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ON A FUNCTIONAL EQUATION OCCURRING IN ASTROPHYSICS

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Abstract: In this paper we describe three methods of constructing the general solution of the functional equation $\overline{f(z)f(-1/\overline{z})} = -1$ and we discuss a few examples. The paper ends with a simple uniqueness theorem.

The functional equation

$$f(z)\overline{f(-1/\overline{z})} = -1 \tag{1}$$

occurs in astrophysics (cf. [4]). Here the unknown function f maps the complex plane punctured at zero $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ into itself and relation (1) ($\overline{z}$ denotes the complex conjugate of z) is assumed to hold for all $z \in \mathbb{C}^*$.

Write

$$h(z) := -1/\overline{z} = -z/|z|^2\,, \quad z \in \mathbb{C}^*\,. \tag{2}$$

The function $h : \mathbb{C}^* \to \mathbb{C}^*$ is an involution

$$h(h(z)) = z\,, \quad z \in \mathbb{C}^*\,, \tag{3}$$

without fixed points (of order 1). With the aid of (2) equation (1) can be written in the form

$$f(h(z)) = h(f(z)), \quad z \in \mathbb{C}^* . \tag{4}$$

Relation (4) expresses the permutability of the functions f and h.

In this paper we describe three methods of constructing the general solution of equation (1) and we discuss a few examples. The paper ends with a simple uniqueness theorem.

1. The first method of solving (1) follows the pattern described in [2; Chapter I] (cf. also [1]). Put

$$\begin{cases} D_1 := \{z \in \mathbb{C}^* \mid \operatorname{Im} z > 0\} \cup \{z \in \mathbb{C}^* \mid \operatorname{Re} z > 0, \ \operatorname{Im} z = 0\}, \\ D_2 := \{z \in \mathbb{C}^* \mid \operatorname{Im} z < 0\} \cup \{z \in \mathbb{C}^* \mid \operatorname{Re} z < 0, \ \operatorname{Im} z = 0\}. \end{cases} \tag{5}$$

We have

$$D_1 \cup D_2 = \mathbb{C}^*, \quad D_1 \cap D_2 = \emptyset, \tag{6}$$

and the function h maps (in a one–to–one manner) D_1 onto D_2 and conversely:

$$h(D_1) = D_2, \quad h(D_2) = D_1 . \tag{7}$$

Let $F : D_1 \to \mathbb{C}^*$ be a quite arbitrary function and define the function $f : \mathbb{C}^* \to \mathbb{C}^*$ by the formula

$$f(z) = \begin{cases} F(z), & z \in D_1, \\ h(F(h(z))), & z \in D_2. \end{cases} \tag{8}$$

Definition (8) is correct in view of (6) and (7). We are going to show that function (8) satisfies equation (4) (i.e., equation (1)) on $\mathbb{C}^*$. Take an arbitrary $z \in \mathbb{C}^*$. According to (6) either $z \in D_1$ or $z \in D_2$. In the former case we have by (8) $f(z) = F(z)$ so that $h(f(z)) = h(F(z))$, and by (7) $h(z) \in D_2$, whence, again by (8),

$$f(h(z)) = h[F(h(h(z)))] = h(F(z)) = h(f(z))$$

(cf. (3)). Consequently relation (4) holds true.

When $z \in D_2$, then $h(z) \in D_1$, and we obtain using (8) and (3)

$$h(f(z)) = h[h(F(h(z)))] = F(h(z)) = f(h(z)) .$$

Thus (4) holds in this case, too.

It is clear that taking in formula (8) all possible functions $F : D_1 \to \mathbb{C}^*$ we obtain all solutions $f : \mathbb{C}^* \to \mathbb{C}^*$ of equation (4). (In order to get a given solution f of (4) one takes $F = f|D_1$.) Thus we have, since equations (1) and (4) are equivalent.

Proposition 1. *With notation* (2) *and* (5), *for every function* $F : D_1 \to \mathbb{C}^*$ *the function* f *defined by* (8) *satisfies functional equation* (1), *and all the solutions* $f : \mathbb{C}^* \to \mathbb{C}^*$ *of* (1) *may be obtained in this manner.*

Thus formula (8) yields the general solution $f : \mathbb{C}^* \to \mathbb{C}^*$ of equation (1), the function F playing the role of a parameter. We say (see [2] or [3]) that *the general solution of* (1) *depends on an arbitrary function.*

It is readily seen from (8) that equation (1) has a lot of very irregular (e.g., discontinuous or nonmeasurable) solutions: to obtain them it is enough to take in (8) an irregular F. We shall return to the problem of the regularity of solutions of (1) later in this paper. Here we observe only that if the function $F : D_1 \to \mathbb{C}^*$ is continuous on D_1 and, moreover, for real negative z_0 it fulfils the condition

$$\lim_{z \to z_0 , z \in D_1} F(z) = h(F(-z_0)),$$

then the solution f of equation (1) obtained from formula (8) is continuous on $\mathbb{C}^*$. Thus also in the class of the continuous functions $f : \mathbb{C}^* \to \mathbb{C}^*$ the solution of equation (1) depends on an arbitrary function.

Remark 1. In this construction instead of sets (5) we could take arbitrary sets fulfilling conditions (6) and (7) (in the argument we use only these properties, the particular shape of sets (5) is irrelevant), e.g. we could take

$$\begin{cases} D_1 := \{z \in \mathbb{C} \,|\, 0 < |z| < 1\} \cup \{z \in \mathbb{C} \,|\, |z| = 1,\ \operatorname{Im} z > 0\} \cup \{1\}, \\ D_2 := \{z \in \mathbb{C} \,|\, |z| > 1\} \cup \{z \in \mathbb{C} \,|\, |z| = 1,\ \operatorname{Im} z < 0\} \cup \{-1\}. \end{cases}$$

The essential thing is that the set D_1 should contain exactly one point of every couple $\{z, h(z)\}$, $z \in \mathbb{C}^*$ (i.e. of every orbit under h contained in $\mathbb{C}^*$) and $D_2 = \mathbb{C}^* \setminus D_1$.

2. The second method of constructing the general solution of (1) is that of the linearization. Its general principles are explained, e.g., in [3; p. 5], but the details must be worked out separately in every particular case.

First we define a function $\sigma : \mathbb{C}^* \to \mathbb{C}^*$ by the formula

$$\sigma(z) = \begin{cases} z\,, & z \in D_1, \\ 1/\overline{z} = z/|z|^2\,, & z \in D_2, \end{cases} \tag{9}$$

where the sets D_1 and D_2 are given by (5). The direct verification shows that σ satisfies for all $z \in \mathbb{C}^*$ the functional relation (the Schrőder equation; cf. [2], [3])

$$\sigma[h(z)] = -\sigma(z)\,. \tag{10}$$

Moreover, σ is invertible. Indeed, suppose that for some $u, v \in \mathbb{C}^*$ we have

$$\sigma(u) = \sigma(v)\,. \tag{11}$$

By (5) and (9) we have $\sigma(D_1) = D_1$, $\sigma(D_2) = D_2$ and (11) implies according to (6) that the points u and v must both lie in the same set D_i. In other words, either $u, v \in D_1$, or $u, v \in D_2$. In the former case, in view of (9), relation (11) turns into

$$u = v\,, \tag{12}$$

while in the latter case (11) yields $1/\overline{u} = 1/\overline{v}$ which again is equivalent to (12). Thus for arbitrary $u, v \in \mathbb{C}^*$ relation (11) implies (12), which means that the function σ is invertible, as claimed.

Consequently there exists the function $\sigma^{-1} : \mathbb{C}^* \to \mathbb{C}^*$, inverse to σ, and by virtue of (10) it satisfies on $\mathbb{C}^*$ the functional equation

$$\sigma^{-1}(-z) = h(\sigma^{-1}(z))\,. \tag{13}$$

Let $\psi : \mathbb{C}^* \to \mathbb{C}^*$ be an arbitrary odd function:

$$\psi(-z) = -\psi(z)\,, \quad z \in \mathbb{C}^*\,, \tag{14}$$

and define the function $f : \mathbb{C}^* \to \mathbb{C}^*$ by the formula

$$f(z) = \sigma^{-1}[\psi(\sigma(z))]\,, \quad z \in \mathbb{C}^*\,. \tag{15}$$

Function (15) satisfies equation (4) (or, equivalently, equation (1)) on $\mathbb{C}^*$. In fact, according to (15), (10), (14) and (13), we have for arbitrary $z \in \mathbb{C}^*$

$$\begin{aligned} f(h(z)) &= \sigma^{-1}[\psi(\sigma(h(z)))] = \sigma^{-1}[\psi(-\sigma(z))] = \\ &= \sigma^{-1}[-\psi(\sigma(z))] = h[\sigma^{-1}(\psi(\sigma(z)))] = h(f(z))\,. \end{aligned}$$

Conversely, if a function $f : \mathbb{C}^* \to \mathbb{C}^*$ satisfies equation (4) (i.e., (1)) on $\mathbb{C}^*$, then it can be written in form (15), where $\psi(z) := \sigma[f(\sigma^{-1}(z))]$

is an odd function by virtue of (13), (4) and (10). Thus we have the following

Proposition 2. *With notations* (5) *and* (9), *for every odd function* $\psi : \mathbb{C}^* \to \mathbb{C}^*$, *the function* f *defined by* (15) *satisfies the functional equation* (1), *and all the solutions* $f : \mathbb{C}^* \to \mathbb{C}^*$ *of* (1) *may be obtained in this manner.*

Remark 2. In this construction function (9) could be replaced by an arbitrary other particular invertible solution $\sigma : \mathbb{C}^* \to \mathbb{C}^*$ of equation (10). There exist many such solutions (the general invertible solution $\sigma : \mathbb{C}^* \to \mathbb{C}^*$ of (10) depends on an arbitrary function) and any one of them can be used here. The argument depends only on (10) and on the invertibility of σ and not on the particular shape of function (9). However, we have been unable to find a more regular invertible particular solution of equation (10) on $\mathbb{C}^*$.

Formula (15) yields the general solution of equation (1) on $\mathbb{C}^*$. Unavoidably, also this formula contains an arbitrary (odd) function as a parameter. Formula (15) is more elegant and looks more agreeable than formula (8) but its disadvantage is that — due to the peculiar shape of the function σ — it is rather difficult to deduce from (15) the regularity properties of f. From this point of view the third method of solving (1), which we are now about to explain, seems most promising.

3. The third method of constructing the general solution of equation (1) is not new either (cf., e.g., [2, p. 148]), but I know of no place where it would be explained in a more general setting.

Let $f_0 : \mathbb{C}^* \to \mathbb{C}^*$ be a particular solution of equation (1) on $\mathbb{C}^*$ and let $g : \mathbb{C}^* \to \mathbb{C}^*$ be an arbitrary function. Put

$$f(z) = f_0(z)g(z)/\overline{g(h(z))}, \quad z \in \mathbb{C}^*. \tag{16}$$

We have by (2) and (3), since f_0 satisfies equation (4),

$$f(h(z)) = f_0(h(z))g(h(z))/\overline{g(z)} = h(f_0(z))g(h(z))/\overline{g(z)} =$$
$$= -\frac{g(h(z))}{\overline{f_0(z)}\,\overline{g(z)}} = h(f(z)),$$

which means that f satisfies equation (4) on $\mathbb{C}^*$. Conversely, let f and f_0 be arbitrary solutions of equation (4) on $\mathbb{C}^*$ and let $\varphi : D_1 \to \mathbb{C}^*$ (where the sets D_1 and D_2 are given by (5)) be an arbitrary function fulfilling the condition

$$[\varphi(z)]^2 = f(z)/f_0(z)\,, \quad z \in \mathbb{C}^*\,. \tag{17}$$

We define the function $g : \mathbb{C}^* \to \mathbb{C}^*$ by the formula

$$g(z) = \begin{cases} \varphi(z)\,, & z \in D_1, \\ 1/\,\overline{\varphi(h(z))}\,, & z \in D_2. \end{cases} \tag{18}$$

For $z \in D_1$ we have by (7) $h(z) \in D_2$, and according to (18) and (3)

$$g(z) = \varphi(z)\,, \quad g(h(z)) = 1/\,\overline{\varphi(z)}\,, \quad \overline{g(h(z))} = 1/\varphi(z)$$

so that $g(z)/\,\overline{g(h(z))} = [\varphi(z)]^2$ and by (17)

$$g(z)/\,\overline{g(h(z))} = f(z)/f_0(z)\,. \tag{19}$$

For $z \in D_2$ we have by (7) $h(z) \in D_1$ and according to (18)

$$g(z) = 1/\,\overline{\varphi(h(z))}\,, \quad g(h(z)) = \varphi(h(z))\,, \quad \overline{g(h(z))} = \overline{\varphi(h(z))}\,,$$

whence we obtain by virtue of (17), and (2), since both f and f_0 satisfy equation (4),

$$g(z)/\,\overline{g(h(z))} = 1/\left[\overline{\varphi(h(z))}\right]^2 = 1/\,\overline{[\varphi(h(z))]^2} = \overline{f_0(h(z))}/\,\overline{f(h(z))} =$$
$$= \overline{h(f_0(z))}/\,\overline{h(f(z))} = f(z)/f_0(z)\,,$$

i.e. again we get (19). Consequently relation (19), and thus also relation (16), holds for all $z \in \mathbb{C}^*$ and we have proved the following

Proposition 3. *With notation* (2), *if* $f_0 : \mathbb{C}^* \to \mathbb{C}^*$ *is a particular solution of equation* (1), *then for every function* $g : \mathbb{C}^* \to \mathbb{C}^*$ *the function* f *defined by* (16) *satisfies the functional equation* (1), *and all the solutions* $f : \mathbb{C}^* \to \mathbb{C}^*$ *of* (1) *may be obtained in this manner.*

Taking as f_0 the simplest possible particular solution $f_0(z) = z$ of (1), we obtain from (16) the formula

$$f(z) = zg(z)/\,\overline{g(h(z))}\,, \quad z \in \mathbb{C}^*\,. \tag{20}$$

Formula (20) yields the general solution of equation (1) on $\mathbb{C}^*$ and, as was to be expected, it contains an arbitrary function in the role of a parameter.

Remark 3. In each method of solving equation (1) we have used sets (5), but in each instance they played a different role. In the first method sets (5) appeared directly in the formula for the solution, in the second method they were used to construct a particular solution σ of equation (10) and so they appear in formula (15) only indirectly. (The same function σ could also be defined in another way, without appealing to sets (5)). In the third method sets (5) were used in the proof, but not in the formulation of Prop. 3.

4. Now we are going to discuss a number of **examples.**

1. Let $f_0 : \mathbb{C}^* \to \mathbb{C}^*$ be an arbitrary solution of equation (1) on $\mathbb{C}^*$. Taking in (16) $g(z) = c = \text{const} \neq 0$ we obtain

$$f(z) = \eta f_0(z), \quad z \in \mathbb{C}^*, \tag{21}$$

where $\eta = c/\overline{c}$ fulfils the condition

$$|\eta| = 1. \tag{22}$$

Thus, together with f_0 also every function f of form (21), where η fulfils (22), is a solution of (1).

2. In (20) take $g(z) = cz^n$, where $c \neq 0$ is a constant and n is an integer. We obtain

$$f(z) = \eta z^{2n+1}, \quad z \in \mathbb{C}^*, \tag{23}$$

where $\eta = (-1)^n c/\overline{c}$ is a constant fulfilling (22). Functions (23) (with arbitrary $n \in \mathbb{Z}$ and η fulfilling (22)) yield a family of analytic solutions of (1) on $\mathbb{C}^*$ which have a removable singularity or a pole at zero, depending on whether $n \geq 0$ or $n < 0$. (For the converse cf. Section 4).

3. In (20) we take $g(z) = ce^z$, $c \neq 0$. Since $\overline{\exp z} = \exp \overline{z}$, we have $\overline{g(h(z))} = \overline{c}e^{-1/z}$ and

$$f(z) = \eta z e^{z+\frac{1}{z}}, \quad z \in \mathbb{C}^*. \tag{24}$$

Functions (24) (with arbitrary η fulfilling (22)) yield a family of analytic solutions of equation (1) on $\mathbb{C}^*$ which have as essential singularity at zero.

4. Let

$$g(z) = cz^{m_0}(z - u_1)^{m_1} \dots (z - u_p)^{m_p}$$

be a polynomial of degree

$$n = m_0 + m_1 + \cdots + m_p, \tag{25}$$

with distinct roots $u_0 = 0$, $u_1, \dots, u_p$ of multiplicity $m_0 \geq 0$, $m_1 > 0$,

$\dots, m_p > 0$ respectively, $p > 0$. Write

$$v_1 = h(u_1), \dots, v_p = h(u_p). \tag{26}$$

Then

$$\overline{g(h(z))} = (-1)^n \bar{c}\,\bar{u}_1^{m_1} \dots \bar{u}_p^{m_p} z^{-n} (z - v_1)^{m_1} \dots (z - v_p)^{m_p}$$

and, according to (20),

$$f(z) = \frac{\eta a z^{m_0+n+1} (z-u_1)^{m_1} \dots (z-u_p)^{m_p}}{(z-v_1)^{m_1} \dots (z-v_p)^{m_p}} \tag{27}$$

where

$$a = (-1)^{m_1+\dots+m_p} \left[\bar{u}_1^{m_1} \dots \bar{u}_p^{m_p}\right]^{-1} = v_1^{m_1} \dots v_p^{m_p} \tag{28}$$

and $\eta = (-1)^{n-m_0} c/\bar{c}$ is a constant fulfilling (22). For arbitrary distinct $u_1, \dots, u_p \in \mathbb{C}^*$, arbitrary integers $m_0 \geq 0$, $m_1 > 0, \dots, m_p > 0$, and arbitrary η fulfilling (22), function (27), where a, n, and $v_1, \dots, v_p$ are given by (28), (25) and (26), respectively, is a meromorphic solution of equation (1) with poles at $v_1, \dots, v_p$. (But if some u_i are equal to some v_j with $m_i \geq m_j$ for the corresponding indices i, j, then function (27) has removable singularities at these points v_j).

As a matter of fact functions (27) are not solutions of equation (1) on $\mathbb{C}^*$ in the spirit of the earlier parts of this paper. They do not map $\mathbb{C}^*$ into $\mathbb{C}^*$: they have zeros and poles on $\mathbb{C}^*$. But they satisfy equation (1) on $\mathbb{C}^* \setminus \{u_1, \dots, u_p, v_1, \dots, v_p\}$, and even on the whole $\mathbb{C}^*$, in the sense that the product $f(z)\,\overline{f(-1/\bar{z})}$ is equal to -1 everywhere on $\mathbb{C}^*$ except at the points $u_1, \dots, u_p$, $v_1, \dots, v_p$, where it has removable singularities.

5. It is easy to check that the functions

$$f(z) = \eta\, \bar{z}^{2n+1}, \quad z \in \mathbb{C}^*, \tag{29}$$

where η fulfils (22) and n is an integer, satisfy equation (1) on $\mathbb{C}^*$. Functions (29) are continuous, but nowhere differentiable on $\mathbb{C}^*$. Similarly the functions

$$f(z) = \eta z^{2n+1}\, \bar{z}^{2m}, \quad z \in \mathbb{C}^*,$$

and

$$f(z) = \eta z^{2n}\, \bar{z}^{2m+1}, \quad z \in \mathbb{C}^*,$$

(obtained from (16) on taking $f_0(z) = (-1)^m \eta z^{2n+1}$, $g(z) = \bar{z}^m$ and $f_0(z) = (-1)^n \eta\, \bar{z}^{2m+1}$, $g(z) = z^n$), where m, n, are integers and η is a constant fulfilling (22), yield families of continuous nondifferentiable solutions of equation (1).

6. Let D_0 denote the set

$$D_0 = \{z \in \mathbb{C}^* \,|\, \operatorname{Re} z\,, \operatorname{Im} z \in \mathbb{Q}\}\,.$$

Both sets D_0 and $\mathbb{C}^* \setminus D_0$ are dense in $\mathbb{C}^*$ and $h(z) \in D_0$ for $z \in D_0$, while for $z \in \mathbb{C}^* \setminus D_0$ also $h(z) \in \mathbb{C}^* \setminus D_0$. Therefore the function $f : \mathbb{C}^* \to \mathbb{C}^*$ defined by

$$f(z) = \begin{cases} f_1(z)\,, & z \in D_0\,, \\ f_2(z)\,, & z \in \mathbb{C}^* \setminus D_0\,, \end{cases} \tag{30}$$

satisfies equation (4) (and thus also equation (1)) on $\mathbb{C}^*$ whenever the functions $f_1 : \mathbb{C}^* \to \mathbb{C}^*$ and $f_2 : \mathbb{C}^* \to \mathbb{C}^*$ do. Taking in particular

$$f_1(z) = z\,, \quad f_2(z) = -z\,, \quad z \in \mathbb{C}^*\,,$$

we obtain from (30)

$$f(z) = \begin{cases} z\,, & z \in D_0\,, \\ -z\,, & z \in \mathbb{C}^* \setminus D_0\,. \end{cases} \tag{31}$$

Function (31) is a measurable, discontinuous (at every point of $\mathbb{C}^*$) solution of equation (1) on $\mathbb{C}^*$.

Such examples could be multiplied. The functions given in examples 3–6 are only a few representatives of solutions of equation (1) in given regularity cases. It is not difficult to show that in each of these classes the general solution of (1) depends on an arbitrary function. The same is true also about nonmeasurable solutions $f : \mathbb{C}^* \to \mathbb{C}^*$ of equation (1). In order to obtain such solutions it is enough to take a nonmeasurable $F : D_1 \to \mathbb{C}^*$ in formula (8).

Therefore the simple uniqueness theorem which we are going to prove in the next section, in spite of the fact that the conditions imposed on f are quite strong, nevertheless seems to be of a considerable interest.

5. As pointed out at the beginning of this paper, the function h given by (2) has no fixed points of order 1. Therefore the conditions ensuring the uniqueness of solutions of equation (1) must have a global character and essential use must be made of the involved structure of the complex plane.

Theorem. *Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a solution of equation* (1) *on $\mathbb{C}^*$ and suppose that f is analytic on $\mathbb{C}^*$ and has a removable singularity or a pole at the origin. Then f has form* (23)*, where n is an integer and η is a constant fulfilling condition* (22).

Proof. Suppose that an $f : \mathbb{C}^* \to \mathbb{C}^*$ fulfils the conditions of the theorem. Thus there exists an entire function $\varphi : \mathbb{C}^* \to \mathbb{C}^*$ and an integer p (positive, negative, or zero) such that

$$f(z) = z^p \varphi(z), \quad z \in \mathbb{C}^*, \tag{32}$$

and

$$\varphi(0) \neq 0. \tag{33}$$

(f, and hence φ, cannot be zero on $\mathbb{C}^*$ because of (1)). The function $|\varphi|$ is continuous at zero, therefore, in view of (33), there exist positive constants a, ε and r such that

$$0 < a - \varepsilon < |\varphi(z)| < a + \varepsilon, \quad |z| < r. \tag{34}$$

Now we insert (32) into (1) to obtain

$$(-1)^p \varphi(z) \overline{\varphi(-1/\overline{z})} = -1, \quad z \in \mathbb{C}^*, \tag{35}$$

that is,

$$\varphi(z) = (-1)^{p-1} / \overline{\varphi(-1/\overline{z})}, \quad z \in \mathbb{C}^*. \tag{36}$$

For $|z| > 1/r$ we have $|-1/\overline{z}| < r$ so that, by virtue of (34),

$$a - \varepsilon < |\varphi(-1/\overline{z})| < a + \varepsilon, \quad |z| > 1/r,$$

and in particular

$$1/\left|\overline{\varphi(-1/\overline{z})}\right| = 1/\left|\varphi(-1/\overline{z})\right| < 1/(a-\varepsilon), \quad |z| > 1/r. \tag{37}$$

Relations (37) and (36) imply that the entire function φ is bounded on $\mathbb{C}$ and thus it must be constant:

$$\varphi(z) = \eta = \text{const}, \quad z \in \mathbb{C} \tag{38}$$

Inserting (38) into (35) we obtain $|\eta|^2 = (-1)^{p-1}$, which implies (22) and, moreover, shows that $p - 1$ must be an even number:

$$p - 1 = 2n. \tag{39}$$

Formula (23) results now from (32), (38) and (39). ◊

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