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# EXTERIOR PARALLELISM FOR POLYHEDRA

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**Abstract:** The aim of this note is to extend the theory of parallel differentiable immersions to the piecewise linear case. Parallelism for differentiable immersions has been established by H.R. Farran and S.A. Robertson [4] and was studied in several subsequent papers ([3], [8] and [9]). It has strong relations to the geometry of the normal bundle ([9]) and to the theory of focal points ([7]).

We mainly shall concentrate on the 1-dimensional case, because there good motivations can be obtained for the study of higher-dimensional polyhedra. The main results obtained in [3] and [8] for the parallelism of differentiable curves can be transferred to the piecewise linear situation. The arguments are rather elementary and therefore proofs are only sketched in these cases.

The behaviour of polyhedral 2-manifolds in $E^3$ and $E^4$ with respect to exterior parallelism is representative for that of higher-dimensional polyhedra. In addition to the 1-dimensional case two local obstructions to the existence of parallel polyhedra to a given one occur. In this context some kind of normal curvature will be developed for polyhedra.

## 1. Parallel polygons

The introduction of the notion of parallelity for polygons can be motivated by the construction of parallel differentiable curves in the plane (or hypersurfaces in $E^n$). There the evolute plays an important role for the regularity of this construction. To see the analogy in the piecewise linear case look at the different situations in Figure 1, where parallel polygons to P are obtained according to our subsequent definition:

1) The polygon Q corresponds to the regular case where the parallel curve is located between the original one and its focal set.

2) The polygon S corresponds to the singular case, where the parallel curve meets the nearest focal point of the original one.

3) The polygon T corresponds to the singular case, where the evolute is met.

4) The polygon R corresponds to the regular case, where the focal set remains between the parallel curve and the original one.

All these cases are exhibited for a rectangle in Figure 2. The regular cases Q and R will be considered as parallel polygons to P while the other cases S and T will not have this property. Also we shall exclude the degenerate situation where the original polygon has angles 0 or $\pi$.

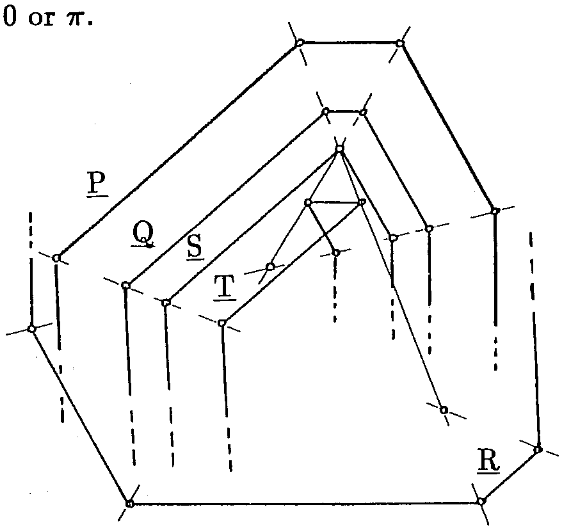


Figure 1

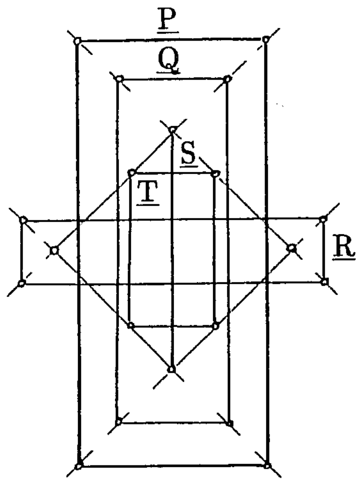


Figure 2
**Parallel rectangles**

For the development of the general theory let P be a *polygon* in Euclidean n-space $E^n$, given by its vertices. $\{p_i | i \in I\}$ and its connecting oriented line segments $s_i = p_i p_{i+1}$ from $p_i$ to $p_{i+1}$, where $I = \mathbb{Z}$, $\mathbb{Z}_k$ or $\{i \in \mathbb{Z} | m \leq i \leq n\}$ for some pair $m, n \in \mathbb{Z}$. We shall restrict our considerations to the *generic* situation where the angle between $s_{i-1}$

and $s_i$ at $p_i$ lies between 0 and $\pi$ for all $i \in I$. At every vertex $p_i$ of $\underline{P}$ there is a hyperplane of local symmetry through $p_i$ which divides the angle between the oriented line segments from $p_i$ to $p_{i-1}$ and $p_{i+1}$ into equal parts. This hyperplane is called the ***symmetric normal*** of $\underline{P}$ at $p_i$, denoted by $N_i$. The intersection between $N_i$ and $N_{i+1}$ is called the *focal* $(n-2)$*-plane* $F_i$ of $\underline{P}$ at $s_i$ (if it exists) (see Figures 3a,b).

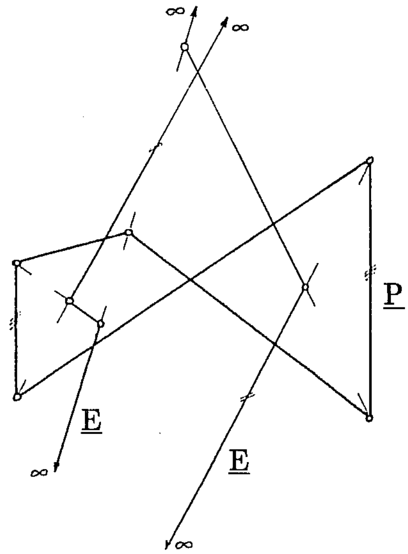


Figure 3a

Focal set and evolute in the planar case

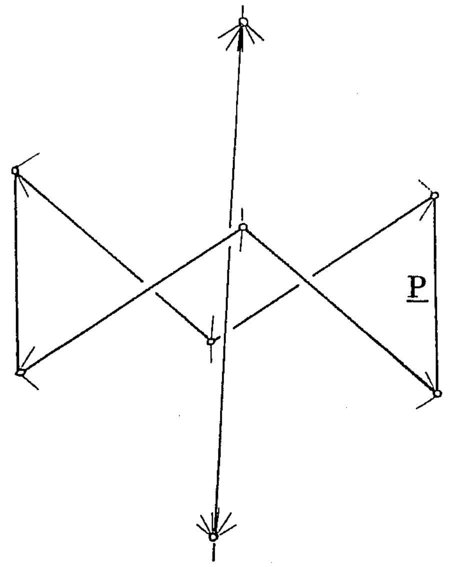


Figure 3b

Degenerated focal set in the spatial case

The focal planes of $\underline{P}$ can be used to construct a kind of *evolute* for $\underline{P}$. The following will show this in the planar case (see Figure 3a) and can be generalized easily to higher dimensions: If $F_{i-1}$ and $F_i$ exist, then take the connecting line segment between $F_{i-1}$ and $F_i$ on $N_i$ if they are located on the same side of $\underline{P}$, and take its closed complement in the other case. If $F_i$ exists and $F_{i-1}$ does not exists then take the closed halfline on $N_i$ which begins at $F_i$ and does not meet $\underline{P}$. The similar procedure is applied, if $F_{i-1}$ exists and $F_i$ does not. The resulting composition of line segments and half lines gives the evolute of $\underline{P}$.

**Definition 1.** Two polygons $\underline{P}= \{p_i | i \in I\}$ and $\underline{Q}= \{q_i | i \in I\}$ of the same combinatorial type are called ***parallel***, if for every $i \in I$ $p_i p_{i+1}$ is parallel to $q_i q_{i+1}$ and the symmetric normal of $\underline{P}$ at $p_i$ coincides with that of $\underline{Q}$ at $q_i$.

**Remark 1.** Parallelism of polygons is an equivalence relation. Furthermore parallel polygons have coinciding focal planes and evolutes.

**Definition 2.** A ***self-parallelism*** of a polygon $\underline{P} = \{p_i | i \in I\}$ is a permutation $\sigma$ of the index set such that $\underline{P \circ \sigma} := \{p_{\sigma(i)} | i \in I\}$ has the same line segments as $\underline{P}$ and is parallel to $\underline{P}$.

**Remark 2.** The self-parallelisms of a polygon form a group under composition of maps, the ***self-parallel group*** $G(\underline{P})$ ***of*** $\underline{P}$. Furthermore it can be seen like in the differentiable case that this group must be cyclic, because $\underline{P}$ is 1-dimensional.

**Remark 3.** If $\underline{P}$ and $\underline{Q}$ are parallel, then the lines which correspond under this parallelism have constant distance from each other, not depending on $i \in I$. This implies that $G(\underline{P})$ acts transitively and isometrically on the point set which is obtained by the intersection of the lines of $\underline{P}$, corresponding to a given one under the operation of $G(\mathrm{P})$, with their common normal hyperplane. This set may be called the *parallel frame* of $\underline{P}$ in our case (see [4] for the differentiable version).

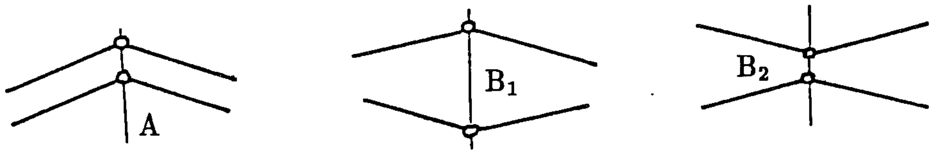


Figure 4

## 2. Polygons in the plane

The study of parallelism for polygons in the plane is rather simple because the choice of the unit normals to the line segments of the polygon is unique up to sign. First it should be observed that in this case there are only three possibilities for the location of the corresponding vertices and neighboring line segments (see Figure 4). Also, looking at the orientations, we see that the cases $A$ and $B$ cannot occur simultaneously for the same polygon. In the case $A$ the focal points on the common symmetric normal lie outside of the segment from $p_i$ to $q_i$ while in the cases $B$ they must be in the interior of that segment.

Now we shall mainly concentrate on self-parallelisms of a closed polygon $\underline{P} = \{p_i | i \in \mathbb{Z}_k\}$. According to Remarks 2 and 3 a parallel frame of $\underline{P}$ admits a transitive isometric operation of the self-parallel group $G(\underline{P})$ which can be assumed to be fixed point free, if multiple coverings are excluded. This implies $G(\underline{P}) = \mathbb{Z}_2$ in the non-trivial case, and thus $k$ must be even and the only non-trivial self-parallelism is given by $\sigma(i) = i + k/2$.

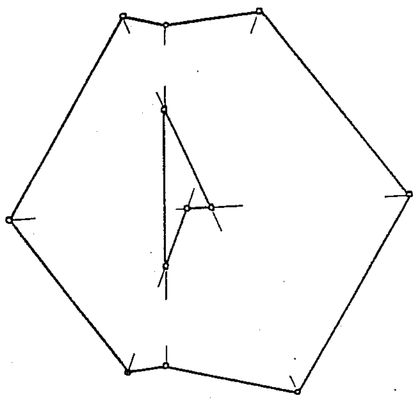

Figure 5
Self-parallel octogon in the plane

An explicit example for this situation is given by Figure 5. The relation to plane curves of constant width [1] is given by the fact that every closed tangent polygon to such a curve serves as an example for a self-parallel polygon, if the set of osculating points always contains both of the intersections of the corresponding binormal with the curve. Similarly examples of self-parallel polygons with self-intersections can be obtained from rosettes of constant width [2], for which explicit constructions have been given in [10]. Since case $A$ of Figure 4 easily can be excluded for closed self-parallel curves, we have

**Theorem 1.** *Let $\underline{P}$ be a closed polygon in the plane admitting a nontrivial self-parallelism. Then $G(\underline{P}) = \mathbb{Z}_2$ and for every $i \in \mathbb{Z}_k$ the sides $s_i$ and $s_{i+k/2}$ have a common focal point given by the intersection of the line segments from $p_i$ to $p_{i+k/2}$ and from $p_{i+1}$ to $p_{i+k/2+1}$.* $\diamond$

**Corollary 1.** *Let $\underline{P}$ be a closed convex polygon admitting a nontrivial self-parallelism. Then the evolute of $\underline{P}$ is contained in the convex domain bounded by $\underline{P}$.* $\diamond$

**Remark 4.** This corresponds to main results in [3] or [8]. Also by a lengthy argument a little more general version of Corollary 1 can be proved avoiding the assumption of convexity. Furthermore it can be shown that in the non-closed case non-trivial self-parallelisms are not possible.

**Remark 5.** Using methods established by P.C. Hammer and A. Sobczyk ([5], [6]) it can be seen that every closed convex polygon with a nontrivial self-parallelism admits an inscribed curve of constant width.

## 3. Normal holonomy of a polygon in space

The aim of this section is to establish some kind of parallel transfer in the normal bundle of a polygon in 3-space and to exhibit its relation to parallelism for polygons. Most constructions and results extend to higher dimensions.

Let $\underline{P} = \{p_i | i \in I\}$ be a generic polygon in Euclidean 3-space with sides $s_i$. Let $A_i$ be the reflection at the symmetric normal plane of $\underline{P}$ at $p_i$. A ***normal vector field*** along the subarc $\underline{P}' = \{p_i | i \in J\}$ of $\underline{P}$ is a choice of normal vectors $\xi_i$ to $s_i$ for every $i \in J$ such that the segment $s_i$ belongs to $\underline{P}$.

**Definition 3.** A normal vector field $\{\xi_i\}_{i \in J}$ along the subarc $\underline{P}'$ of $\underline{P}$ is called *parallel*, if $\xi_{i+1} = A_{i+1}(\xi_i)$ for all $i, i+1 \in J$. The *parallel transfer* of the normal vector $\xi_{i_0}$ at $s_{i_0}$ to $s_{i_1}$ is given by the value of the parallel vector field along $\underline{P}' = \{p_i | i_0 \leq i \leq i_1 + 1\}$ at $s_{i_1}$ which is uniquely determined by its initial value $\xi_{i_0}$ at $s_{i_0}$.

**Remark 6.** In the closed case $\underline{P} = \{p_i | i \in \mathbb{Z}_k\}$ the parallel transfer of normal vectors along one period of $\underline{P}$ is a proper linear isometry of the normal space of $\underline{P}$ at $s_i$ onto itself, given by $\prod_{v=1}^{k} A_{i+v}$. Hence it is a rotation around an angle $\alpha(\underline{P})$ which does not depend on $i \in \mathbb{Z}_k$. This angle is called the ***normal rotation angle*** of $\underline{P}$.

**Remark 7.** Smoothing the vertices of $\underline{P}$ by small circles tangent to the corresponding adjacent sides of $\underline{P}$, we get a $C^1$-curve having the same differential geometric normal holonomy as $\underline{P}$.

**Example 1.** a) Every closed polygon contained in a plane in 3-space has normal rotation angle 0. The same is true for closed tangent polygons to a sphere and in particular for closed edge polygons on Platonic solids.

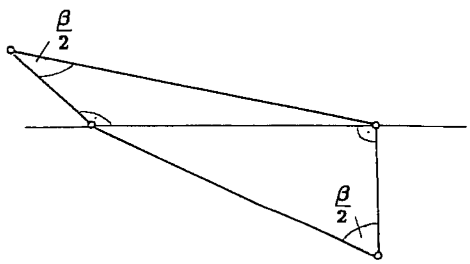


Figure 6

b) For a given $\beta \in (0, \pi)$ a closed quadrangle with normal rotation angle $\beta$ is demonstrated in Figure 6. There the planes spanned by $p_1, p_2, p_3$ and by $p_2, p_3, p_4$ are assumed to be perpendicular to each other.
c) The center polygon in Figure 7 is a closed hexagon with normal rotation angle $\pi$. We conjecture that there is no quadrangle or pentagon with this property.

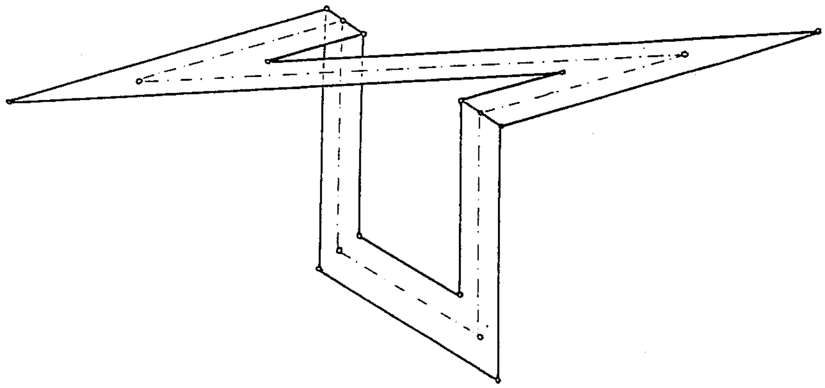

Figure 7
Möbius strip bounded by a self-parallel 12-gon with a center hexagon.

**Proposition 1.** *Two generic polygons* $\underline{P} = \{p_i | i \in I\}$ *and* $\underline{Q} = \{p_i | i \in I\}$ *are parallel if and only if the segments* $p_i q_i$ *and* $p_{i+1} q_{i+1}$ *have equal normal parts* $\xi_i$ *with respect to* $s_i$ *for every* $i \in I$ *and if* $\{\xi_i | i \in I\}$ *is a parallel normal vector field to* $\underline{P}$.
**Proof.** If $\underline{P}$ and $\underline{Q}$ are parallel then their sides with equal subscript are parallel and the local symmetry with respect to the common symmetric normal implies parallelism of the normal vector field given above. Conversely, the assumed equality of the normal parts implies that the corresponding sides are parallel. Since they constitute a parallel normal vector field to $\underline{P}$ it is easily seen that the symmetric normal planes of $\underline{P}$ and $\underline{Q}$ coincide at corresponding vertices. ◊

As in [8] this shows that non-vanishing normal rotation angle constitutes an obstruction to the existence of parallel polygons.
**Corollary 2.** *A closed generic polygon admits a (non-identical) parallel polygon if and only if it has vanishing normal rotation angle.*

The sufficiency of the second condition is obtained as a special case from the following
*construction:* Let $\alpha$ be the normal rotation angle of $\underline{P} = \{p_i | i \in \mathbb{Z}_k\}$.

Assume $\alpha$ is a rational multiple of $2\pi$; $\alpha = 2\pi l/m$ with $(l, m)$ relatively prime, $m = 1$ for $\alpha = 0$. Let $\varepsilon_i > 0$ be such the $\varepsilon_i$-tube around $p_i \vee p_{i+1}$ does not meet the focal line of $\underline{P}$ at $s_i$, and take $\varepsilon > 0$ as the minimum of these $\varepsilon_i$, $i \in \mathbb{Z}_k$. Choose some unit normal $\xi_1$ to $s_1$ and extend it by parallel transfer of normal vectors to the $m$-fold covering of $\underline{P}$. By the assumption on the normal holonomy of $\underline{P}$ this gives a parallel normal vector field $\{\xi_i | i \in \mathbb{Z}_{km}\}$ along the $m$-fold covering of $\underline{P}$. The line $l_{i+\nu k}$, $i = 1, \ldots, k$, $\nu = 0, \ldots, m-1$, is obtained from $p_i \nu p_{i+1}$ by parallel displacement about $\varepsilon \xi_{i+\nu k}$. Let $q_j := l_{j-1} \wedge l_j$, $j \in \mathbb{Z}_{km}$, which is non-empty by our construction. Then $\underline{Q}= \{q_j | j \in \mathbb{Z}_{km}\}$ defines a closed polygon which is parallel to the $m$-fold covering of $\underline{P}$ and has self-parallel group $\mathbb{Z}_m$ (see Figures 8,9). ◇

**Corollary 3.** *For every natural number $m$ there exists a generic polygon with self-parallel group $\mathbb{Z}_m$.*

This follows directly from the given examples together with the construction described above. ◇

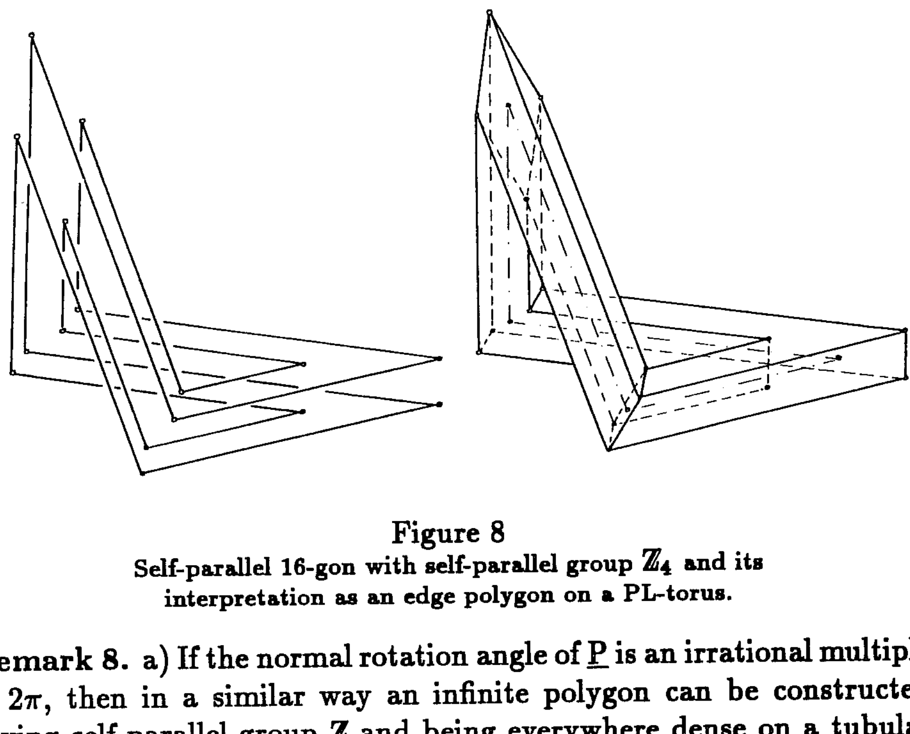

Figure 8
**Self-parallel 16-gon with self-parallel group $\mathbb{Z}_4$ and its interpretation as an edge polygon on a PL-torus.**

**Remark 8.** a) If the normal rotation angle of $\underline{P}$ is an irrational multiple of $2\pi$, then in a similar way an infinite polygon can be constructed having self-parallel group $\mathbb{Z}$ and being everywhere dense on a tubular surface around $\underline{P}$.

b) Considering higher dimensions than three, it can be observed that only in the odd case obstructions to the existence of parallel polygons may occur. In even dimensions there always exists a parallel polygon to a given closed one because the corresponding normal holonomy map has at least one fixed direction.

**Theorem 2.** *Let $\underline{P}$ be a self-parallel polygon with $k$ vertices and self-parallel group $\mathbb{Z}_m$ satisfying $m \geq 3$. Then $k$ is an integer multiple of $m$ and there is a polygon $\underline{C}$ with $k/m$ vertices, the center of $\underline{P}$, from which $\underline{P}$ can be reconstructed by the construction given above.*

**Proof.** According to Remark 3 the set of line segments of $\underline{P}$ which correspond to a given one are located in a regular way on a circular cylinder. Intersecting the axes of these cylinders appropriately we shall get a polygon $\underline{C}$ with $k/m$ vertices. As in [8] it can be seen that the focal lines of $\underline{P}$ remain outside of the convex hull of every $m$ related parallel segments of $\underline{P}$. This leads to the conclusion that $\underline{P}$ is parallel to the $m$-fold covering of $\underline{C}$. ◇

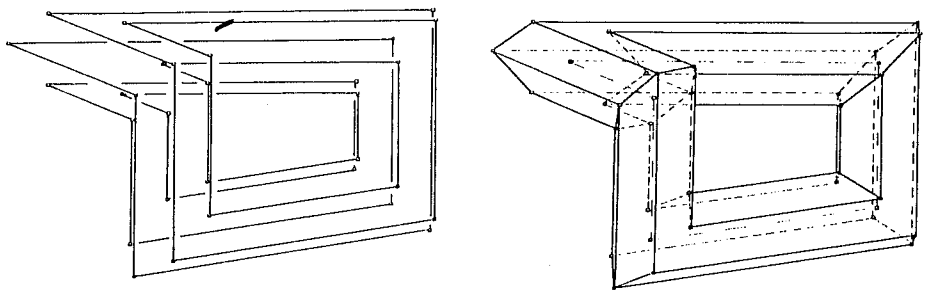

Figure 9
Self-parallel 30-gon with self-parallel group $Z_6$ and its interpretation as an edge polygon on a PL-torus.

**Remark 9.** The same result can be shown in the case $m = 2$, if no focal line meets the strip between the associated parallel lines of $\underline{P}$. Then $\underline{P}$ bounds a piecewise linear immersion of the Möbius strip (see Figure 7). The case where this condition is not valid is also possible (see Figure 3).

Using arguments from the preceding proof and elementary geometry we also get

**Theorem 3.** *Let* P *be a self-parallel polygon with self-parallel group* $\mathbb{Z}_k$ *and center polygon* $\underline{Q}$*. Then length* $(\underline{P}) = k\cdot$ *length*$(\underline{Q})$

## 4. Obstructions to exterior parallelism in higher dimensions

A theory of exterior parallelism for piecewise linear submanifolds of dimension greater than one in $E^n$ can be developed only for special types. There are two local obstructions which will be sketched by the following considerations:

Let $\underline{P} = \{\mathcal{V}, \mathcal{E}, \mathcal{S}\}$ be a polyhedral 2-manifold in $E^3$ (possibly with self-intersections) where $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{S}$ denote the sets of vertices, edges and sides respectively. The existence of a polyhedral 2-manifold $\underline{Q}$ of the same combinatorial type such that the corresponding sides of $\underline{P}$ and $\underline{Q}$ are parallel and have constant distance from each other implies that for all vertices of $\underline{P}$ (and $\underline{Q}$) the following is satisfied:

**Definition 4.** A vertex $p \in \mathcal{V}$ of the polyhedral 2-manifold $\underline{P}$ in $E^3$ is called *pa-admissible*, if for all edges $l \in \mathcal{E}$ ending at $p$ the planes, which intersect the angle between the corresponding adjacent sides of $\underline{P}$ into equal parts, have a common line. This line is uniquely determined and called the *symmetric normal* of $\underline{P}$ at $p$.

For polyhedral 2-manifolds, having *pa*-admissible vertices only, parallelism can be defined in the same way as in Definition 1. Examples for polyhedra possessing non-*pa*-admissible edges can be obtained easily. Clearly, if only three edges end at some vertex, then this vertex is *pa*-admissible. Sufficient and necessary for the *pa*-admissibility of a vertex is that every four consecutive unit normals (suitably oriented and labelled) of the sides of $\underline{P}$ meeting at this vertex lie in a common plane. Examples for polyhedra having *pa*-admissible vertices only are given by the boundaries of the Platonic solids or the polyhedral tori obtained by suitable connections of the vertices of a self-parallel polygon (see Figures 8 and 9).

Using symmetric normals, focal points can be introduced as previously. These can be used to develop criteria for the construction of parallel polyhedra along fields of unit normals as above. Thus the focal points of a suitably constructed polyhedral torus show some similarity to the corresponding situation for the standard torus.

If we consider polyhedral 2-manifolds $\underline{P}$ in $E^4$, we get an additional obstruction to the existence of parallel polyhedra. This corresponds to the fact that in the case of differentiable 2-manifolds in $E^4$ sometimes parallel sections of the normal bundle do not exist, i.e., the normal connection is not flat. The visualization in the piecewise linear case is prepared by the following

**Definition 5.** Let $\underline{P} = \{\mathcal{V}, \mathcal{E}, \mathcal{S}\}$ be a polyhedral 2-manifold in $E^4$. For a given $p \in \mathcal{V}$ let $k$ be the number of sides of $\underline{P}$ meeting at $p$, and choose a cyclic labelling of these sides $\{s_i | i \in \mathbb{Z}_k\}$, such that two consecutive sides have a common edge at $p$. Let $A_i$ denote the linear map given by the reflection at the 3-plane which divides the angle between $s_i$ and $s_{i+1}$ into equal parts. Then the *normal curvature* of $\underline{P}$ at $p$ is given by $\Gamma(p) = \prod_{i=1}^{k} A_i$.

**Remark 10.** The normal curvature is a linear orientation preserving isometry of the normal plane of $s_1$, i.e. it is given by a rotation about an angle $\alpha$, the normal curvature angle of $\underline{P}$ at $p$. This angle is uniquely determined up to sign. The definition of a parallel normal vector field along some part of $\underline{P}$ can be given in the obvious way, but for the existence of such a field on the simplex star around $p$, the vanishing of the normal curvature angle at $p$ is necessary and sufficient. Clearly this represents another local obstruction to the existence of parallel polyhedra.

That the normal curvature angle can attain many values can be seen from

**Example 2.** Take a quadrangle in 3-space with normal rotation angle $\beta \in (0, \pi)$ (see Figure 6 and Example 1). Consider the line $l$ in 4-space which is orthogonal this 3-space and passes through the center of gravity of the quadrangle. Look at a point $p$ on $l$ as a vertex of a polyhedral 2-manifold $\underline{P}$ having the simplex star around $p$ bounded by the given rectangle. If $p$ lies in the 3-space of the quadrangle, then the normal curvature angle of $\underline{P}$ at $p$ vanishes. But if $p$ tends to infinity, then the normal curvature angle tends to $\pm\beta$.

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