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ROBOT-MANIPULATORS AS SUB-MANIFOLDS

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Abstract: Robot-manipulators with less than six degrees of freedom are considered as submanifolds of the pseudo-Riemannian Lie group C_6 of all orientation preserving congruences of the Euclidean space. They are generalisations of quadratical ruled surfaces in Euclidean geometry to the geometry of C_6. In the paper we discuss the problem of existence of one more "straight" line of such a submanifold and describe relations of this problem to the geometry of the motion of robot-manipulators.

The paper is a straightforward continuation of [1] and [2] and therefore we shall use results and denotations from [1] and [2] without special reference. In the presented paper we shall limit ourselves to robot-manipulators with less then 6 degrees of freedom and we shall treat them as submanifolds of the pseudo-Riemannian homogeneous space C_6.

Let us consider a p-parametric robot-manipulator g as the mapping

$$g\colon \mathbb{R}^p \to C_6 : [u_1, \dots, u_p] \to g(u_1, \dots, u_p) = g_1(u_1). \dots .g_p(u_p),$$

where $g_i(u_i) = \exp(u_i X_i)$ and $X_1, \dots, X_p$ are linearly independent vectors from L. g determines an imbedded submanifold of C_6 on some neighbourhood $U(0)$ of $0 \in \mathbb{R}^p$;let us denote $g(U) = M$.

Robot manipulators with $p < 6$ were locally characterized as submanifolds of C_6 in [1]: Let us notice that geodesic lines in C_6 are left or right translates of screw-motions (including rotations and translations as special cases). From the definition of the robot manipulator we see that M is a submanifold such that it has p independent geodesic lines passing through each of its points. It was proved in [1], Th. 1, that this property is also a local characterisation of robot-manipulators in the general case – if a submanifold of C_6 has the above mentioned property, it is locally a robot-manipulator (if we consider only rotations and translations, we have to take isotropic geodesic lines only). This shows that a robot-manipulator is a generalisation of a quadratical ruled surface in the Euclidean space to the geometry of the pseudo-Riemannian space C_6 – a p-parametric robot-manipulator is characterized as a submanifold of C_6 with p systems of straight lines (geodesics) on it similarly as quadratical ruled surfaces in E_3 are the only 2-dimensional submanifolds with two systems of straight lines on them.

Similarly as in the 6-parametric case we have the coordinate system $u = [u_1, \dots, u_p]$ given on g in a neighbourhood $U(0)$ of $0 \in \in \mathbb{R}^p$. The induced pseudo-Riemannian metric is given by the same formula as in the 6-parametric case, $h_{ij} =< Y_i, Y_j >$, where $Y_i = = Adg_p^{-1} \dots Adg_{i+1}^{-1} X_i$. The affine connection induced by h_{ij} is the Levi–Civita connection and therefore it is given by the same formula as in [2],

$$\Gamma_{ij,m} = \frac{1}{2}\varepsilon_{ij} < [Y_i, Y_j], Y_m > \quad i, j, m = 1, \dots, p.$$

The same is true for equations of geodesic lines, which are

$$u_i'' + \Gamma^i_{jk} u_j' u_k' = 0 \quad i, j, k = 1, \dots, p,$$

as in any submanifold of the pseudo-Riemannian space.

From now on we shall consider robot-manipulators with rotational axes only. In this case we have $h_{ii} = 0$. As the values of $g_{ij}, h_{ij}, \Gamma_{ij,k}$ depend only on the instantaneous position of axes of the robot-manipulator, it is not difficult to find the geometrical meaning of $g_{ij}, h_{ij}, \Gamma_{ij,k}$ for a given robot-manipulator.

Let Y_q, Y_r, Y_s be three pairwise different axes of the p-parametric robot-manipulator g, (i, j, k) be an even (cyclic) permutation of (q, r, s). Let us denote α_k the angle of Y_i, Y_j, a_k the distance of Y_i and Y_j, $C_k = = \cos \alpha_k, S_k = \sin \alpha_k$. Let $Y_i = (x_i; y_i), Y_j = (x_j; y_j), Y_k = (x_k; y_k), \delta =$

$= |x_i, x_j, x_k| > 0$, where $x_i^2 = x_j^2 = x_k^2 = 1, (x_i, y_i) = (x_j, y_j) = (x_k, y_k) = 0$.

Lemma 1. $g_{ij} = C_k, h_{ij} = \frac{1}{2} S_k a_k, \Gamma_{ij,k} = \frac{1}{2\delta}\varepsilon_{ij} S(a_i S_i(C_j C_k - C_i))$, *where S denotes the cyclic sum over (i, j, k).*

Proof. Let us write $y_i = m_i^a x_a$. We have $(x_i, x_j) = C_k, (x_i, y_j) + (x_j, y_i) = a_k S_k, <[Y_i, Y_j], Y_k> = |x_i, x_j, y_k| + |x_i, y_j, x_k| + |y_i, x_j, x_k| = \delta Tr(m)$. $a_i S_i = (x_j, y_k) + (x_k, y_j) = (x_j, m_k^i x_i + m_k^j x_j + m_k^k x_k) + (x_k, m_j^i x_i + m_j^j x_j + m_j^k x_k) = m_k^i C_k + m_k^j + m_k^k C_i + m_j^i C_j + m_j^j C_i + m_j^k$. $S[a_i S_i(C_j C_k - C_i)] = S[m_k^i C_j C_k^2 + m_j^i C_j^2 C_k + (m_k^j + m_j^k) C_j C_k + (m_k^k + m_j^j) C_i C_j C_k - m_k^i C_i C_k - m_j^i C_j C_i - (m_k^j + m_j^k) C_i - (m_k^k + m_j^j) C_i^2] = S[-m_k^i C_j S_k^2 - m_j^i C_k S_j^2 + (m_k^k + m_j^j)(C_i C_j C_k - C_i^2)] = S[m_j^j S_j^2 + (m_k^k + m_j^j)(C_i C_j C_k - C_i^2)] = \delta^2 Tr(m)$, because $(x_j, y_j) = (x_j, m_j^i x_i + m_j^j x_j + m_j^k x_k) = m_j^i C_k + m_j^k C_i + m_j^j = 0, \delta^2 = 2C_i C_j C_k + S_j^2 - C_i^2 - C_k^2$ and the formula follows. ◊

To give more insight into the formula for $\Gamma_{ij,k}$, let us consider a 3-parametric robot-manipulator with axes $Y_1 = (x_1; y_1), Y_2 = (x_2; y_2)$ and $Y_3 = (x_3; y_3)$. Let Z_3 be the axis of Y_1 and Y_2, Z_1 be the axis of Y_2 and Y_3, u_2 be the angle between Z_3 and Z_1, d_2 be the distance between Z_3 and Z_1. a_1 be the distance of Y_2 and Y_3, a_3 be the distance of Y_1 and Y_2. We have the following ($c_2 = \cos u_2, s_2 = \sin u_2$)

Lemma 2. $4\Gamma_{12,3} = -s_2(a_1 C_1 S_3 + a_3 S_1 C_3) - d_2 c_2 S_1 S_3$.

Proof. We have $Z_3 = S_3^{-1}(x_1 \times x_2; x_1 \times y_2 + y_1 \times x_2 + C_3 a_3 S_3^{-1} x_1 \times x_2)$, $Z_1 = S_1^{-1}(x_2 \times x_3; x_2 \times y_3 + y_2 \times x_3 + C_1 a_1 S_1^{-1} x_2 \times x_3)$, $(x_1 \times x_2, x_2 \times x_3) = c_2 S_1 S_3 = C_1 C_3 - C_2$, $d_2 s_2 = S_1^{-1} S_3^{-1}(C_3 a_1 S_1 + C_1 a_3 S_3 - a_2 S_2 + C_1 a_1 c_2 S_3 + C_3 a_3 c_2 S_1)$ and hence

$$a_2 S_2 = C_3 a_1 S_1 + C_1 a_3 S_3 + C_1 a_1 c_2 S_3 + C_3 a_3 c_2 S_1 - d_2 s_2 S_1 S_3.$$

This yields

$$4\Gamma_{12,3} = \frac{\partial}{\partial u_2}(h_{13}) = \frac{\partial}{\partial u_2}(a_2 S_2) = -s_2(a_1 C_1 S_3 + a_3 S_1 C_3) - d_2 c_2 S_1 S_3. \ \Diamond$$

Now we are going to study the relation between the Levi–Civita connection on M, which is determined by the induced scalar product on M and the Cartan connection on C_6. For the Cartan connection ∇ we have the following splitting:

$$\nabla_{Y_i} Y_j = \frac{1}{2}\varepsilon_{ij}([Y_i, Y_j]_1 + [Y_i, Y_j]_2), \tag{1}$$

where $[Y_i, Y_j]_1$ denotes the component into the tangent space of the submanifold M into the space $Y = \{Y_1, \dots, Y_p\}$ generated by vectors

$Y_1, \ldots, Y_p$. $[Y_i, Y_j]_2$ denotes the component into the orthogonal complement $Z = Y^\perp$ of Y in L with respect to the Klein form.

Remark. It is convenient to translate the tangent space of C_6 at the point g to the Lie algebra L by left translations, $Y_i = L_{g^{-1}}(\frac{\partial}{\partial u_i})_g$. The splitting (1) is invariant, because the Klein form is invariant. We have to suppose that M is a submanifold of the pseudo-Riemannian manifold C_6, which requires the tangent space of M to be non-degenerated in the induced metric. In this case L is the direct sum of Y and $Z = Y^\perp$.

Let $Z_a, a = p+1, \ldots, 6$, be a basis in $Z = Y^\perp, Y = \{Y_1, \ldots, Y_p\}$. We have

$$\frac{1}{2}\varepsilon_{ij} = [Y_i, Y_j]_1 = \tilde{\Gamma}^k_{ij} Y_k, \frac{1}{2}\varepsilon_{ij}[Y_i, Y_j]_2 = H^a_{ij} Z_a$$
$$i, j, k = 1, \ldots, p; a = p+1, \ldots, 6.$$

(1) now reads as

$$\frac{1}{2}\varepsilon_{ij}[Y_i, Y_j] = \tilde{\Gamma}^k_{ij} Y_k + H^a_{ij} Z_a. \tag{2}$$

Scalar multiplication of (2) by Y_m yields $\Gamma_{ij,m} = \tilde{\Gamma}_{ij,m}, i, j, m = 1, \ldots, p$, which gives the relation between the Cartan connection on C_6 and the Levi–Civita connection on M. $\Gamma_{ij,m}$ are Christoffel symbols of the Cartan connection for any 6-parametric robot-manipulator, which has axes of the given p-parametric robot-manipulator as its first p axes. Scalar multiplication of (2) by Z_b yields

$$\frac{1}{2}\varepsilon_{ij} < [Y_i, Y_j], Z_b >= H^a_{ij} < Z_a, Z_b >,$$

which determines H^a_{ij}, because the matrix $< Z_a, Z_b >$ is nonsingular. Coefficients $H^a_{ij}, i, j = 1, \ldots, p; a = p+1, \ldots, 6$ determine the so called *second metric tensor* of the submanifold M. The second metric tensor is a bilinear form on $T(M)$ with values in $T(M)^\perp$. With respect to the second metric tensor there is a fundamental difference between the classical geometry of submanifolds of multi-dimensional Euclidean spaces and geometry of submanifolds of C_6:

The geometry of C_6 gives the possibility to define a canonical orthonormal basis in each space $T(M)^\perp$ for $p = 2, \ldots, 5$ in a way, which is independent of the geometry of the submanifold M. Such a construction is in the Euclidean geometry possible only for codimension 1.

To construct the canonical basis in $T(M)^\perp$, we have to know whether the tangent space $T_m(M)$ at the point $m \in M$ is degenerated or not. Let us discuss this problem at first. The tangent space $T_m(M)$

is non-degenerated iff the matrix h_{ij} of the fundamental metric tensor is non-singular, $\det h_{ij} \neq 0$.

Let an instantaneous position of a p-parametric robot-manipulator be determined by axes $Y_1, \dots, Y_p \in L$. Then $h_{ij} = < Y_i, Y_j >; i, j = 1, \dots, p$. $h_{ij} = \frac{1}{2} a_{ij} \sin \alpha_{ij}$, where a_{ij} is the distance of Y_i and Y_j, α_{ij} is the angle of Y_i and Y_j. For instance for $p = 2$ we have

$$\det h_{ij} = \begin{vmatrix} 0 & h_{12} \\ h_{12} & 0 \end{vmatrix} = -h_{12}^2,$$

for $p = 3$ we have

$$\det h_{ij} = \begin{vmatrix} 0 & h_{12} & h_{13} \\ h_{12} & 0 & h_{23} \\ h_{13} & h_{23} & 0 \end{vmatrix} = 2h_{12}h_{23}h_{13}.$$

For the description of a p-parametric robot-manipulator we shall use the so called Denavit-Hartenberg parameters: Let $X_1, \dots, X_p$ be axes defining the robot-manipulator, $Y_1, \dots, Y_p$ be any instantaneous position of $X_1, \dots, X_p$. Let a_i be the distance of X_i, X_{i+1}, α_i be the angle of X_i, X_{i+1}, d_{i+1} be the distance of the axis of lines X_i, X_{i+1} from the axis of lines X_{i+1}, X_{i+2} (the offset), u_i be the angle of those axes. We write $S_i = \sin \alpha_i, C_i = \cos \alpha_i, s_i = \sin u_i, c_i = \cos u_i$.

Lemma 3. *A 2-parametric robot-manipulator is a submanifold of C_6 iff $a_1 S_1 \neq 0$ and it has index 1. A 3-parametric robot-manipulator is a submanifold of C_6 iff*

$$a_1 a_2 S_1 S_2 \neq 0, d_2^2 + (a_1^2 - a_2^2)^2 + (\cot^2 \alpha_1 - \cot^2 \alpha_2)^2 \neq 0.$$

It has index 1 or 2 according to the sign of $h_{12}h_{23}h_{13}$.

Proof. We use the expression for h_{13} in the proof of Lemma 2. ◊

Remark. Any p-parametric robot-manipulator has nonzero index because it has isotropic lines (rotations).

For $p > 3$ we have more complicated situation, but similarly as in Lemma 3 we can see that the equation $\det h_{ij} = 0$ is algebraic in $\cos u_i$ and $\sin u_i$. Such an equation can be changed into an algebraic equation by a suitable substitution. This means that either the equation $\det h_{ij} = 0$ is identically satisfied or the set of positions, for which $\det h_{ij} \neq 0$ is dense and open. This justifies our assumption that the induced metric is defined on M and we can write (1).

Now we shall construct the canonical basis in $T_m(M)^\perp$. Let $p = 3$ and let $Z = T_m(M)^\perp = \{X_1, X_2, X_3\}$ be a nondegenerated 3-dimensional subspace in L, let us write $X_i = (y_i; z_i)$. Let us consider only the general case, for which vectors y_i are linearly independent. We can choose the basis $\{X_i\}$ in such a way that y_i are orthonormal. Then $z_i = m_i^j y_j$. Let $\{y_i'\} = \gamma\{y_i\}$ be another orthonormal triple, $\gamma \in O(3)$. Then

$$\{z_i'\} = \gamma\{z_i\} = \gamma m\{y_i\} = \gamma m \gamma^T \{y_i'\}, m = (m_i^j).$$

We obtain a new matrix $m' = \gamma m \gamma^T$. This means that we can choose the basis in the space Z in such a way that the symmetric part of m is diagonal (the symmetric and skew-symmetric parts of m transform separately). This procedure means geometrically the transformation of the ruled hyperboloid determined by Z to the main axes. We also see that the canonical system of coordinates in $T_m(M)^\perp$ yields immediately a canonical basis in $T_m(M)$, because $T_m(M)$ determines the other system of straight lines on the same hyperboloid.

For $p = 4$ we obtain a similar situation. The space $T_m(M)$ determines a linear congruence and $T_m(M)^\perp$ determines axes of this congruence. These axes determine the canonical basis – see for instance [1].

The 5-dimensional case is obvious – the orthogonal complement is one-dimensional. Geometrically it means that we have to find the axis of a linear complex. The degeneration of the induced metric has obvious geometrical meaning in this case: the induced metric in $T_m(M)$ is degenerated iff all axes of the robot-manipulator intersect one straight line. The linear complex determined by $T_m(M)$ is special in this case and the orthogonal complement $Z = T_m(M)^\perp$ is a one-dimensional isotropic subspace and the induced metric degenerates.

The most interesting case is the case $p = 2$. Let X_1, X_2 be two straight lines such that $< X_1, X_2 > \neq 0$.Let Y be the 3-dimensional subspace in L generated by vectors $X_1, X_2, X_3 = [X_1, X_2]$. Let $X = = m_i X_i \in Y$ be an arbitrary vector. Then $< X, X > = a_1 S_1 (m_1 m_2 - m_3^2 C_1)$, because $< [X_1, X_2], [X_1, X_2] > = -a_1 C_1 S_1$. This shows that Y is nondegenerated iff $a_1 C_1 S_1 \neq 0$. In this case we obtain a 3-dimensional complement $Y^\perp$ in which we construct the canonical basis in the same way as for the case $p = 3$. Because $< X_1, X_3 > = < X_2, X_3 > = 0$ we have $X_3 \in \{X_1, X_2\}^\perp$ and we have a canonical basis in $T_m(M)^\perp$.

A submanifold M of a pseudo-Riemannian manifold is called *flat* at the point $m \in M$ iff the second metric tensor $H = 0$ at m. M is called *totally geodesic* iff it is flat at all its points.

Theorem 1. *Let $g: \mathbb{R}^p \to C_6$ be a p-parametric robot-manipulator* *with only rotational axes. If g is flat at one of its points, it is totally geodesic. Then $S_i = 0$ for $i = 1, \dots, p-1$ or $a_i = 0$ for $i = 1, \dots, p-1$ and $d_j = 0$ for $j = 2, \dots, p-1$; $p = 3, 4, 5$. A 2-parametric robot manipulator has no flat points.*

Proof. Let g be determined by vectors $X_1, \dots, X_p \in L$. $H = 0$ at the point $0 \in \mathbb{R}^p$ shows that the component of $[X_i, X_j]$ into the orthogonal complement $\{X_1, \dots, X_p\}^\perp$ is equal to zero. This means that $[X_i, X_j]$ must be a linear combination of $X_1, \dots, X_p$. This shows that $\{X_1, \dots, X_p\}$ must be a subalgebra generated by rotations. There are only two such subalgebras, the algebra SO(3) of the group of all spherical motions and the Lie algebra of the group of all congruences of the plane E_2. This follows that all axes of the robot-manipulator pass through one point or all of them are parallel. ◊

Remarks. 1. Strictly speaking we have to suppose $p = 3$, because for $p = 4$ and $p = 5$ vectors $X_1, \dots, X_p$ are not linearly independent. In that case we consider the manifold $M = g(\mathbb{R}^p)$ as a subset of C_6.

2. If we admit robot-manipulators with translational (prismatic) and screw joints, we obtain all connected subgroups of C_6 as totally geodesic submanifolds generated by robot-manipulators. Their list can be found for instance in [3].

3. Th. 1 shows that robot-manipulators have one more property of ruled quadratical surfaces in E_3 – if such a quadratical surface has one flat point, it splits into planes. Let us remark that robot-manipulators with only rotational axes are generalisations of the one-sheet hyperboloid, robot-manipulators with prismatic joints are generalisations of the hyperbolic paraboloid.

Robot-manipulators with $a_i = 0$, $i = 1, \dots, p-1$ will be called *spherical* robot-manipulators, robot-manipulators with $S_i = 0$, $i = 1, \dots, p-1$ will be called *planar* robot-manipulators. A curve $c(t)$ on a pseudo-Riemannian submanifold M is called *asymptotic* iff $H(c'(t)) = 0$, where $c'(t)$ is the tangent vector of $c(t)$. This shows that a geodetic curve on M is asymptotic iff it is a geodetic curve of the enveloping space of M. This means that a p-parametric robot-manipulator is characterized as a submanifold with p independent asymptotic geodesic curves passing through each of its points. (Independent means independent tangent vectors and the statement is true only for regular points.)

In the next part of the paper we shall consider some special properties of robot-manipulators. For instance we may ask if there exist robot-manipulators which have one more asymptotic geodesic line (apart from those given above). The answer is positive because totally geodesic robot-manipulators have all geodesic lines asymptotic. Interesting question is whether there are some other solutions. In the Euclidean geometry of E_3 the answer is negative – if a ruled quadratical surface contains one more straight line it must be flat. For robot-manipulators we have nontrivial solutions of this problem – for instance the so called Bennets mechanism is one of them and some other cases are known. The general solution of this problem is not known. The above mentioned problem is not uninteresting from the practical point of view, because an asymptotic geodesic curve on a robot-manipulator with less then six degrees of freedom means a translation, rotation or a screw-motion. This means that we ask whether the end-effector of such a robot-manipulator can perform a rotation different from the rotation around one of its axes or if it can perform a translation or a screw motion. We shall see that this problem is closely connected with some other problems concerning robot-manipulators. To simplify our language we shall introduce some definitions:

Definition 1. A p-parametric robot-manipulator is called *singular* iff $\dim\{Y_1, \dots, Y_p\} < p$ for all positions of the robot-manipulator. We say that a p-parametric robot-manipulator has an *additional degree of freedom* iff there exists such a location of the end-effector (uncertainty position) that its joints can move with the end-effector fixed.

Remark. For instance the spherical and planar robot-manipulators are singular for $p > 3$. Robot-manipulators with $p > 6$ have additional degree of freedom at most of their positions; therefore we shall suppose $p \leq 6$.

Because we can identify the coordinate system in the moving space with the one in the fixed space for any fixed location of the end-effector, we can write the equation for the additional degree of freedom in the form

$$g_1(u_1(t)). \dots .g_p(u_p(t)) = e, \tag{3}$$

where $u_i(t)$ are functions of one parameter t and at least one of these functions is not constant.

Let us denote by S_p the subgroup of all permutations of numbers $(1, \dots, p)$, which is generated by the cyclic permutation $(1, \dots, p) \to \to (p, 1, \dots, p-1)$ and by the permutation $(1, \dots, p) \to (p, p-1, \dots, 1)$. The group S_p operates on vectors $X_1, \dots, X_p$ in a natural manner.

Lemma 4. *The group S_p preserves the property of additional degree of freedom.*

Proof. Let $g_1 \dots g_p = e$. Then $g_1 \dots g_{p-1} = g_p^{-1}$ and $g_p g_1 \dots g_{p-1} = e$. Similarly $g_p^{-1}(u_p) \dots g_1^{-1}(u_1) = e$ and $g_p(-u_p) \dots g_1(-u_1) = e$. $\Diamond$

Lemma 5. *Let the p-parametric robot-manipulator g have an additional degree of freedom. Then there exists a robot-manipulator $g' S_p$ equivalent with g, such that the $(p-1)$- parametric robot-manipulator obtained from g' by leaving out the last axis has an additional asymptotic geodesic curve.*

Proof. Let $g_1(u_1(t)) \dots g_p(u_p(t)) = e$ for functions $u_1(t), \dots, u_p(t)$. Then there exists a, $1 \le a \le p$, such that $u_a(t)$ is not constant. This yields

$$g_a(u_a(t)).g_{a+1}(u_{a+1}(t)) \dots g_p(u_p(t)).g_1(u_1(t)) \dots g_{a-1}(u_{a-1}(t)) = e$$

and therefore

$$g_{a+1}(u_{a+1}(t)) \dots g_p(u_p(t)).g_1(u_1(t)) \dots g_{a-1}(u_{a-1}(t)) = g_a(-u_a(t)).$$

We change the parameter to $w = -u_a(t)$ and obtain a geodesic asymptotic line $g_a(w)$ on the $(p-1)$-parametric robot-manipulator

$$g_{a+1}(u_{a+1}) \dots g_p(u_p).g_1(u_1) \dots g_{a-1}(u_{a-1}). \ \Diamond$$

Lemma 6. *A $(p-1)$-parametric robot-manipulator with additional asymptotic geodesic curve determines a p-parametric robot-manipulator with additional degree of freedom.*

Proof. Let $u_i(t)$ be the parametric expression of the additional asymptotic geodesic curve, $i = 1, \dots, p-1$. This yields $g_1(u_1(t)) \dots g_{p-1}(u_{p-1}(t)) = g(t)$, where $g(t)$ is a rotation or translation. This means that the p-parametric robot-manipulator $g_1(u_1) \dots g_{p-1}(u_{p-1}) \cdot g^{-1}(u_p)$ has an additional degree of freedom ($g(u_p)$ must be different from $g_{p-1}(u_{p-1})$, because $g = g_{p-1}$ leads to $g_1(u_1(t)) \dots g_{p-2}(u_{p-2}(t)) \cdot g_{p-1}(u_{p-1}(t) - t) = e$, which shows that the given robot-manipulator has an additional degree of freedom and the problem is trivial). $\Diamond$

Remark. The p-parametric robot-manipulator from Lemma 6 can be singular.

As the dimension of the vector space generated by vectors $Y_1, \dots, Y_p$ is constant on an open and dense set in $\mathbb{R}^p$, we can define this dimension as the rank of the robot-manipulator. It is the maximal dimension of the tangent space of the robot-manipulator. Obviously, singular robot-manipulators have rank less then the number of parameters.

Theorem 2. *Let g be a p-parametric robot-manipulator of rank $p - s$. Then g has s independent additional degrees of freedom.*

Proof. Let at first $s = 1$. Vectors $Y_1, \dots, Y_p$ are linearly dependent for all $u \in \mathbb{R}^p$. This means that there exist functions $m_1(u_1, \dots, u_p), \dots, m_p(u_1, \dots, u_p)$ such that

$$\sum_{i=1}^{p} m_i(u_a)Y_i(u_a) = 0,$$

where we can suppose that m_i are differentiable on an open subset of $\mathbf{R}^p$. Let us consider the following system of ordinary differential equations

$$\frac{du_i}{dt} = m_i(u_a). \tag{4}$$

Let $u_i(t)$ be an arbitrary solution of (4). Let us consider the following expression:

$$[g_1(u_1(t)). \dots .g_p(u_p(t))]'.[g_1(u_1(t)). \dots .g_p(u_p(t))]^{-1} =$$
$$= \frac{dg_1}{du_1}g_1^{-1}.u_1' + g_1\frac{dg_2}{du_2}g_2^{-1}g_1^{-1}u_2' + \dots + g_1 \dots g_{p-1}\frac{dg_p}{du_p}g_p^{-1} \dots g_1^{-1}u_p' =$$
$$= X_1m_1 + \dots + Ad(g_1 \dots g_{p-1})X_pm_p = 0$$

because $\sum_{i=1}^{p} m_iY_i = 0$. This yields $[g_1(u_1(t)) \dots g_p(u_p(t))]' = 0$ and $g_1(u_1(t)) \dots g_p(u_p(t)) = \gamma$, where $\gamma \in C_6$ is constant. We can change the representation of the robot-manipulator to have $\gamma = e$. (If $g_1(u_1(t)) \dots g_p(u_p(t)) = \gamma$, we have $\gamma = g_1(u_1(t_o)) \dots g_p(u_p(t_o))$ for some t_o, let us denote $g_i(u_i(t_o)) = \gamma_i$.) Let for simplicity $p = 3$, other cases are similar. We obtain

$$e = g_1g_2g_3\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1} = g_1\gamma_1^{-1}.\gamma_1(g_2\gamma_2^{-1}).\gamma_1\gamma_2(g_3\gamma_3^{-1})(\gamma_1\gamma_2)^{-1}.$$

Because $g_i\gamma_i^{-1} = g_i(u_i(t) - u_i(t_o))$, we obtain an another position of the same robot-manipulator and we have a solution for the additional degree of freedom. Let us remark that the choice of initial conditions $u_i(t_o) = u_i^o$ for (4) shows that any position of the robot-manipulator leads to an additional degree of freedom.

The proof for $s > 1$ is similar, but we have to show at first that the corresponding system of partial differential equations satisfies the integrability conditions. To simplify denotations, let $s = 2, p = 6$. This means that the dimension of the vector space generated by $Y_1, \dots, Y_6$ is equal to 4 on some open subset of $\mathbf{R}^4$. This yields

$$\sum_{i=1}^{6} Y_im_i(u_a) = 0, \sum_{i=1}^{6} Y_in_i(u_a) = 0, \tag{5}$$

where we can suppose $m_1 = 1$, $m_2 = 0$, $n_1 = 0$, $n_2 = 1$. Let us consider the following system of partial differential equations

$$\frac{\partial u_i}{\partial t} = m_i(u_a), \frac{\partial u_i}{\partial r} = n_i(u_a). \tag{6}$$

Differentiation of (6) yields

$$\frac{\partial^2 u_i}{\partial t \partial r} = \sum_{a=1}^{6} \frac{\partial m_i}{\partial u_a} \cdot \frac{\partial u_a}{\partial r} = \sum_{a=1}^{6} \frac{\partial m_i}{\partial u_a} . n_a = \sum \frac{\partial n_i}{\partial u_a} . m_a .$$

Integrability conditions are

$$\sum_{a=1}^{6} \Big(\frac{\partial m_i}{\partial u_a} n_a - \frac{\partial n_i}{\partial u_a} m_a \Big) = 0 \text{ for } i = 1, \dots, 6.$$

Differentiation of (5) yields

(7)

$$\sum_{i=1}^{6} \Big(\frac{\partial Y_i}{\partial u_a} m_i + Y_i \frac{\partial m_i}{\partial u_a} \Big) = 0; \sum_{i=1}^{6} \Big(\frac{\partial Y_i}{\partial u_a} n_i + Y_i \frac{\partial n_i}{\partial u_a} \Big) = 0 \text{ for } a = 1, \dots, 6.$$

We multiply the first equation of (7) by n_a, the second one by m_a, add them over a and subtract the results. The result is

$$\sum_{i=1}^{6} Y_i \Big\{ \sum_{a=1}^{6} \Big(\frac{\partial m_i}{\partial u_a} n_a - \frac{\partial n_i}{\partial u_a} m_a \Big) \Big\} + \sum_{i,a=1}^{6} \frac{\partial Y_i}{\partial u_a} (m_i n_a - n_i m_a) = 0.$$

From [2] we know that

$$\frac{\partial Y_i}{\partial u_a} = 0 \,\text{for}\, a \leq i, \frac{\partial Y_i}{\partial u_a} = [Y_i, Y_a] \,\text{for}\, i > a.$$

This yields

$$\sum_{i,a=1}^{6} \frac{\partial Y_i}{\partial u_a} (m_i n_a - n_i m_a) =$$

$$= \sum_{i>a} [Y_i, Y_a] m_i n_a + \sum_{i<a} [Y_i, Y_a] m_i n_a = \sum_{i,a=1} [Y_i, Y_a] m_i n_a = 0,$$

because

$$0 = \Big[\sum_{i=1}^{6} Y_i m_i, \sum_{a=1}^{6} Y_a n_a \Big] = \sum_{i,a=1}^{6} [Y_i, Y_a] m_i n_a .$$

We have obtained

$$\sum_{i=1}^{6} Y_i \Big\{ \sum_{a=1}^{6} \Big(\frac{\partial m_i}{\partial u_a} n_a - \frac{\partial n_i}{\partial u_a} m_a \Big) \Big\} = 0.$$

The dimension of the vector space generated by $Y_i, \dots, Y_6$ is four and therefore there exist functions $\lambda(u_a), \mu(u_a)$ such that

$$\sum_{a=1}^{6}\left(\frac{\partial m_i}{\partial u_a}n_a - \frac{\partial n_i}{\partial u_a}m_a\right) = \lambda m_i + \mu n_i.$$

Substitution for $i = 1$ and $i = 2$ yields $\lambda = \mu = 0$. This finishes the proof. ◊

So far we have proved the following implications for properties of robot-manipulators:

singular $\to$ additional degree of freedom at each position;

totally geodesic $\to$ additional asymptotic geodesic curves;

totally geodesic $\to$ rank equal to 3;

additional degree of freedom $\equiv$ additional asymptotic geodesic curve in a partial robot-manipulator.

It is not difficult to show that the only singular robot-manipulators of rank 3 (and therefore $p > 3$) are spherical and planar robot-manipulators. The problem of the classification of all robot-manipulators with additional degree of freedom remains open. (It includes the classification of all singular robot-manipulators and the classification of robot-manipulators with additional geodesic asymptotic curve.)

There is a 1-1 correspondence between motions of robot-manipulators with additional degrees of freedom and closed kinematical chains, which have possibility to move. To show it we prove the following

Lemma 7. *Let $g_1(u_1(t))\dots g_p(u_p(t)) = e$ for functions $u_i(t), i = 1,\dots,p$. Let us denote d_p the offset between axes Y_{p-1}, Y_p and Y_p, Y_1 and similarly for d_1. Then $<Y_1(t), Y_p(t)>, K(Y_1(t), Y_p(t)), d_p$ and d_1 are constant.*

Proof. We have $Y_k = Ad(g_1\dots g_{k-1})X_k$, where X_k is the initial position of $Y_k, k = 1,\dots,p$. This yields $Y_1 = X_1$, $Y_p = Ad(g_1\dots g_{p-1})X_p = Ad(g_p^{-1})X_p = X_p$ and therefore $<Y_1, Y_p> = <X_1, X_p>$, $K(Y_1, Y_p) = K(X_1, X_p)$. Let now $S_p = \sin\alpha_p$, $C_p = \cos\alpha_p$, where α_p is the angle between Y_p and Y_1. Calculation yields

$$-d_pS_{p-1}^2S_p^2 = S_p^2[|z_{p-1}, v_{p-1}, z_p| + |z_p, v_p, z_{p-1}|C_{p-1}] - S_{p-1}^2[|z_1, v_1, z_p| + |z_p, v_p, z_1|C_p],$$

where $Y_i = (z_i; v_i)$ for $i = 1,\dots,p$. As $Y_{p-1} = Ad(g_1\dots g_{p-2})X_{p-1} = Ad(g_1\dots g_{p-1})X_{p-1} = Ad(g_p^{-1})X_{p-1}$, it is easy to show that $|z_{p-1}, v_{p-1}, z_p| + |z_p, v_p, z_{p-1}|C_{p-1}$ is invariant with respect to u_p. ◊

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