $m(K + L) \cong O$. Thus $-K$ is ample by Kleiman numerical critertion of ampleness. Thus V is a Fano 3-fold. Therefore Pic(V) has no torsion; hence $L = -K$.

First we assume $b_2(V) = 1$, i.e. (for Fano 3-folds) Pic(V) $\cong \mathbf{Z}$. Let r be the index of V. By 1.4 and [11], 2.3, we have $r \geq 2$. Such V are classified, and we have to check all the possible V as was done in [1].

(α) (case (1) in [1], §1) Now assume $(V, L) = (\mathbf{P}^3, O(3))$. By 1.2 for every line A with $A \cap S = \emptyset$, we have $E|A \cong O_A(2) \oplus O_A(1)$, while for every line D with $D \cap S \neq \emptyset$, $(E|D)/\text{Tors}(E|D) \cong 2\,O(1)$ and Tors($E|D$) has length 1. By the proof of 1.2 we have card(S) $= 1$; set $\{P\} := S$. Fix a two-dimensional linear subspace M with $M \cap S = \emptyset$. By [10] either $E|M \cong T\mathbf{P}^2$ or $E|M = O(2) \oplus O(1)$. In the second case we get $c_2(E) = 2$, i.e. C is a conic; thus by (2) $h^0(E(-2)) \neq 0$; but every section of $E(-2)$ must vanishes identically on every line trough P, contradiction. Thus we may assume $E|M \cong T\mathbf{P}^2$ for every plane M with $P \notin M$. Thus $c_2(E) = 3$, i.e. $\deg(C) = 3$; since C has no trisecant line by (2) and 1.3, $g \leq 0$; by (1) we have $g = 0$ and $c_3 = 1$. This implies that E has a very mild singularity at P (it is called "convenable" or "suitable": see [7]); suffice to say that this implies "curvilinear" and thus that we may take C smooth without assuming a priori that E is curvilinear). Thus C is a rational normal curve in $\mathbf{P}^3$. Since all the possible configurations of such pairs (C, P) are projectively equivalent, we get the uniqueness of E, up to the action of $\text{Aut}(\mathbf{P}^3)$; the sheaf given in 0.1 is a solution. For this sheaf we have $h^0(E) = 11$, $h^0(E(-1)) = 2$. Thus we see that given 5 general points $P_i \in V$, there is $s \in H^0(E)$ with $\{P, P_1, \ldots, P_5\} \subset (s)_0$; since six points of $\mathbf{P}^3$ in linear general position are contained in a unique rational normal curves, we see that for E all C can occur. Thus we see that E is uniquely determined by $\mathbf{P}^3$. In particular the set of solutions is parametrized by $\mathbf{P}^3$.

(β) Now we assume $(V, L) = (\mathbf{P}^3, O_V(4))$ (hence $c_3(E) = 2g - 2$); fix (2). By (1) we have $2g - 2 = c_3 > 0$. By 1.3 C has no line D, D no component of C, with $\text{length}(D \cap C) \geq 4$ (hence no quadrisecant in the sense of [5]; in particular, since $g > 1$, C spans $\mathbf{P}^3$.

First assume C does not contain a line as irreducible component. We will show that this case not only gives solutions (ii) and (iii) in the statement of 0.1 but also gives in natural way a few (known) classes of interesting (from our point of view) reflexive sheaves. By [5], Prop. 2.4, the fact that there is no quadrisecant line implies that $[(d-2)(d-3)^2(d-4)/12] = [g(d^2 - 7d + 13 - g)/2]$. Since $g > 1$, we get easily (e.g. using some bounds for the arithmetic genus of (reducible) curves) that (d, g) has one of the following values: (5,2), (6,3), (6,4), (7,5), (9,10), (9,21); the last case cannot occur since there C has no plane component of degre ≥ 4, hence its genus cannot be so large. Note that if $h^0(I_C(2)) \neq 0$, we get infinitely many quadrisecant lines, except maybe if $(d, g) = (6, 4)$ or (5,2). If $h^0(I_C(3)) \neq 0$ and $h^0(I_C(2)) = 0$, by (2) we have $h^0(E(-1)) \neq 0$ and $h^0(E(-2)) = 0$; thus there is $t \in H^0(E(-1))$ with $\dim((t)_0) = 1$. Set $B := (t)_0$; B may be unreduced; we get an exact sequence

$$0 \to O(1) \to E \to I_B(3) \to 0 \tag{3}$$

Thus we see that E is not ample if B has a trisecant line not contained in B. Note that if $h^0(I_C(3)) > 1$, we have $h^0(I_B(2)) \neq 0$, hence B has a trisecant line (not contained in B!) if $\deg(B) \geq 5$, except maybe if B is union of (multiple) lines on a quadric cone (if $\deg(B) \geq 6$

this case will be checked in cases 3) and 4)). By [6], 2.2 and 4.1, $\deg(B) = c_2(E(-1)) = c_2(E) - c_1(E) + 1^2 = c_2(E) - 3 = \deg(C) - 3$, and $p_a(B)$ is given (in term of (d, g)). Now we check separately each case. We assume always C connected, leaving for part (β2) the discussion of what happens in the disconnnected case.

1): $(d, g) = (5, 2)$. By Riemann-Roch C is contained in a quadric. By (2) $h^0(E(-2)) \neq 0$ and $h^0(E(-3)) = 0$. Thus there is $m \in H^0(E)$ with $\dim((m)_0) = 1$. By [6], Cor. 2.2, we have

$$0 \to O(2) \to E \to I_D(2) \to 0 \tag{4}$$

We want to show that $E|D$ is not ample. By [6], Th. 4.1, (4) corresponds to a choice (up to a constant) of $h \in H^0(h, O_T(2))$ i.e. to a degree 2 positive divisor $\mathbf{a}$ on T. Two possibilities: $\mathbf{a}$ is reduced or not. Furthermore, up to the action of $\mathrm{Aut}(\mathbf{P}^3)$, these are the only possibilities for $(D, \mathbf{a})$ hence for $E(-2)$. Thus we see that there are two families of reflexive sheaves, such that for any two elements E and E' of each family, there is $g \in \mathrm{Aut}(\mathbf{P}^3)$ with $E' \cong g^*(E)$; furthermore, since the unreduced divisor is the limit of a flat family of reduced ones, we see that the second family is a limit of the first one. By (2) every sheaf E with $E(-2)$ given by (4) is spanned and its restriction $E|T$ to any curve $T \neq D$ is ample (see the proof of case 5) below). We want to check that $E|D$ is not ample. Assume the contrary. By (4) $E|D$ has a factor $O_D(2)$. To obtain a contradiction, it is sufficient (for degree reason) to check that $E|D$ has a torsion part $\mathrm{Tors}(E|D)$ with $\mathrm{length}(\mathrm{Tors}E|T)) \geq 2$. Indeed this length is exactly 2 and the torsion is isomorphic to $\mathbf{a}$; however by semicontinuity to obtain the inequality it is sufficient to check the case "$\mathbf{a}$ reduced" and show in that case that there is some torsion at each of the points in the support of $\mathbf{a}$. This is obtained tensoring (4) by O_D and make a local homological calculation.

2): $(d, g) = (6, 4)$. C must be a canonical curve, complete intersection of a cubic and a quadric; thus we have again (4) with, now, $\deg(D) = 2$. Thus D has (many) secant lines (even if it is not reduced) and each of them is an obstruction (by (4)) to the ampleness of E.

3): $(d, g) = (9, 10)$. By Riemann-Roch $h^0(I_C(3)) \geq 2$; thus we have (3) with $h^0(I_B(2)) \neq 0$, $\deg(B) = 6$. We claim that B has infinitely many trisecant lines, and in particular a trisecant line not in B, hence an obstruction to the ampleness of E. The claim is obvious except if the quadric A containing B is a quadric cone and B is a union (may

be unreduced) of lines. But in this case, looking at the minimal desingularization $F_2 \to A$ of A, we see that B is the intersection of A with a cubic surface (hence all the lines of A are trisecant to B). If E is "convenable" in the sense of [7] (sometime translated as "suitable"), i.e. card$(S) = c_3$), there is another proof for this case.
4): $(d, g) = (7, 8)$. Since $h^1(O_C(2)) \leq 1$, we would have $h^0(I_C(2)) \neq 0$. Look at the proof of case 3); if now B is a union of lines in a quadric cone A, B contains the complete intersection of A with a cubic surface; hence there is no such E.
5): $(d, g) = (7, 5)$. By Riemann-Roch we have $h^0(I_C(3)) \geq 3$ thus there is (3) with $h^0(I_B(2)) \geq 2$; if there is E, B has no plane component of degree ≥ 3; since $p_a(B) = 1$ by (1) (i.e. [6], 2.2 and 4.1), we get that B must be the complete intersection of 2 quadrics. Vieceversa, starting with such B and $m \in H^0(\omega_B(2)) \cong H^0(O_B(2))$, m vanishing only at finitely many points, by [6], Th. 4.1, we get a reflexive sheaf $E(-1)$ with E given by (3). We get an irreducible family of such bundles and the choice of B and (3) give that any such E is spanned. We check that they are ample, at least for general B, m; we assume that B is irreducible. Fix an integral curve $T \neq B$. Fix a general quadric with $B \subset A$, with T not in A. Note that $T \cap B$ (as scheme) is contained in the scheme $T \cap A$ which is a Cartier divisor on T with degree 2deg(T); by (3) and 1.6, $E|T$ is ample. We have to check the ampleness of $E|B$. Given B, we get E, hence C with $h^0(I_C(3)) \geq 3$; hence given E we may find B' instead of B giving the same E; since we know that $E|B'$ is ample, we know that $E|B$ is ample, too.
6): $(d, g) = (6, 3)$. Exactly the same proof as in case (5) shows how to get the family claimed by 0.1. To get the sheaf for simplicity start from an irreducible B, i.e. from a rational normal curve. We note only that, for C smooth, a necessary and sufficient condition for the spannedness of the corresponding E is that $h^0(I_C(2) = 0$; this is known to be equivalent to the fact that C is not hyperelliptic.

(β2) Now we assume that C is not connected; if C contains no line, the quotation of [5] works again and can reduce very much the possible cases. But it is easier to consider all the case simultaneously. If $h^0(V, I_{C,V}(2)) \neq 0$ and $d = 5$, again we have (3) and conclude. Thus, since $g > 1$, we may assume $d > 5$. Again we do not have quadrisecant lines (hence the plane components have low degrees). The trick is to fix one or more components which together have a 1-dimensional family of trisecant lines (one can take 3 disjoint lines or two disjoint conics or an irreducible curve of degree d' and genus g' with $(d', g') \neq (3,0)$ and $(4,1)$, or ...). Then look at the intersection of the other components with the surface union of these trisecant lines. This work (details left to the reader) unless there are exactly two irreducible components both with (degree, genus)=(3,0) or (4,1). But by (1) this implies $c_3 \leq 0$, contradiction.

(γ) (case (2) in [1], §2). Assume that V is a smooth quadric $Q \subset \mathbf{P}^4$ and $L = \mathcal{O}(3)$ (hence $c_3(E) = 2g - 2$ by (1)). By (2) E is not ample if C has a trisecant line. Note that every trisecant line to C in $\mathbf{P}^4$ is contained in V. First assume that C is contained in a hyperplane. By (2) this means $h^0(E(-2)) \neq 0$; the morphism $\mathcal{O}(2) \to E$ is an obstruction to the ampleness of E (much easier that case 1) in (β)). Thus we will assume that C spans $\mathbf{P}^4$.

Assume C connected. The smoothness of C, $g > 1$, and all page 533 in [2] give that either $g = 2$, $\deg(C) = 6$, or $g = 5$, $\deg(C) = 8$ (and C is the complete intersection of 3 quadrics in $\mathbf{P}^4$ in the latter case). In both cases C is projectively normal and $h^0(\mathcal{I}_{C,V}(2)) \geq 2$, $h^0(\mathcal{I}_{C,V}(1)) = 0$; thus by (2) we get $h^0(E(-1)) \geq 2$ and $h^0(E(-2)) = 0$; fix $t \in H^0(E(-1))$ with $\dim((t)_0) = 1$. Set $B := (t)_0$; by [6], 2.2, $\deg(B) = \deg(C) - 4$. We get the following exact sequence:

$$(5) \qquad 0 \to \mathcal{O}(1) \to E \to \mathcal{I}_B(2) \to 0.$$

By [6], Th. 4.1 (i.e. by (1)) if $d = 6$ we have $p_a(B) = 0$, while if $d = 8$ we have $p_a(B) = 1$. In both cases any sheaf E fitting in (5) is spanned. Since $h^0(E(-1)) > 1$, $h^0(\mathcal{I}_B(1)) \neq 0$; we get that if $d = 8$ there are lines $D \subset V$ with $\text{length}(B \cap D) \geq 2$; by (5) the line D prevents the ampleness of E for $d = 8$. Now we assume $d = 6$, hence $h^0(\mathcal{I}_{C,V}(2)) = 3$ and $h^0(\mathcal{I}_B(1)) = 2$. Thus B is the intersection of V with a plane Π. As in (β), case 5), we get from (5) the ampleness of E. By [6], Th. 4.1, E is uniquely determined when we fix B and a degree 2 positive divisor $\mathbf{a}$ on B with support S. A dimensional count shows that the orthogonal group $\text{Aut}(V)$ acts transitively on the pairs $(B, \mathbf{a})$ with B smooth conic and $\mathbf{a}$ reduced positive divisor of degree 2 on B, and on the pairs $(B, \mathbf{a})$ with B smooth conic and $\mathbf{a}$ double point on it. Thus we get exactly two irreducible families of solutions (since [6], Th. 4.1, gives an equivalence between (E, s) and $(B, \mathbf{a})$), the second one being a specialization of the first one. Furthermore two sheaves in the same family differ by an element of the orthogonal group Aut(V). Since $h^0(E(-1)) > 1$ a dimensional count shows also that for each E there is a subgroup $G \subset$Aut(V) with dim(G) $= 1$ and such that $g^*(E) \cong E$ for every $g \in G$. The two irreducible families are distinguished exactly by the condition: "card(S) = 2" or "card(S) = 1". In the case "card(S) = 2" (i.e. card(S) $= c_3$), we get a priori only convenable sheaves in the sense of [7] (hence curvilinear sheaves, without making a priori this assumption).

To handle the other cases and get further we need a lemma.

Lemma 2.1. *With the usual notations, V is not covered by a flat family of smooth rational curves $\{T\}$ with $LT = 2$.*

Proof. Fix $P \in S$. Assume by contradiction there is $T \cong \mathbf{P}^1$ with $P \in T$ and fix a general C. If length($C \cap T$) ≥ 2, the contradiction comes from 1.3. Assume length($C \cap T$) $= 1$ (and in particular C has embedding dimension at most 2 at $\mathbf{P}$). A local calculation shows that the torsion part of $I_C \otimes O_T$ has length 1. From the restriction of (2) to T we get $h^0(T, E|T) = 4$, contradicting 1.2. ◇

By 2.1 we get at once all the cases in [1], §3 (i.e. the cases with $b_2(V) \geq 2$) and cases (3), (4), (5) (since the case left was done in the "safe" §3), and (8) of [1], §2. Now we will check how 2.1 gives cases (6) and (7) of [1], §2; in these cases V is respectively the intersection of 2 quadrics in $\mathbf{P}^5$ and a cubic hypersurface in $\mathbf{P}^4$ and $L = O(2)$; by 12.1, since $S \neq \emptyset$, it is sufficient to check that every point of V is contained in a line contained in V; a general hyperplane section contains a line (by the explicit theory of Del Pezzo surfaces); thus V contains a two-dimensional family of lines; they cover all the points of V by the properness of the Hilbert scheme (here of the Grassmannians).

Now look at case (10) of [1], §2; again we find $h(C)$ a line; now there is no contradiction to the spannedness of E, but, as in the remark just after that case we get $g = 1$ by the Riemann-Hurwitz formula, contradicting (1). The proof of 0.1 is over.

Now we want to spend a few lines for the case "E not curvilinear". The reduction (as in [1]) to the very few cases considered in §2 does not use the curvilinear assumption. To handle the single cases, however more care than I have is needed. At some point (in particular in (α)) we stressed that we never used the curvilinear assumption. Care for case (9) of [1], §2; but this is not a big problem. Care with the search for the quadrisecant line in (β); however [5] works for singular reducible curves with no line as component; essentially the reducible case looks easy (as was the disconnected one) and reduces the problem to subcases (1),. . .,(6). In (γ) we used heavily the curvilinear assumption when we used [1], p. 533; there it was used in an essential way the enumerative formula for the number of trisecant lines to a curve in $\mathbf{P}^4$; this formula is proved in [9] only for smooth curves. Summary: we do not claim, even for (β), to have checked all possible configurations, and we do not claim that in the cases giving the families (ii), (iii), and (iv) of the statements of 0.1 the non curvilinear sheaves arise only as limit of curvilinear solutions. But there is a case in which both problems about multisecant lines could be answered very easily, showing that no new solution can arise; and this happens exactly if the singularities of E are bad i.e. there is $P \in S$ such that even for general $s \in H^0(E)$, $(s)_0$ has embedded dimension 3 at P; for instance this is the case if $(s)_0$ is not locally a complete intersection at P and by (2) this condition means that the fiber of E at P has dimension > 3. Assume that E has such a bad point P; and consider cases (β) or (γ); every line T in V through P intersects C at P in a scheme length ≥ 2; in (γ) take as T a line through P and another point of C (the only trouble arises if C is union of lines through P); in (β) to find the quadrisecant line it should be sufficient to project from P and apply the genus formula for plane curves and one of the available (even a very weak one) bound for reducible space curves whose plane components have low degrees (but we have not made all the numerical checkings). Exactly for the same reasons it should be very easy to handle the case in which E is assumed to be not curvilinear at two different points (or more).

References

[1] BALLICO, E.: On vector bundles on threefolds with sectional genus 1, *Trans. A.M.S.* (to appear).

[2] BALLICO, E.: A characterization of the Veronese surface, *Proc. A.M.S.* **105** (1989), 531 – 534.

[3] BANICA, C. and COANDA, I.: Existence of rank 3 vector bundles with given Chern classes on homogeneous rational 3-folds, *Manusc. math.* **51** (1985), 121 – 143.

[4] GIESKER, D.: p-ample bundles and their Chern classes, *Nagoya Math. J.* **43** (1971), 91 – 116.

[5] GRUSON, L. and PESKINE, C.: Courbes de l'espace projectif: variétés de sécant; in: *Enumerative Geometry and Classical Algebraic Geometry*, p. 1-31, Progress in Math. **24**, Birkhäuser, 1982.

[6] HARTSHORNE, R.: Stable reflexive sheaves, *Math. Ann.* **254** (1980), 121 – 176.

[7] HIRSCHOWITZ, A.: Existence de fasceaux réflexifs de rang deux sur $\mathbf{P}^3$ à bonne cohomologie, *I.H.E.S.* **66** (1988), 105 – 137.

[8] IONESCU, P.: Generalized adjunction and applications, *Math. Proc. Camb. Phil. Soc.* **99** (1986), 457 – 472.

[9] LE BARZ, P.: Formule multi-secants pour les courbes gauche quelconques; in: *Enumaritive Geometry and Classical Algebraic Geometry*, p. 165 – 197, Progress in Math. **24**, Birkhäuser, 1982.

[10] VAN DE VEN, A.: On uniform vector bundles, *Math. Ann.* **195** (1972), 245 – 248.

[11] WISNIEWSKI, J.: Length of extremal rays and generalized adjunction, *Math. Z.* **200** (1989), 409 – 427.