A CHARACTERIZATION OF EVOLU-TIONARILY STABLE STRATEGIES

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Abstract: Let n be an integer > 1 and let A be a real $n \times n$-matrix. Put $S := \{(x_1, \dots, x_n) \in [0, \infty)^n \mid x_1 + \dots + x_n = 1\}$. An element p of S is called an evolutionarily stable strategy (ESS) (with respect to the payoff matrix A) if $pAp^T \geqq xAp^T$ for all $p \in S$ and if $pAx^T > xAx^T$ for all $x \in S \setminus \{p\}$ with $xAp^T = pAp^T$. For all $x \in \mathbb{R}^n$ the set $\operatorname{supp} x := \{i \in \{1, \dots, n\} \mid x_i \neq 0\}$ is called the support of x. The aim of this paper is to provide a criterion for the existence of an ESS with given support and to give a concrete description of such an ESS. By means of this characterization the case $n = 3$ is completely settled.

When applying game-theoretical methods to biological situations, Maynard Smith and Price ([15]) introduced the notion of an evolutionarily stable strategy (ESS). For basic literature on ESS's cf. e.g. [5], [10] and [14]. It is well-known that (for given payoff matrix) there exists at most one ESS with given support. We characterize the existence of such an ESS by means of inequalities between certain determinants

and by the strict copositiveness of a certain matrix. We also describe the unique ESS in this case. As an application, we give a complete description of all ESS's in the case of three pure strategies. For literature concerning the characterization of ESS's cf. e.g. [1] (correcting [9]), [4], [6], [7], [11], [12] and [17].

Let n be an arbitrary fixed integer > 1 and let $A = (a_{ij})_{i,j=1,\dots,n}$ be a real $n \times n$-matrix. Let $\mathbf{R}^n$ denote the set of all n-dimensional real row vectors and for any $a \in \mathbf{R}^n$ and any $i \in \{1,\dots,n\}$ let a_i denote the i-th component of a. If not explicitly stated otherwise, all summation indices run from 1 to n. Put

$$S := \left\{ x \in [0,\infty)^n \mid \sum_i x_i = 1 \right\},$$

$$N := \{ x \in S \mid xAx^T \geqq yAx^T \text{ for all } y \in S \},$$

$$E := \{ x \in N \mid xAy^T > yAy^T \text{ for all } y \in S \setminus \{x\} \text{ with } yAx^T = xAx^T \}.$$

The elements of S are called *(mixed) strategies*, those of N Nash equilibria (with respect to the payoff matrix A) and those of E evolutionarily stable strategies (ESS's) (with respect to the payoff matrix A). For all $i,j,k,l \in \{1,\dots,n\}$ put $b_{ij}^{(k)} := a_{ik} + a_{kj} - a_{ij}$, $c_{ij}^{(k,l)} := b_{ij}^{(k)}$ if $j \neq l$ and $c_{ij}^{(k,l)} := a_{ik}$ if $j = l$. For $I \subseteq \{1,\dots,n\}$ the matrix A is called *strictly I-copositive* if $xAx^T > 0$ for all $x \in \mathbf{R}^n \setminus \{(0,\dots,0)\}$ with $x_i \geq 0$ for all $i \in I$. A is called *strictly copositive* if it is strictly $\{1,\dots,n\}$-copositive. For literature concerning strict copositiveness cf. e.g. [4], [8], [12] and [16]. For all $i \in \{1,\dots,n\}$ let e_i denote the element $(0,\dots,0,1,0,\dots,0)$ of $\mathbf{R}^n$ with 1 on the i-th place and 0 on the other places. For all $x \in \mathbf{R}^n$ put $\operatorname{supp} x := \{i \in \{1,\dots,n\} \mid x_i \neq 0\}$ and $J(x) := \{i \in \{1,\dots,n\} \mid e_i Ax^T = xAx^T\}$. $\operatorname{supp} x$ is called the *support* of x. Finally, let (T) denote the following condition:

(T) a,b,c,d,e,f are real numbers such that $a+c, b+e, d+f > 0$ and such that $(a+c)^{1/2}, (b+e)^{1/2}$ and $(d+f)^{1/2}$ are the lengths of the sides of a triangle of positive area.

Lemma 1 (cf. [2]). *Let $p \in N$ and $q \in E$ and assume $p \neq q$. Then* $\operatorname{supp} p \not\subseteq J(q)$.

Proof. $\operatorname{supp} p \subseteq J(q)$ would imply $pAq^T = \sum_{i \in \operatorname{supp} p} p_i(e_i Aq^T) = \sum_{i \in \operatorname{supp} p} p_i(qAq^T) = qAq^T$ which together with $q \in E \setminus \{p\}$ would lead to $qAp^T > pAp^T$ contradicting $p \in N$. $\Diamond$

Lemma 2 (cf. [13]). *Let $p \in N$. Then $\operatorname{supp} p \subseteq J(p)$.*

Proof. $pAp^T = \sum\limits_{i \in \operatorname{supp} p} p_i(e_i Ap^T) \leq \sum\limits_{i \in \operatorname{supp} p} p_i(pAp^T) = pAp^T.$ $\Diamond$

Lemma 3. *Let $p \in N$ and $q \in E$ and assume $p \neq q$. Then $\operatorname{supp} p \not\subseteq \operatorname{supp} q$.*

Proof. $\operatorname{supp} p \subseteq \operatorname{supp} q$ would imply $\operatorname{supp} p \subseteq J(q)$ according to Lemma 2. But this contradicts Lemma 1. $\Diamond$

Theorem 4. *A possesses at most one ESS with given support.*

Proof. Lemma 3. $\Diamond$

Lemma 5 (cf. [3]). *Let $b \in \mathbb{R}^n$. Then A and $(a_{ij} + b_j)_{i,j=1,\dots,n}$ have the same ESS's.*

Proof. Put $B := (a_{ij} + b_j)_{i,j=1,\dots,n}$. Then for all $x, y, z \in S$ we have
$(x - y)Bz^T = (x - y)Az^T + \sum\limits_j b_j z_j \sum\limits_i (x_i - y_i) = (x - y)Az^T.$ $\Diamond$

Theorem 6. *A and $(a_{ij} - a_{jj})_{i,j=1,\dots,n}$ have the same ESS's.*

Proof. Lemma 5. $\Diamond$

In the following we assume (without loss of generality) $a_{ii} = 0$ for all $i = 1, \dots, n$.

Lemma 7. *Let $p \in S$. Then the following are equivalent:*

  (i) $p \in N$;

  (ii) $pAp^T \geq e_i Ap^T$ for all $i = 1, \dots, n$.

Proof. (i) $\Rightarrow$ (ii): Trivial.

    (ii) $\Rightarrow$ (i): For all $x \in S$ we have $xAp^T = \sum\limits_i x_i(e_i Ap^T) \leq$
$\leq \sum\limits_i x_i(pAp^T) = pAp^T.$ $\Diamond$

Lemma 8. *Let $i, k \in \{1, \dots, n\}$ and $p \in S$. Then the following are equivalent:*

  (i) $e_i Ap^T \left\{ \begin{smallmatrix} \leq \\ = \\ > \end{smallmatrix} \right\} e_k Ap^T$;

  (ii) $\sum\limits_{j \in (\operatorname{supp} p) \setminus \{k\}} b_{ij}^{(k)} p_j \left\{ \begin{smallmatrix} \geq \\ = \\ < \end{smallmatrix} \right\} a_{ik}$.

Proof. The following are equivalent:

    (i),

$$\sum\limits_j a_{ij} p_j \left\{ \begin{smallmatrix} \leq \\ = \\ > \end{smallmatrix} \right\} \sum\limits_j a_{kj} p_j,$$

$$\sum\limits_{j \neq k} a_{ij} p_j + a_{ik} \left(1 - \sum\limits_{j \neq k} p_j \right) \left\{ \begin{smallmatrix} \leq \\ = \\ > \end{smallmatrix} \right\} \sum\limits_{j \neq k} a_{kj} p_j,$$

$$\sum\limits_{j \neq k} b_{ij}^{(k)} p_j \left\{ \begin{smallmatrix} \geq \\ = \\ < \end{smallmatrix} \right\} a_{ik} \text{ and}$$

    (ii). $\Diamond$

Lemma 9 (cf. [1] and [4]). *Let $p \in N$ and $y \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\}$ with* $\operatorname{supp} y \subseteq J(p)$, $y_i \geq 0$ *for all* $i \in J(p) \setminus \operatorname{supp} p$ *and* $\sum_i y_i = 0$. *Then there exists an $x \in S \setminus \{p\}$ and a positive real number c such that $x - p = cy$.*

Proof. Put $c := \min\{-p_i/y_i \mid i \in \{1, \ldots, n\}, y_i < 0\}$ and $x := p + cy$. Then c is well-defined and positive. Let $j \in \{1, \ldots, n\}$. If $y_j < 0$ then $p_j > 0$ and $c \leq -p_j/y_j$ and hence $x_j = p_j + cy_j \geq 0$. If $y_j \geq 0$ then also $x_j \geq 0$. The rest of the proof is obvious. $\Diamond$

Proposition 10 (cf. [1] and [4]). *Let $p \in N$ and $k \in \operatorname{supp} p$. Then the following are equivalent:*

(i) $p \in E$;

(ii) $|J(p)| = 1$ *or* ($|J(p)| > 1$ *and* $(b_{ij}^{(k)})_{i,j \in J(p) \setminus \{k\}}$ *is strictly* $(J(p) \setminus \operatorname{supp} p)$*-copositive*).

Proof. For all $x \in S$ we have $xAp^T = \sum_i x_i(e_i Ap^T) \leq \sum_i x_i(pAp^T) = pAp^T$. Hence, for $x \in S$ the equality $xAp^T = pAp^T$ is equivalent to $\operatorname{supp} x \subseteq J(p)$. Put $U := \{x \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\} \mid \operatorname{supp} x \subseteq J(p), x_i \geq 0$ for all $i \in J(p) \setminus \operatorname{supp} p$ and $\sum_i x_i = 0\}$. Using Lemma 9 we see that the following are equivalent:

(i),

$(x - p)A(x - p)^T < 0$ for all $x \in S \setminus \{p\}$ with $\operatorname{supp} x \subseteq J(p)$, $xAx^T < 0$ for all $x \in U$,

$\sum_{i,j} a_{ij} x_i x_j < 0$ for all $x \in U$,

$\sum_{i,j \neq k} a_{ij} x_i x_j + \sum_{i \neq k} a_{ik} x_i \left(-\sum_{j \neq k} x_j\right) + \sum_{j \neq k} a_{kj}\left(-\sum_{i \neq k} x_i\right) x_j < 0$ for all $x \in U$,

$\sum_{i,j \neq k} b_{ij}^{(k)} x_i x_j > 0$ for all $x \in U$,

$\sum_{i,j \in J(p) \setminus \{k\}} b_{ij}^{(k)} x_i x_j > 0$ for all $x \in U$ and

(ii). $\Diamond$

(Observe that $U = \emptyset$ if $|J(p)| = 1$.)

Theorem 11. *Let $k \in \{1, \ldots, n\}$ and put $J := \{i \in \{1, \ldots, n\} \setminus \{k\} \mid a_{ik} = 0\}$. Then the following are equivalent:*

(i) $e_k \in E$;

(ii) (1) *and* (2) *hold:*

(1) $a_{ik} < 0$ *for all* $i \in \{1, \ldots, n\} \setminus (J \cup \{k\})$,

(2) $J = \emptyset$, *or* $J \neq \emptyset$ *and* $(a_{kj} - a_{ij})_{i,j \in J}$ *is strictly copositive.*

Proof. Follows from Lemma 7 and Prop. 10 by observing that $b_{ij}^{(k)} = a_{kj} - a_{ij}$ for all $i, j \in J$. $\Diamond$

Lemma 12. *Let $B = (b_{ij})_{i,j=1,\dots,n}$ be an arbitrary (not necessarily symmetric) real positive definite $n \times n$-matrix. Then $|B| > 0$.*

Proof. For all $x \in \mathbf{R}^n \setminus \{(0,\dots,0)\}$ we have $x((B + B^T)/2)x^T = xBx^T > 0$ and hence $|(B + B^T)/2| > 0$. Now assume $|B| \leq 0$. Put $B(t) := tB + (1 - t)(B + B^T)/2$ for all $t \in \mathbf{R}$. Then $t \mapsto |B(t)|$ is a continuous function, $|B(0)| > 0$ and $|B(1)| \leq 0$ and hence there exists a $t_0 \in (0, 1]$ with $|B(t_0)| = 0$. Therefore there exists a $d \in \mathbf{R}^n \setminus \{(0,\dots,0)\}$ with $B(t_0)d^T = (0,\dots,0)^T$. Now we would obtain $dBd^T = dB(t_0)d^T = 0$ contradicting the positive definiteness of B. Hence $|B| > 0$. $\Diamond$

Theorem 13. *Let $I \subseteq \{1,\dots,n\}$ with $|I| > 1$, let $k \in I$ and put $J := I \setminus \{k\}$, $D := |(b_{ij}^{(k)})_{i,j \in J}|$, $D_l := |(c_{ij}^{(k,l)})_{i,j \in J}|$ for all $l \in J$ and $K := \left\{ i \in \{1,\dots,n\} \setminus I \mid \sum_{j \in J} b_{ij}^{(k)} D_j = a_{ik}D \right\}$. Then the following are equivalent:*

(i) *A possesses an ESS with support I;*

(ii) *(1)–(4) hold:*

    (1) *$D_j > 0$ for all $j \in J$,*

    (2) *$\sum_{j \in J} D_j < D$,*

    (3) *$\sum_{j \in J} b_{ij}^{(k)} D_j > a_{ik}D$ for all $i \in \{1,\dots,n\} \setminus (I \cup K)$,*

    (4) *$(b_{ij}^{(k)})_{i,j \in J \cup K}$ is strictly K-copositive.*

If this is the case then the corresponding ESS p is given by $p_j := D_j/D$ for all $j \in J$, $p_k := 1 - \sum_{j \in J} p_j$ and $p_i := 0$ otherwise.

Proof. (i) $\Rightarrow$ (ii): Let p denote the ESS with support I. Because of Lemma 2 and Prop. 10 $(b_{ij}^{(k)})_{i,j \in J}$ is positive definite. According to Lemma 12 we have $D > 0$. Because of Lemmas 2 and 8 it holds $\sum_{j \in J} b_{ij}^{(k)} p_j = a_{ik}$ for all $i \in J$. Applying Cramer's rule we obtain $p_j = D_j/D$ for all $j \in J$. Now (1) and (2) follow immediately. Observe that $J(p) = I \cup K$ (because of Lemmas 2 and 8). (3) follows from Lemmas 2 and 8. (4) finally follows from Prop. 10.

(ii) $\Rightarrow$ (i): Define $p \in S$ by $p_j := D_j/D$ for all $j \in J$, $p_k := 1 - \sum_{j \in J} p_j$ and $p_i := 0$ otherwise. Because of (1) and (2) $\operatorname{supp} p = I$

and $D > 0$. According to Cramer's rule $\sum\limits_{j \in J} b_{ij}^{(k)} p_j = a_{ik}$ for all $i \in$

$\in J$. Because of (3) we have $\sum\limits_{j \in J} b_{ij}^{(k)} p_j \geq a_{ik}$ for all $i = 1, \ldots, n$. Now Lemmas 2, 7 and 8 imply $p \in N$. Using (3) and Lemmas 2 and 8 we obtain $J(p) = I \cup K$. Now $p \in E$ follows from (4) and Prop. 10. $\lozenge$

Theorem 14 (cf. [17]). *Let* $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ *with* $a, b \in \mathbf{R}$. *Then* (i)–(v) *hold:*

  (i) *If* $a = b = 0$ *then* $E = \emptyset$,

  (ii) *if* $b \leq 0 \leq a \neq b$ *then* $E = \{e_1\}$,

  (iii) *if* $a \leq 0 \leq b \neq a$ *then* $E = \{e_2\}$,

  (iv) *if* $a, b > 0$ *then* $E = \{(a, b)/(a + b)\}$,

  (v) *if* $a, b < 0$ *then* $E = \{e_1, e_2\}$.

Proof. We apply Theorems 11 and 13. $e_1 \in E$ if and only if $b < 0$ or $b = 0 < a$. $e_2 \in E$ if and only if $a < 0$ or $a = 0 < b$. A has an ESS with support $\{1, 2\}$ if and only if $a, b > 0$. If this is the case then this ESS reads $(a, b)/(a + b)$. $\lozenge$

Lemma 15. *Let* $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $a, b, c, d \in \mathbf{R}$. *Then* (i) *and* (ii) *hold:*

  (i) B *is positive definite if and only if* $a, d > 0$ *and* $|b + c| < 2(ad)^{1/2}$,

  (ii) B *is strictly copositive if and only if* $a, d > 0$ *and* $b + c > -2(ad)^{1/2}$.

Proof. (i) If B is positive definite then $a = e_1 B e_1^T > 0$, $d = e_2 B e_2^T > 0$ and $a(4ad - (b + c)^2) = (b + c, -2a) B (b + c, -2a)^T > 0$ whence $|b + c| < 2(ad)^{1/2}$. Conversely, assume $a, d > 0$ and $|b + c| < 2(ad)^{1/2}$. Then $4ax B x^T = (2ax_1 + (b + c)x_2)^2 + (4ad - (b + c)^2)x_2^2 > 0$ for all $x \in \mathbf{R}^2 \setminus \{(0, 0)\}$.

    (ii) If B is strictly copositive then $a = e_1 B e_1^T > 0$, $d = e_2 B e_2^T > 0$ and $(ad)^{1/2}(2(ad)^{1/2} + b + c) = (d^{1/2}, a^{1/2}) B (d^{1/2}, a^{1/2})^T > 0$ whence $b + c > -2(ad)^{1/2}$. Conversely, assume $a, d > 0$ and $b + c > -2(ad)^{1/2}$. Then $x B x^T = (a^{1/2} x_1 - d^{1/2} x_2)^2 + (2(ad)^{1/2} + b + c)x_1 x_2 > 0$ for all $x \in [0, \infty)^2 \setminus \{(0, 0)\}$. $\lozenge$

Lemma 16. *Let* $a, b, c, d, e, f \in \mathbf{R}$ *with* $a + c, b + e > 0$. *Then the following are equivalent:*

  (i) $(a + c)^2 + (b + e)^2 + (d + f)^2 - 2(a + c)(b + e) - 2(a + c)(d + f) -$
     $- 2(b + e)(d + f) < 0$;

  (ii) (T).

Proof. Put $g := (a + c)^2 + (b + e)^2 + (d + f)^2 - 2(a + c)(b + e) - 2(a + c)(d + f) - 2(b + e)(d + f)$. Then $g = (a + c - b - e - d - f)^2 - 4(b + e)(d + f)$.

Hence (i) implies $d+f > 0$. Under the additional assumption $d+f > 0$ the following are equivalent:

$g < 0$,

$(a+c-b-e-d-f)^2 < 4(b+e)(d+f)$,

$|a+c-b-e-d-f| < 2(b+e)^{1/2}(d+f)^{1/2}$,

$a+c-b-e-d-f < 2(b+e)^{1/2}(d+f)^{1/2}$ and $-a-c+b+e+d+f < 2(b+e)^{1/2}(d+f)^{1/2}$,

$a+c < ((b+e)^{1/2}+(d+f)^{1/2})^2$ and $((b+e)^{1/2}-(d+f)^{1/2})^2 < a+c$,

$(a+c)^{1/2} < (b+e)^{1/2}+(d+f)^{1/2}$ and $|(b+e)^{1/2}-(d+f)^{1/2}| < (a+c)^{1/2}$,

$(a+c)^{1/2} < (b+e)^{1/2}+(d+f)^{1/2}$, $(b+e)^{1/2}-(d+f)^{1/2} < (a+c)^{1/2}$ and $(d+f)^{1/2}-(b+e)^{1/2} < (a+c)^{1/2}$,

(T). ◊

Theorem 17 (cf. [17]). *Assume* $A = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$ *with* $a,b,c,d,e,f \in \mathbb{R}$ *and put* $D_1 := ad+bf-df$, $D_2 := bc+de-be$ *and* $D_3 := ae+cf-ac$. *Then* (i)–(vii) *hold:*

(i) $e_1 \in E$ *if and only if* $c,e < 0$ *or* $c < 0 = e < b$ *or* $e < 0 = c < a$ *or* $(c = e = 0 < a,b$ *and* $d+f < (a^{1/2} + b^{1/2})^2)$,

(ii) $e_2 \in E$ *if and only if* $a,f < 0$ *or* $a < 0 = f < d$ *or* $f < 0 = a < c$ *or* $(a = f = 0 < c,d$ *and* $b+e < (c^{1/2} + d^{1/2})^2)$,

(iii) $e_3 \in E$ *if and only if* $b,d < 0$ *or* $b < 0 = d < f$ *or* $d < 0 = b < e$ *or* $(b = d = 0 < e,f$ *and* $a+c < (e^{1/2} + f^{1/2})^2)$,

(iv) *A possesses an ESS with support* $\{1,2\}$ *if and only if* $D_3 < 0 < a,c$ *or* $(D_3 = 0 < a,c$ *and* (T)). *In this case the corresponding ESS reads* $(a,c,0)/(a+c)$,

(v) *A possesses an ESS with support* $\{1,3\}$ *if and only if* $D_2 < 0 < b,e$ *or* $(D_2 = 0 < b,e$ *and* (T)). *In this case the corresponding ESS reads* $(b,0,e)/(b+e)$,

(vi) *A possesses an ESS with support* $\{2,3\}$ *if and only if* $D_1 < 0 < d,f$ *or* $(D_1 = 0 < d,f$ *and* (T)). *In this case the corresponding ESS reads* $(0,d,f)/(d+f)$,

(vii) *A possesses an ESS with support* $\{1,2,3\}$ *if and only if* $D_1, D_2, D_3 > 0$ *and* (T). *In this case the corresponding ESS reads* $(D_1, D_2, D_3)/(D_1 + D_2 + D_3)$.

Proof. (i)–(iii) follow from Th. 11 and Lemma 15. (iv)–(vii) follow from Th. 13 and from Lemmas 15 and 16. ◊

Theorem 18. *Let* $A = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$ *with* $a, b, c, d, e, f \in \mathbb{R}$. *Then*
$E = \{e_1, e_2, e_3\}$ *if and only if* $a, b, c, d, e, f < 0$.
Proof. Theorem 17. $\Diamond$

Theorem 19. *Let* $A = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$ *with* $a, b, c, d, e, f \in \mathbb{R}$. *Then* (i)
and (ii) *hold:*

(i) *If* $a, d, e \geq 0$ *and* $b, c, f \leq 0$ *and* $\min(a + c, b + e, d + f) \leq 0$ *then*
$E = \emptyset$,

(ii) *if* $a, d, e \leq 0$ *and* $b, c, f \geq 0$ *and* $\min(a + c, b + e, d + f) \leq 0$ *then*
$E = \emptyset$.

Proof. Theorem 17. $\Diamond$

Theorem 20 (cf. [17]). *Let* $i, j, k \in \{1, \ldots, n\}$ *with* $i \neq j \neq k \neq i$.
Then $\{\{i, j\}, \{j, k\}, \{k, i\}\} \not\subseteq \{\operatorname{supp} p \mid p \in E\}$.

Proof. Assume $\{\{i, j\}, \{j, k\}, \{k, i\}\} \subseteq \{\operatorname{supp} p \mid p \in E\}$. Then there
exist $p, q, r \in E$ with $\operatorname{supp} p = \{i, j\}$, $\operatorname{supp} q = \{j, k\}$ and $\operatorname{supp} r =$
$= \{k, i\}$. Because of Th. 13 $a_{ij}, a_{ik}, a_{ji}, a_{jk}, a_{ki}, a_{kj} > 0$ and $a_{kj}/a_{ij} +$
$+ a_{ki}/a_{ji}, a_{ik}/a_{jk} + a_{ij}/a_{kj}, a_{ji}/a_{ki} + a_{jk}/a_{ik} \leq 1$. Hence $a_{kj}/a_{ij} \leq 1/2$
or $a_{ki}/a_{ji} \leq 1/2$. In the first case $a_{ij}/a_{kj} \geq 2$ contradicting $a_{ik}/a_{jk} +$
$+ a_{ij}/a_{kj} \leq 1$. Similarly, the second case leads to a contradiction. $\Diamond$

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