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MEROMORPHIC FUNCTIONS SHARING THREE VALUES

Gerhard **Jank**

Lehrstuhl II für Mathematik, RWTH Aachen, Templergraben 55, D-5100 Aachen, Federal Republic of Germany.

Norbert **Terglane**

Lehrstuhl II für Mathematik, RWTH Aachen, Templergraben 55, D-5100 Aachen, Federal Republic of Germany.

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Abstract: A well known theorem of R. Nevanlinna [5] states that there are at most two distinct meromorphic functions sharing three distinct values CM (counting multiplicities). This does not hold, if one only demands sharing two values CM and one IM (ignoring multiplicities). But in this case we are able to show that there are at most three distinct functions. This result is sharp in the sense that its conclusion does not hold, if one only demands sharing one value CM and two IM. Besides, we will present some other extensions of Nevanlinna's theorem dealing with the case that there are only few zeros and poles of the functions, or that there exists a nonempty set, which is "shared" by the functions.

1. Introduction

Given $n \geq 2$ meromorphic functions $f_1, \ldots, f_n$ on $\mathbb{C}$. We say that

$f_1, \ldots, f_n$ *share* a value $c \in \mathbb{C} \cup \{\infty\}$ if all the sets $C_j = \{z \in \mathbb{C} : f_j(z) = c\}$, $j = 1, \ldots, n$ are equal. In the following it will be helpful to make a distinction between sharing a value CM (counting multiplicities) and IM (ignoring multiplicities). In the first case we have a k-fold c-point of f_i exactly at the same points of the complex plane where f_j takes a k-fold c-point for $i, j = 1, \ldots, n$. Particularly, if c is a value not taken by $f_1, \ldots, f_n$, this value is CM-shared. In the second case we allow the multiplicities of the c-points to be different.

We assume that the reader is familiar with the notations and standard results of Nevanlinna theory (see e.g. [3], [4]).

In this paper, $S(r, f)$ denotes a quantity which is $o(T(r, f))$ as $r \to \infty$, possibly outside a set of finite Lebesgue measure. A frequently used lemma is the following: *If two nonconstant meromorphic functions f and g share three values IM, then $S(r, f) = S(r, g)$.* (One can easily prove this using the second fundamental theorem, see e.g. [4], p. 72.)

Beside the standard notations we will use the following: $\overline{N}(r, f)$ denotes the counting function of the poles, where each pole will be counted only once without regard to multiplicity. $N_0(r, c, f, g, h)$ is the counting function of the only once counted common c-points of f, g and h, again without regard to multiplicity. $N_1(r, f)$ counts the multiple poles of f, that is, $N_1(r, f) := N(r, f) - \overline{N}(r, f)$. $\mathcal{E}$ is the set of all $E \subset [0, \infty)$ of finite Lebesgue measure.

2. Results

The following is a well known result of R. Nevanlinna ([5], p. 125).

Theorem 1. *If three nonconstant meromorphic functions f, g, h share three distinct values CM, then at least two of them are equal.*

It is well-known, too, that the functions in general need not be Möbius transformations of each other. Theorem 1 is sharp in the sense that it is not correct for sharing two values CM and one value IM. To see this, consider an entire function β and define

$$f = \frac{e^{3\beta}}{e^{\beta}+1-e^{2\beta}};\ g = \frac{e^{\beta}}{e^{\beta}+1-e^{2\beta}};\ h = \frac{e^{-\beta}}{e^{\beta}+1-e^{2\beta}}.$$

Since $e^z \neq 0, \infty$ for all complex z it is obvious that there are no zeros of the three functions, which means, they share the 0-points CM, and they share the poles CM, since the denominator is exactly the same for all of them. A short calculation gives:

$$\begin{aligned} f=1 &\Leftrightarrow (e^\beta-1)(e^\beta+1)^2=0;\\ g=1 &\Leftrightarrow (e^\beta-1)(e^\beta+1)\ \ =0;\\ h=1 &\Leftrightarrow (e^\beta-1)^2(e^\beta+1)=0. \end{aligned}$$

This means f,g,h share 1 IM (not CM). (It is easily seen that none of the three functions is a Möbius transformation of another.)

Another direction to get results of the above kind without the assumption of sharing three values gives the following theorem (whose proof uses some ideas due to G. Brosch [1]).

Theorem 2. *If there are three distinct meromorphic nonconstant functions f,g and h with*

$$\overline{N}(r,f),\ \overline{N}(r,1/f)=S(r,f); \tag{2.1}$$

$$\overline{N}(r,g),\ \overline{N}(r,1/g)=S(r,g); \tag{2.2}$$

$$\overline{N}(r,h),\ \overline{N}(r,1/h)=S(r,h); \tag{2.3}$$

then there exists a set $E\in\mathcal{E}$ such that

$$\tau := \limsup_{r\to\infty, r\notin E} \frac{N_0(r,1,f,g,h)}{T(r,f)+T(r,g)+T(r,h)} \le 1/4.$$

An immediate consequence is the following result.

Corollary 3. *If there are three nonconstant meromorphic functions f,g and h that share 1-points IM and for which* (2.1), (2.2), (2.3) *hold, then at least two of them are equal.*

The inequality in Theorem 2 is sharp. This means that there are three functions for which (2.1), (2.2), (2.3) and $\tau=1/4$ hold. Put for example

$$f(z)=e^z;\ g(z)=e^{-z};\ h(z)=e^{2z}.$$

After an easy calculation one obtains

$$\begin{gathered} T(r,f)=r/\pi+O(1)=T(r,g);\ T(r,h)=2r/\pi+O(1);\\ f=1\Leftrightarrow e^z=1\Leftrightarrow g=1;\ h=1\Leftrightarrow e^z=\pm1. \end{gathered}$$

Since there do not exist zeros or poles of the three functions, it follows by the second fundamental theorem of R. Nevanlinna that τ equals 1/4.

With the Corollary 3 we are able to prove the following

Theorem 4. *If there are four nonconstant meromorphic functions f,g,h and k that share three distinct values of $\mathbb{C}\cup\{\infty\}$, two of them CM and one IM, then at least two of them are equal.*

Theorem 4 is sharp in the same sense mentioned after Theorem 1. To see this, let α be an entire function and define

$$f = 2e^{\alpha} - 1;\ g = e^{-\alpha}(2e^{\alpha} - 1);\ h = e^{-2\alpha}(2e^{\alpha} - 1);\ k = (2e^{\alpha} - 1)^2.$$

Similarly to the above reasoning it is seen that this four functions have no poles, i.e., they share ∞ CM, and that they share 0 IM (not CM). An easy computation gives:

$$\begin{aligned} f = 1 &\Leftrightarrow 2(e^{\alpha} - 1) &= 0;\\ g = 1 &\Leftrightarrow (e^{\alpha} - 1) &= 0;\\ h = 1 &\Leftrightarrow (e^{\alpha} - 1)^2 &= 0;\\ k = 1 &\Leftrightarrow 4e^{\alpha}(e^{\alpha} - 1) &= 0. \end{aligned}$$

This shows that the functions share 1-points IM (not CM).

For further details about the construction of such functions see [6]. Other conditions on the functions were given by F. Gross and C.F. Osgood [2].

Definition 5. (*Preimage sharing*) Let M be a finite, nonempty set in $\mathbb{C} \cup \{\infty\}$. Two meromorphic functions f, g *share the set* M if it follows from $f(z) \in M$ that $g(z) \in M$ and vice versa, with regard to multiplicity.

With this definition they gave

Theorem 6. *If there are two nonconstant entire functions f, g of finite order, which share 0 CM and the set $\{-1, +1\}$, one of the following equalities holds: $f \equiv \pm g$ or $fg \equiv \pm 1$.*

It was shown independently by G. Brosch ([1], p. 48) and K. Tohge ([7], p.251) that this result remains true for functions of infinite order. They proved that this even holds if f and g are meromorphic functions which share ∞ CM. We strengthen their result as follows.

Theorem 7. *If there are two nonconstant meromorphic functions f, g sharing ∞ IM, 0 CM and the set $\{-1, +1\}$, one of the following equalities holds: $f \equiv \pm g$ or $fg \equiv \pm 1$.*

For further results concerning unicity problems of meromorphic functions see [6].

3. Proofs

Proof of Theorem 2. Given three distinct nonconstant meromorphic functions f, g, h with (2.1) – (2.3), we have to show that $\tau \le 1/4$. Let us define auxiliary functions

$$\begin{aligned} \alpha_1 &= (\tfrac{f''}{f'} - 2\tfrac{f'}{f-1}) - (\tfrac{g''}{g'} - 2\tfrac{g'}{g-1}); \\ \alpha_2 &= (\tfrac{g''}{g'} - 2\tfrac{g'}{g-1}) - (\tfrac{h''}{h'} - 2\tfrac{h'}{h-1}); \\ \alpha_3 &= (\tfrac{h''}{h'} - 2\tfrac{h'}{h-1}) - (\tfrac{f''}{f'} - 2\tfrac{f'}{f-1}); \\ \beta_1 &= f'/f; \beta_2 = g'/g; \beta_3 = h'/h. \end{aligned}$$

It follows from the lemma of the logarithmic derivative (see e.g. [4], p. 65) that

$$\begin{aligned} m(r, \alpha_1) &= S(r, f) + S(r, g); \\ m(r, \alpha_2) &= S(r, g) + S(r, h); \\ m(r, \alpha_3) &= S(r, h) + S(r, f). \end{aligned} \tag{3.1}$$

Because of (2.1) – (2.3) and the lemma of the logarithmic derivative it follows that

$$T(r, \beta_1) = S(r, f);\ T(r, \beta_2) = S(r, g);\ T(r, \beta_3) = S(r, h). \tag{3.2}$$

Since f, g, h are nonconstant we have $\beta_i \not\equiv 0$ for $i = 1, 2, 3$.
Let z_0 be a zero of f' but not of f. With $\overline{N}(r, 1/f) = S(r, f)$ and (3.2) we get

$$\begin{aligned} \overline{N}(r, 1/f') &= \overline{N}(r, 1/f') - \overline{N}(r, 1/f) + S(r, f) \\ &\le N(r, 1/\beta_1) + S(r, f) \\ &\le T(r, \beta_1) + S(r, f) = S(r, f). \end{aligned}$$

This and a similar argument leads to

$$\overline{N}(r, 1/f') = S(r, f);\ \overline{N}(r, 1/g') = S(r, g);\ \overline{N}(r, 1/h') = S(r, h). \tag{3.3}$$

Noting that β_1 vanishes in multiple 1-points of f and using a similar argument for 1-points of g and h, we get

$$N_1(r, \tfrac{1}{f-1}) = S(r, f);\ N_1(r, \tfrac{1}{g-1}) = S(r, g);\ N_1(r, \tfrac{1}{h-1}) = S(r, h). \tag{3.4}$$

This shows that "most" of the 1-points of the functions are simple ones.

By expanding the α_i into their Laurent series it is easily shown that

(3.5) *α_1 (α_2 and α_3 respectively) vanishes at simple common 1-points of f, g (g, h and h, f respectively).*

If $\alpha_1 \not\equiv 0$, then (3.2) – (3.5), together with the first fundamental theorem and the assumptions (2.1) and (2.2), give

$$\begin{aligned}N_0(r,1,f,g,h) &\le N(r,1/\alpha_1)+S(r,f)+S(r,g)\\&\le T(r,\alpha_1)+S(r,f)+S(r,g)\\&=m(r,\alpha_1)+N(r,\alpha_1)+S(r,f)+S(r,g)\\&\le N(r,1/(f-1))-N_0(r,1,f,g,h)+\overline{N}(r,f)+\\&\quad+\overline{N}(r,1/f')+N(r,1/(g-1))-N_0(r,1,f,g,h)+\\&\quad+\overline{N}(r,g)+\overline{N}(r,1/g')+S(r,f)+S(r,g)\\&\le T(r,f)+T(r,g)+S(r,f)+S(r,g)-2N_0(r,1,f,g,h).\end{aligned}$$

Thus we have proved (with a similar argument for $\alpha_1,\alpha_2 \not\equiv 0$)

$$\begin{aligned}&3N_0(r,1,f,g,h)\le(1+o(1))(T(r,f)+T(r,g)) \text{ for } \alpha_1\not\equiv 0;\\&3N_0(r,1,f,g,h)\le(1+o(1))(T(r,g)+T(r,h)) \text{ for } \alpha_2\not\equiv 0;\\&3N_0(r,1,f,g,h)\le(1+o(1))(T(r,h)+T(r,f)) \text{ for } \alpha_3\not\equiv 0.\end{aligned}\tag{3.6}$$

Now we distinguish the following four cases.

Case 1. $\alpha_1,\alpha_2,\alpha_3 \not\equiv 0$;
Case 2. $\alpha_1 \equiv 0$; $\alpha_2,\alpha_3 \not\equiv 0$;
Case 3. $\alpha_1,\alpha_2 \equiv 0$; $\alpha_3 \not\equiv 0$;
Case 4. $\alpha_1,\alpha_2,\alpha_3 \equiv 0$.

We proceed to obtain $\tau \le 1/4$ in each one of them.

Case 1. In this case (3.6) leads to

$$9N_0(r,1,f,g,h)\le(2+o(1))(T(r,f)+T(r,g)+T(r,h)).$$

Hence we get $\tau \le 2/9$. This shows $\tau \le 1/4$.

Case 2. An easy calculation shows that $\alpha_1 \equiv 0$ is equivalent to $f = L\circ g$ with a Möbius transformation L. Because of (2.1) and (2.2) we get $f \equiv g$ or $f \equiv 1/g$. Since $f \equiv g$ gives a contradiction, we assume that $f \equiv 1/g$. This means $T(r,g)=T(r,f)+S(r,f)$. $N_0(r,1,f,g,h) = S(r,f)+S(r,g)+S(r,h)$ means $\tau = 0$, which gives (2.4). Now let $N_0(r,1,f,g,h) \ne S(r,f)+S(r,g)+S(r,h)$. This, together with (3.6) for $\alpha_3 \not\equiv 0$ and $N(r,1/(f-1)) \ge N_0(r,1,f,g,h)$, yields

$$\begin{aligned}\tau &= \limsup_{r\to\infty, r\notin E} \tfrac{N_0(r,1,f,g,h)}{T(r,f)+T(r,g)+T(r,h)}\\&= \limsup_{r\to\infty, r\notin E} \tfrac{N_0(r,1,f,g,h)}{2T(r,f)+T(r,h)+S(r,f)}\\&\le \limsup_{r\to\infty, r\notin E} \tfrac{N_0(r,1,f,g,h)}{T(r,f)+3N_0(r,1,f,g,h)+S(r,f)+S(r,h)}\\&\le \limsup_{r\to\infty, r\notin E} \tfrac{N_0(r,1,f,g,h)}{4N_0(r,1,f,g,h)+S(r,f)+S(r,h)}\\&= 1/4.\end{aligned}$$

This gives $\tau \leq 1/4$.
Cases 3/4. $\alpha_1, \alpha_2 \equiv 0$ yields

$$g \equiv h \text{ or } g \equiv 1/h$$

and

$$g \equiv f \text{ or } g \equiv 1/f,$$

which is a contradiction to our assumption that the three functions are pairwise distinct. ◇

Proof of Corollary 3. Assume there are three such functions f, g, h. The second fundamental theorem in the $\overline{N}$-version gives

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,1/f) + \overline{N}(r,1/(f-1)) + \overline{N}(r,f) + S(r,f) \\ &= \overline{N}(r,1/(f-1)) + S(r,f) \\ &\leq T(r,f) + S(r,f). \end{aligned}$$

This means $\overline{N}(r,1/(f-1)) = T(r,f) + S(r,f)$. In the same way $\overline{N}(r,1/(g-1)) = T(r,g)+S(r,g)$ and $\overline{N}(r,1/(h-1)) = T(r,h)+S(r,h)$ hold. Because of sharing 1-points, we conclude $\tau = 1/3$. Therefore Theorem 2 gives a contradiction. ◇

Proof of Theorem 4. General assumption: There are four nonconstant, distinct functions f, g, h, k sharing $0, \infty$ CM and 1 IM. Since the functions share three values the equations $S(r,f) = S(r,g) = S(r,h) = = S(r,k) =: S(r)$ hold. It is convenient to define

$$\overline{N}(r,0) = \overline{N}(r,1/f); \ \overline{N}(r,1) = \overline{N}(r,1/(f-1)).$$

Here it is of no interest whether these counting functions are defined with f or g, because the functions f and g share the zeros and 1-points. Without loss of generality we can suppose that

$$\overline{N}(r,0), \overline{N}(r,1) \neq S(r). \tag{3.7}$$

This is valid because of Corollary 3.
We define the entire functions α, β, γ by

$$f/g = e^{\alpha}; \ f/h = e^{\beta}; \ f/k = e^{\gamma}. \tag{3.8}$$

Since there are 1-points, we get

$$\alpha, \beta, \gamma, \alpha - \beta, \ \alpha - \gamma \not\equiv \textit{constant}. \tag{3.9}$$

Further we define the meromorphic functions A, B, C by

$$\tfrac{f-1}{g-1} = A; \ \tfrac{f-1}{h-1} = B; \ \tfrac{f-1}{k-1} = C. \tag{3.10}$$

Since there are zeros, we get

(3.11) $$A, B, C, A/B, A/C, B/C \not\equiv \textit{constant}.$$

We get the following representations for the function g:

(3.12) $$g = \frac{A-1}{A-e^{\alpha}};$$

(3.13) $$g = \frac{B/A-1}{B/A-e^{\beta-\alpha}} e^{\beta-\alpha};$$

(3.14) $$g = \frac{C/A-1}{C/A-e^{\gamma-\alpha}} e^{\gamma-\alpha}.$$

We equate (3.12) with (3.13) and (3.12) with (3.14) and get

(3.15) $$e^{\alpha-\beta} = \frac{e^{\alpha}(B-A)+(A-AB)}{B-AB};$$

(3.16) $$e^{\alpha-\gamma} = \frac{e^{\alpha}(C-A)+(A-AC)}{C-AC}.$$

A short computation gives the following equivalent representations:

(3.15') $$e^{\beta-\alpha} = \frac{e^{\beta}(A-B)+(B-AB)}{A-AB};$$

(3.16') $$e^{\gamma-\alpha} = \frac{e^{\gamma}(A-C)+(C-AC)}{A-AC}.$$

From (3.15) and (3.15') we conclude that e^{α} and e^{β} share the 1-points. So we have three functions which have neither poles nor zeros, sharing 1-points, and they are nonconstant and distinct because of (3.9). Corollary 3 shows that this can not be true. So two of the exponential functions have to be equal and this means with (3.7) that two of the functions f, g, h, k have to be equal. This yields the expected contradiction and completes the proof. ◇

Proof of Theorem 7. Since f, g share ∞ IM, 0 CM and the set $\{-1, 1\}$, the functions $F := f^2$, $g := g^2$ share 0,1 CM and ∞ IM. Now it is easily seen that

(3.17) $$S(r, F) = S(r, G) = S(r, f) = S(r, g) =: S(r).$$

Case 1. $\overline{N}(r, f) \neq S(r)$.

At any pole of f, F and G have a multiple pole. Define $N_2^{F,G}(r)$ as the counting function of the common multiple poles of F and G counted only once. Thus we get

(3.18) $$N_2^{F,G}(r) \neq S(r).$$

Now we consider the following auxiliary function

$$\beta := \left(\frac{F'}{F} - \frac{F'}{F-1}\right) - \left(\frac{G'}{G} - \frac{G'}{G-1}\right) = -\frac{F'}{F(F-1)} + \frac{G'}{G(G-1)}.$$

Because of sharing 0,1 CM and ∞ IM we conculde from (3.17) and the lemma of the logarithmic derivative

$$T(r,\beta) = S(r). \tag{3.19}$$

The second representation of β shows that it vanishes at common multiple poles, so we get from (3.18) and (3.19) for $\beta \not\equiv 0$

$$N_2^{F,G}(r) \leq N(r, 1/\beta) \leq T(r,\beta) + S(r) = S(r).$$

This is a contradiction and therefore we conclude $\beta \equiv 0$. An easy calculation shows that this is equivalent to $F \equiv G$. This yields $f \equiv \pm g$.
Case 2. $\overline{N}(r,f) = S(r)$.
Since ∞ is shared by the two functions we have $\overline{N}(r,g) = S(r)$.
Subcase 2.a. $\overline{N}(r,1/f) \neq S(r)$.
At any zero of f, F and G have a multiple zero. So in this case there are many multiple zeros of F and G. An analogous consideration as in Case 1, here with the auxiliary function

$$\gamma := \frac{F'}{F-1} - \frac{G'}{G-1},$$

yields $\gamma \equiv 0$, which is in this case equivalent to $f \equiv \pm g$.
Subcase 2.b. $\overline{N}(r,1/f) = S(r)$.
Hence

$$\overline{N}(r,F),\ \overline{N}(r,1/F),\ \overline{N}(r,G),\ \overline{N}(r,1/G) = S(r) \tag{3.20}$$

is valid. From (3.20) we get $F \equiv G$ or $FG \equiv 1$.
Otherwise assume that $F \not\equiv G$ and $FG \not\equiv 1$. Now define the function $H := 1/G$. It is clear that F, G, H share 1-points and with (3.20) we get

$$\overline{N}(r,H),\ \overline{N}(r,1/H) = S(r). \tag{3.21}$$

Since F, G, H share 1-points and because of (3.20) and (3.21) Corollary 3 shows that two of the functions have to be equal. This gives a contradiction to the above assumption that $F \not\equiv G$ and $FG \not\equiv 1$ or to the assumption that f, g are nonconstant. So we have shown $F \equiv G$ or $FG \equiv 1$ and this leads to $f \equiv \pm g$ or $fg \equiv \pm 1$ and therefore our proof is complete. $\diamondsuit$
Remark. K. Tohge has strengthened Theorem 6 in the following way:

If f, g share 0, ∞ CM and the set $M = \{c \in \mathbb{C} : c^n = 1\}$ for a given integer $n \geq 2$, then $f \equiv d_1 g$ or $fg \equiv d_2$ with complex constants d_1, d_2 such that $d_1^n = 1$ and $d_2^n = 1$ holds.
In a similar way as in the proof of Theorem 7 with $F := f^n$ and $G := g^n$ instead of $F := f^2$ and $G := g^2$ one can easily prove that Tohge's result still remains true for sharing the poles IM instead of CM. It seems that this can not be obtained by Tohge's method.

References

[1] BROSCH, G.: Eindeutigkeitssätze für meromorphe Funktionen, Dissertation, RWTH Aachen, 1989.

[2] GROSS, F. and OSGOOD, C.F.: Entire functions with common preimages in Factorization theory of meromorphic functions, *Lecture notes in pure and applied mathematics* **78**, 19 – 24, Marcel Dekker Inc., 1982.

[3] HAYMAN, W.K.: Meromorphic Functions, Clarendon Press, Oxford, 1964.

[4] JANK, G. und VOLKMANN, L.: Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen, Birkhäuser Verlag, Basel-Boston-Stuttgart, 1985.

[5] NEVANLINNA, R.: Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Gauthiers-Villars, Paris, 1929.

[6] TERGLANE, N.: Identitätssätze in $\mathbb{C}$ meromorpher Funktionen als Ergebnis von Werteteilung, Diplomarbeit, RWTH Aachen, 1989.

[7] TOHGE, K.: Meromorphic functions covering certain finite sets at the same points, *Kodai Math. J.* **11** (1988), 249 – 279.