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AROUND THE ROGERS-SHEPARD INEQUALITY

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Abstract: We prove a stability version of the Rogers-Shepard difference body inequality. In addition, we consider possible generalizations with respect to mixed volumes, introducing various types of parametric bodies.

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1. Introduction

In the late 50's, Rogers and Shepard proved their celebrated inequality about the volume of the difference-body; namely, if C is a convex body in $\mathbf{E}^d$ then

$$V(C-C) \leq \binom{2d}{d} V(C), \tag{1}$$

with equality if and only if C is a simplex. The problem about the stability of the Rogers–Shepard inequality has been recently posed independently by Peter Gruber and by Károly Bezdek. We give a solution

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in a small neighbourhood of the simplex (see Th. 4.3). The estimates are based on the Minkowski measure of symmetry whose basic properties are reviewed in Section 3. Note that we even provide a stability version of the well-known fact that this measure is d if and only if C is simplex.

An earlier approach towards to the proof of (1) was suggested by Godbersen in [4]. He conjectured the inequalities

$$V(C,-C;i) \leq \binom{d}{i} V(C) \text{ for } i=1,\dots,d-1, \tag{2}$$

which would readily yield (1). For the definition and basic properties of the mixed volumes see Section 2. Equality holds in (2) if C is a simplex, and the inequalities are verified in [4] if C is a body of constant width.

In the second half of the paper we consider various generalizations of the Rogers–Shepard inequality and the Godbersen inequalities. In Sections 5-6, the meanprojections and the affine surface-area of Busemann of the difference body are investigated.

The topic of the last two sections originated from the theory of packings (actually, similar connections brought Rogers' and Shepard's attention to this problem). The question is what ρ ensures that the volume of $K+(C-C)$ is at most $V(K+\rho C)$. For such a ρ, the parametric density of any finite packing of C is at most the packing density of C (see [1]).

2. Mixed volumes

Denote the family of compact, convex sets by $\mathcal{K}^d$. According to the theorem of Minkowski (see [2]), for $K_1,\dots,K_m \in \mathcal{K}^d$ the function $V(\lambda_1 K_1+\ldots+\lambda_m K_m)$ is a homogeneous polynomial of degree d for non-negative $\lambda_1,\dots,\lambda_m$. The mixed volume $V(K_1,\dots,K_d)$ (where we allow repetition) is defined so that the coefficient of $\lambda_1\cdot\ldots\cdot\lambda_d$ in this polynomial is $d!V(K_1,\dots,K_d)$. The mixed volume is actually well-defined. It is non-negative and symmetric, monotonic, positive linear and continuous in its variables. In addition, it is invariant under volume-preserving affine transformations (applied simultaneously to $K_1,\dots,K_d$). We have $V(K_1,\dots,K_d)\neq 0$ if and only if there exist segments $s_i\subset K_i$ whose directions span $\mathbf{E}^d$.

As it is frequently done, we use the breviation

$$V(K,\dots,K,\overbrace{C,\dots,C}^{i}) = V(K,C;i).$$

Note that $V(K,\dots,K) = V(K)$ and if B is the unit ball then $V(K,B;i)$ is proportional to the mean $(d-i)$-projection of K (say $V(K,B;1)$ to the surface-area).

Let $K \in \mathcal{K}^d$. We denote its support function by $h_K(u)$ and its face with outer normal u by $F_K(u)$. In addition, we use $v(\cdot,\dots,\cdot)$ for the $(d-1)$-dimensional mixed volume. Then for polytopes $P_1,\dots,P_{d-1}$ the mixed volumes can be calculated with the help of the formula (see [2])

(3) $$V(P_1,\dots,P_{d-1},K) = \frac{1}{d}\sum_{u\in\mathcal{S}^{d-1}} v(F_{P_1}(u),\dots,F_{P_{d-1}}(u))\;h_K(u).$$

Observe that the summand on the right hand side is non-zero only for finitely many $u \in \mathbf{S}^{d-1}$.

3. About the Minkowski-measure of symmetry

Denote by $q(C)$ the minimal $\lambda > 0$ such that $-C \subset x + \lambda C$ for some x. It is well-known that $q(C) = q(-C)$ is a measure of symmetry for C in the sense of Grünbaum (cf. [6]), and satisfies (see [7])

Theorem 3.1. *For any convex body C, we have $1 \le q(C) \le d$ where $q(C) = 1$ if and only if C is centrally symmetric and $q(C) = d$ if and only if C is a d-simplex.*

We always assume, unless otherwise stated that C translated so as to satisfy

(4) $$-C \subset q(C)\,C.$$

Note that this way the origin is contained in int$C$. We deduce by the definition of $q(C)$ and by (4) the existence of some $v_0,\dots,v_m \in \partial C$ such that $o \in \operatorname{conv}\{v_0,\dots,v_m\}$ and

(5) $$-q(C)\,v_i \in \partial C$$

for $i = 0,\dots,m$. Observe that there exist parallel supporting hyperplanes in v_i and $-q(C)\,v_i$ and denote by u_i the outer unit-normal vector to C at v_i. It also follows that $o \in \operatorname{conv}\{u_0,\dots,u_m\}$ and

(6) $$h_{-C}(u_i) = q(C)\,h_C(u_i)$$

for $i = 0,\dots,m$.

Let T be some d simplex whose centroid is the origin. Denote by $\sigma(C)$ the minimum of λ such there exists some affine transformation A satisfying $A(T) \subset C \subset A(\lambda T)$. Taking $A(T)$ to be the d-simplex of maximal volume shows that $\sigma(C)$ satisfies $1 \leq \sigma(C) \leq d + 2$ (cf. [8], Th. 3). Note that $\sigma(C) = d$ if $C = -C$, and so most probably $\sigma(C) \leq d$ with equality if and only if C is centrally symmetric.

Theorem 3.2. *Let C be a convex body satisfying $d - \frac{1}{d} < q = q(C) \leq d$. Then*

$$\frac{d}{q} \leq \sigma(C) \leq 1 + (d+1)\frac{d-q}{1-d(d-q)}.$$

Remark. The lower bound holds for any C (no need for the restriction on $q(C)$) and can not be improved in general.

Proof. Since $-T$ is covered by a translate of $\rho\,\sigma(C)T$, the relation $\rho\,\sigma(C) \geq d$ readily follows. The fact that this estimate is the best possible shown by the example of $C = \operatorname{conv}(-\frac{\sigma}{d}T \cup T)$ for some $1 \leq \sigma \leq d$. Here readily $\sigma(C) \leq \sigma$ and $q(C) \leq d/\sigma$, which in turn yields that $\rho\,\sigma(C) = d$.

Let $v_0, \dots, v_m \in \partial C$ be the points appearing in (5). Turning to the upper bound for $\sigma(C)$, first we prove by induction on k ($k \leq d$) that if $o \in \operatorname{conv}\{v_0, \dots, v_k\}$ then $q(C) \leq k$. Set $w_i = -q(C)v_i$. The claim readily holds for $k = 1$, so assume that $T = \operatorname{conv}\{w_0, \dots, w_k\}$ is a k-simplex and $k \geq 2$. Denote H_i^+ the half space in aff$T$ whose bounding hyperplane divides the distance between w_i and the opposite facet of T in the ratio $q(C)$ to 1. Then $o \in \cap H_i^+$ since v_i is not contained in relint$T$, which in turn yields that $q(C) \leq k$.

We may assume by Charatheodory's theorem that $o \in \operatorname{conv}\{v_0, \dots, v_d\}$, and so conclude that $q(C) \leq d$ (see also Th. 3.1). Assume in addition, that $q = q(C) > d - \frac{1}{d}$, and hence $T = \operatorname{conv}\{w_0, \dots, w_d\}$ is a d-simplex. Translate C so as that the center of mass of T is the origin and denote by s the common point $\frac{q}{q+1}v_i + \frac{1}{q+1}w_i$. Recall that $s \in T_0$ where $T_0 = \cap_{i=0,\dots,d} H_i^+ = -\frac{d-q}{q+1}T$. We deduce that $v_0 \in T_1 = w_0 + \frac{q+1}{q}(T_0 - w_0)$, and in particular,

$$v_0 \in \operatorname{conv}\{y, w_1, \dots, w_d\} \tag{7}$$

where $y = -\frac{d+1}{q}w_0$ is the vertex of T_1 outside of T.

Let x be a point of C contained, say, in pos$\{w_1, \dots, w_d\}$. We deduce by (7) that $y \notin \operatorname{int}\operatorname{conv}\{x, w_1, \dots, w_d\}$, and hence $x - y \notin \operatorname{int}\operatorname{pos}\{y - w_1, \dots, y - w_d\}$. On the other hand, some simple calculations show that the line aff $\{o, w_1\}$ intersects aff $\{y, v_2, \ldots, v_d\}$ in a point contained in $\frac{(d+1)(d-q)}{1-d^2+dq}\,T$. It follows that $x \in \frac{(d+1)(d-q)}{1-d^2+dq}\,T$, which in turn yields the upper bound for $\sigma(C)$. $\Diamond$

Remark. If $q(C) = d - \varepsilon$ and $\varepsilon < \frac{1}{2d}$ then we deduce the estimate

$$1 + \frac{\varepsilon}{d} \leq \sigma(C) \leq 1 + 2(d+1)\,\varepsilon. \tag{8}$$

4. The stability of the Rogers-Shepard inequality

Before giving an estimate of the right order for the stability of the Rogers–Shepard inequality (see Th. 4.3) in a neighbourhood of the simplex let us see what to expect. The relation $-C \subset q(C)\,C$ yields that

$$V(C, -C; 1) \leq q(C)V(C). \tag{9}$$

Let $1 \leq q \leq d$ and let T be some simplex having the origin as its center of mass. Then for $C = \operatorname{conv}(T \cup -qT)$, $q(C) = q$ and equality holds in (9). Observe that the conjecture (2) of Godbersen would yield the following estimate: If $q(C) = d - \varepsilon$ then

$$V(C - C) \leq \left(\binom{2d}{d} - 2d\varepsilon\right) V(C).$$

In particular, we have an exact upper estimate if $d = 3$, and hence $\binom{2d}{d} = 20$.

Proposition 4.1. *Let C be a convex body in the 3-space and set $q(C) = q = 3 - \varepsilon$. Then $V(C - C) \leq (20 - 6\varepsilon)V(C)$, and equality holds if $C = \operatorname{conv}(T \cup -qT)$ where T is a simplex having the origin as its center of mass.*

For general d, we use the original ideas of C. A. Rogers and Shepard. In their paper [9] (see also Section 8, assuming that K is a point), they essentially establish a formula of the type $V(C - C) = (1 - \Delta(C))\binom{2d}{d}V(C)$. Letting $\lambda(x) = \min\{\lambda > 0 \,|\, x \in \lambda(C - C)\}$, we have

$$\Delta(C) = \frac{1}{V(C)^2} \int_{C-C} \{V[(x + C) \cap C] - V[(1 - \lambda(x))C]\}\, dx. \tag{10}$$

In other words, $\Delta(C)$ measures that how much "smaller" is $(1 - \lambda(x))C$ than $(x + C) \cap C$ in average.

Set $x = \lambda(b - c)$ for some $b, c \in C$ and $0 < \lambda < 1$. In addition, assume that for a point y outside of C, we have $z = \lambda b + (1 - \lambda)y \in C$ and $\lambda c + (1 - \lambda)y \in C$. Then some simple consideration show that

$$z \in [(x + C) \cap C] \setminus b + (1 - \lambda)(C - b). \tag{11}$$

This observation, taking λ to be close to $\lambda(x)$, is applied to give a lower bound for the right hand side of (10).

Proposition 4.2. *Let $d \geq 4$ and $q(C) = d - \varepsilon$ for some convex body C. There exist some $\varepsilon_0, c_0 > 0$ such that if $0 \leq \varepsilon < \varepsilon_0$ then*

$$V(C - C) \leq (1 - c_0\varepsilon)\binom{2d}{d}V(C).$$

Proof. The idea of the proof runs as follows: "Normalize" C so that the simplex T in the proof of Th. 3.2 is regular having edge-length one, and construct a certain cylinder M, independent of C (up to congruency). For $x \in M$ and λ be close to $\lambda(x)$, find a small cone N in $(x + C) \cap C$ not in $(1 - \lambda)C$ satisfying $V(N) \geq \omega\,\varepsilon$ for some constant ω. Finally, verify that the difference between the volumes of $(1 - \lambda(x))C$ and of $(1 - \lambda)C$ is at most $\frac{1}{2}\omega\,\varepsilon$, and so (10) yields the proposition. Since the constant c_0 resulting from our method is small anyway, we estimate rather generously the constants arising along the way.

Assume that

$$\varepsilon < \frac{1}{2d(d+1)}, \tag{12}$$

and hence (8) holds. We use the notation of the proof of Th. 3.2. According to this, $v_0, \dots, v_d$ and $w_0, \dots, w_d$ are points of ∂C such that there exists an $s \in \mathrm{conv}\{v_0, \dots, v_d\}$ satisfying $w_i - s = q(C)(s - v_i)$ for $i = 0, \dots, d$. In addition, for each pair v_i and w_i there exists supporting hyperplanes to C with opposite outer-normals. Since our problem is affine invariant, we may assume that the d-simplex $T = \mathrm{conv}\{w_0, \dots, w_d\}$ is regular having edge-length one and the origin is the center of mass of T. Note that $V(T) = \frac{\sqrt{d+1}}{\sqrt{2}^d d!}$, the height of T is $\sqrt{\frac{d+1}{2d}}$ and by the proof of Th. 3.2,

$$C \subset [1 + 2(d+1)\varepsilon]T \subset \left(1 + \frac{1}{d}\right) T.$$

Denote by F_i the facet of T opposite to w_i and by n_i an outer-normal to C at v_i such that $-n_i$ is an outer-normal to C at w_i. Assume that v_0 is the farthest from T among $v_0, \dots, v_d$ and let L be the linear $(d-1)$-space parallel to F_0. Among $w_1, \dots, w_d$, there exists one, say w_1, so that the angle of $-\frac{1}{d}w_0 - w_1$ (which is in L) and the projection of n_0 onto L is at most $\frac{\pi}{2} - \arcsin\frac{1}{d-1}$. Let K be the right circular $(d-1)$-cone in L with apex o, whose height is $\frac{1}{2d}$ and the maximal angle of any $x \in K$ and $-\frac{1}{d}w_0 - w_1$ is $\arcsin\frac{1}{d-1}$. The condition on w_1 yields that $v_0 + K$ does not intersect $\text{int}C$ (see Fig. 1).

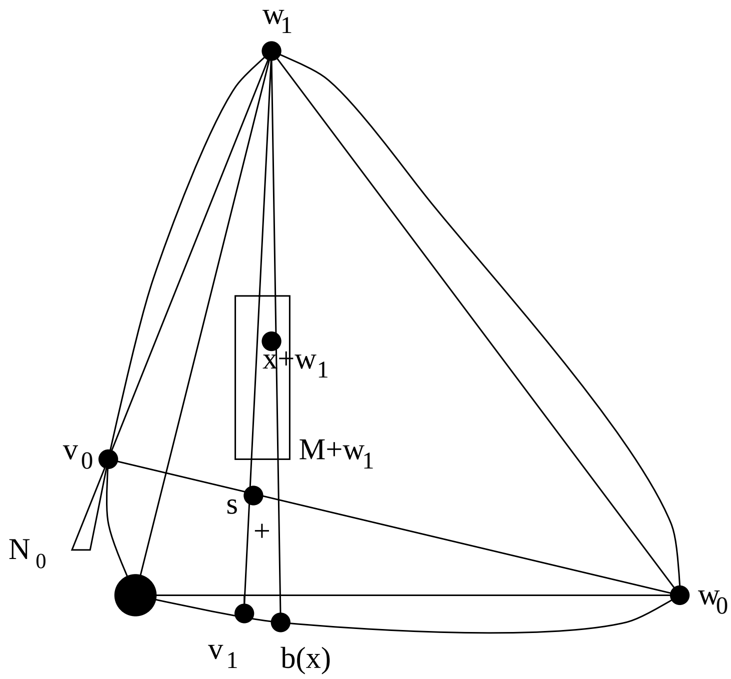

Fig. 1

Denote the volume of the unit d-ball by κ_d, and set

$$\omega = \frac{\kappa_{d-2}}{8^d\, d^{2d+2}}. \tag{13}$$

The right circular cylinder M has height $\frac{1}{8}\sqrt{2}$ and the radius of its base is $\frac{1}{32d}\omega$. We assume that the axis of rotation for M is parallel to $v_1 - w_1$ and the centers of the bases of M are distance $\frac{1}{4}\sqrt{2}$ and $\frac{3}{8}\sqrt{2}$ from o along the halfline $(o, v_1 - w_1)$.

Let $x \in M$. Denote by $b(x)$ the point of ∂C such that $x = \widetilde{\lambda}(x)(b(x) - w_1)$ for certain $\widetilde{\lambda}(x) > 0$. The definition of M yields that $\frac{1}{4} < \widetilde{\lambda}(x) < \frac{3}{4}$. The "excess cone" N is defined with the help of the cone

$$N_0 = v_0 + \text{conv}\left(\frac{1}{d}(v_0 - w_1), K\right).$$

If $y \in N_0$ then $y \notin \text{int}\, C$, and we deduce by the bounds on $\widetilde{\lambda}(x)$ and

some elementary calculations that $\widetilde{\lambda}(x)b(x)+(1-\widetilde{\lambda}(x))y$ and $\widetilde{\lambda}(x)w_1+$ $+(1-\widetilde{\lambda}(x))y$ are contained in C. It follows by (11) that
$N=\widetilde{\lambda}(x)b(x)+(1-\widetilde{\lambda}(x))N_0\subset[(x+C)\cap C]\setminus b(x)+(1-\widetilde{\lambda}(x))(C-b(x))$. The maximality property of v_0 yields that the distance of v_0 from F_0 is at least

$$\left(\frac{1}{q(C)}-\frac{1}{d}\right)\sqrt{\frac{d}{2(d+1)}}>\frac{\varepsilon}{2d^2}.$$

We deduce that $V(N_0)>\frac{\kappa_{d-2}}{2^d d^{2d+2}}\,\varepsilon$, and hence $V(N)>V(\frac{1}{4}N_0)>\omega\,\varepsilon$.

Our final estimate is to compare the volume of $(1-\widetilde{\lambda}(x))C$ to the volume of $(1-\lambda(x))C$. Denote the point $\frac{1}{\lambda(x)}\,x$ of $\partial(C-C)$ by $u(x)$. Then $V(C)<(1+\frac{1}{d})^dV(T)<1$ yields that

$$V((1-\lambda(x))C)-V((1-\widetilde{\lambda}(x))C)<\left(\frac{1-\lambda(x)}{1-\widetilde{\lambda}(x)}\right)^d-1=$$

$$=\left(\frac{\|u(x)\|}{\|b(x)-w_1\|}\right)^d-1.$$

Now observe that $\mathrm{pos}(C-v_1)$ contains a cone with axis $\mathrm{pos}\{w_1-v_1\}$ such that the maximal angle of its points and w_1-v_1 is $\mathrm{arc\,tg}\varepsilon$. It follows that the angle $(b(x),v_1,w_1)$ is at least $\mathrm{arc\,ctg}\varepsilon$, and since the segment $\mathrm{conv}\,\{w_1,w_1+u(x)\}$ is contained in the strip bounded by the parallel supporting hyperplanes at v_1 and w_1, the angle $(w_1+u(x),v_1,w_1)$ is at most $\pi-\mathrm{arc\,ctg}\varepsilon$. We deduce by $x\in M$ that the angle $(b(x),w_1,v_1)$ is at most $\mathrm{arc\,tg}\frac{1}{4\sqrt{2}d}\omega$, which in turn yields that

$$\left(\frac{\|u(x)\|}{\|b(x)-w_1\|}\right)^d<\left(\frac{1+\frac{1}{4\sqrt{2}d}\omega\,\varepsilon}{1-\frac{1}{4\sqrt{2}d}\omega\,\varepsilon}\right)^d<1+\frac{\omega}{2}\,\varepsilon.$$

Combining all these together implies by (10) that

$$\Delta(C)>\frac{\omega}{2}\,V(M)\,\varepsilon,$$

and the Proposition follows. $\Diamond$

Now we are ready to state the stability version of the Rogers-Shepard inequality.

Theorem 4.3. *Let $d\geq 3$. There exist some $\varepsilon_1,c_1>0$ such that if $\sigma(C)=1+\varepsilon$ for some convex body C and $0\leq\varepsilon<\varepsilon_1$ then*

$$(1-d\varepsilon)\binom{2d}{d}V(C)\leq V(C-C)\leq(1-c_1\varepsilon)\binom{2d}{d}V(C).$$

Proof. Assume that T is a simplex such that $T \subset C \subset (1+\varepsilon)T$. Then
$$V(C-C) \geq V(T-T) \geq \frac{1}{(1+\varepsilon)^d}\binom{2d}{d}V(C),$$
which in turn yields the lower bound for $V(C-C)$. On the other hand, we deduce the existence of c_1 by Prop. 4.2 and by (8). $\Diamond$

Remark. Since in the Rogers-Shepard inequality equality holds only for the simplex, we have the following result related to Th. 4.3: If $V(C-C) = (1-\varepsilon)\binom{2d}{d}V(C)$ and ε is small then there exist a simplex T and an x such that $T \subset C - x \subset (1+c\,\varepsilon)T$ for some constant c depending only on d.

5. Inequalities with parameter-bodies

In this section, we introduce some "parameter-bodies" for the Rogers–Shepard-type inequalities. If these bodies are zonoids then we use the fact that the mixed volumes can be represented *via* projections.

Theorem 5.1. *Let $C \in \mathcal{K}^d$ and let $Z_{n+1}, \ldots, Z_d$, $n = 1, \ldots, d-1$, be zonoids. Then*
$$V(C-C, \ldots, C-C, Z_{n+1}, \ldots, Z_d) \leq \binom{2n}{n}V(C, \ldots, C, Z_{n+1}, \ldots, Z_d),$$
and equality holds if C is an n-simplex.

Proof. By the continuity of mixed volumes, it is sufficient to consider the case where all the Z_i's are zonotopes, and by linearity we may assume that the Z_i's are segments in general position. Let L be the linear n-space orthogonal to each Z_i. Since $V(C, \ldots, C, Z_{n+1}, \ldots, Z_d)$ is proportional to the n-volume of the projection of C onto L, the theorem follows by the original Rogers-Shepard inequality. $\Diamond$

As the unit ball B is a zonoid, the corresponding result for the mean projections is a direct consequence.

Corollary 5.2. *For any $C \in \mathcal{K}^d$ satisfying $\dim C \geq n$, $n = 1, \ldots, d-1$, we have*
$$V(C, B; d-n) \leq \binom{2n}{n}V(C-C, B; d-n)$$
and the inequality is the best possible.

Remark. Here equality holds if and only if C is an n-simplex. If C ($\dim C \geq n$) is not an n-simplex, then its projection to some open neighbourhood of linear n-spaces is again not an n-simplex. Since the mean projections can be represented as the average of the n-volumes of the n dimensional projections of C with respect the Haar-measure on the linear n-spaces, it follows that strict inequality holds in Cor. 5.2.

Now we allow the parameter-bodies to be any centrally symmetric body. In this case much less can be said. Surprisingly, if we do not have too many centrally symmetric parameter-bodies than the trivial estimate coming from $-C \subset q(C)$ is optimal:

Proposition 5.3. *Let C be a convex body and $n < d/2$. Then for any centrally symmetric $K_1, \dots, K_n \in \mathcal{K}^d$ and for any family of convex bodies $C_{n+1}, \dots, C_{d-1}$, the inequality*

$$V(K_1, \dots, K_n, C_{n+1}, \dots, C_{d-1}, -C) \leq$$
$$\leq q(C)\, V(K_1, \dots, K_n, C_{n+1}, \dots, C_{d-1}, C)$$

holds. For a given C, $q(C)$ can not be replaced by a smaller constant in general.

Proof. The inequality readily holds by $-C \subset q(C)$.

Assume that replacing $q(C)$ by some ρ, the inequality holds for any suitable choice of K_i and C_i. We deduce by (6) the existence of $u_0, \dots, u_k \in \mathbf{S}^{d-1}$ such that $o \in \mathrm{conv}\,\{u_0, \dots, u_k\}$, the family $u_0, \dots, u_k$ is minimal with respect to this property and $h_{-C}(u_i) = q(C)\, h_C(u_i)$.

Define T be a k-simplex whose outer $(k-1)$-face-normals in aff$T$ are $u_0, \dots, u_k$ so that the origin is the centroid of T, and set $M = T \cap \cap (-T)$. As our problem is affine invariant, assume that T is a regular simplex.

First consider the case when $k = d$, and let $K_1 = \ldots = K_n = M$ and $C_{n+1} = \ldots = C_{d-1} = T$. Assume that $v(F_T(u), F_K(u); n) \neq 0$ for some $u \in \mathbf{S}^{d-1}$. Then the relation $\dim F_T(u) \geq d - 1 - n \geq \frac{1}{2}(d - - 1)$ implies that $F_T(u)$ is not father from the origin than $F_T(-u)$, and hence $F_K(u) \subset F_T(u)$. We deduce that $\dim F_T(u) = d - 1$, or in other words, that $u = u_i$ for some $i = 0, \dots, d$. Now (3) and (4) yield that $\rho \geq q(C)$.

Therefore assume $k < d$ and let P be some centrally symmetric $(d-k)$-polytope orthogonal to T. It will be convenient to set $Q_i = M$ if $i \leq n$ and $Q_i = T$ if $i = n+1, \dots, d-1$. Now define $C_i = Q_i + \lambda^d P$ if $i \leq d-1-k$ and $C_i = Q_i + \frac{1}{\lambda} P$ if $i = d-k, \dots, d-1$ for some positive λ, and finally let $K_i = C_i$ for $i = 1, \dots, n$.

Observe that the bodies C_i's are defined so that $V(C_1, \dots, C_{d-1}, C)$ is a Laurant polynomial in λ whose highest order term is $\lambda^{d(d-1-k)}$ and

$$\lim_{\lambda\to\infty} \frac{V(C_1,\dots,C_{d-1},-C)}{V(C_1,\dots,C_{d-1},C)} = \frac{V(P,\dots,P,Q_{d-k},\dots,Q_{d-1},-C)}{V(P,\dots,P,Q_{d-k},\dots,Q_{d-1},C)}.$$

(If $k = d-1$ then P naturally does not appear on the right hand side.) Now applying a similar argument in aff$T$ as for the case $k = d$, one concludes that again $\rho \geq q(C)$. $\Diamond$

Corollary 5.4. *Let $n < d/2$. Then for any centrally symmetric $K_1,\dots,K_n \in \mathcal{K}^d$ and for any family of convex bodies $C_{n+1},\dots,C_{d-1},C$, we have the inequality*

$$V(K_1,\dots,K_n,C_{n+1},\dots,C_{d-1},-C) \leq$$
$$\leq d\,V(K_1,\dots,K_n,C_{n+1},\dots,C_{d-1},C).$$

Here equality holds if C is the d-simplex whose centroid is the origin, $K_1 = \dots, K_n = C \cap -C$ and $C_{n+1} = \dots, C_{d-1} = C$.

Remark. The condition $n < d/2$ is needed to ensure that d is the best possible constant. Say if $n = d-1$ then $V(K_1,\dots,K_{d-1},-C) = = V(K_1,\dots,K_{d-1},C)$. Let $d = 4$ and $n = 2$. Define $C = C_3$ to be 4-simplex whose centroid is the origin and $K_1 = K_2 = C \cap -C$. This cast is a candidate to be the optimal one, and $V(K_1,K_2,C_3,-C) = = 3.2\,V(K_1,K_2,C_3,C)$.

6. The affine surface area

We have already seen that there exist a Rogers–Shepard-type inequality for the mean projections. We now consider the case of the so called Minkowski-surface area $S(C)$ with respect to a centrally symmetric convex body N (see [3]). Denote by N^* the intersection body assigned to N. It is a centrally symmetric convex body, and $S(C) = = V(C,N^*;1)$. So we want to have an upper bound for the quotient $V(C-C,K;1)/V(C,K;1)$ where C and K are any convex bodies with $K = -K$. In the rest of the section, we present results supporting the following conjecture:

Conjecture 6.1. *For any convex bodies C and K with $K = -K$, we have the inequality*

$$\frac{V(C-C,K;1)}{V(C,K:1)} \leq \sum_{i=0}^{d-1} \binom{d-1}{i} \binom{d}{\min\{i,d-i\}}.$$

With the help of (3), it is easy to establish that equality holds if C is a d-simplex whose centroid is the origin and $K = C \cap -C$. Note that the expression on the right hand side is asymptotically the same as $\frac{1}{2}\binom{2d}{d}$.

With respect to lower bounds, the Alexandrov–Fenchel inequality yields that

$$\frac{V(C-C,K;1)}{V(C,K:1)} \geq 2^{d-1}, \tag{14}$$

with equality if $C=-C$. Note that one can have equality in (14) also in some other cases. For example, in $\mathbf{E}^3$ take K to be the cube $[-1,1]^3$ and C to be regular simplex whose vertices are the vertices of K having even number of 1 as coordinates.

Turning to Conj. 6.1, the case $d \leq 4$ follows right away by Cor. 5.4.

Theorem 6.2. *If $d=3$ or 4 then Conjecture 6.1 holds for any convex bodies C and K with $K=-K$. We have equality if C is a d-simplex and $K=C\cap -C$ where we assume that the origin is the centroid of C. If $d=3$ then this is the only case for equality up to a homothety of K.*

Proof. The only statement to prove is the necessary condition for $d = 3$. So let C and K be convex bodies which satisfy $K=-K$ and $V(K,C,-C)=3V(K,C,C)$. Then C is a 3-simplex by Prop. 5.3 and Th. 3.1. We may assume that C is a regular simplex and $h_K(u_0) = h_C(u_0)$ where $u_0,\dots,u_3 \in \mathbf{S}^3$ are the outer normals of the faces of C. Denote by G the symmetry group of C.

First assume that G is also contained in the symmetry group of K. The convex body $M=C\cap -C$ is a regular octahedron containing K which satisfies $V(M,C,C)=V(K,C,C)$ by (3), and hence also $V(M,C,-C)=V(K,C,-C)$. Note that $v(F_C(u),F_{-C}(u))\neq 0$ if u is perpendicular to a pair of opposite edges of C, in which case $F_M(u)$ is a vertex of M. Now $K\subset M$ and $V(M,C,-C)=V(K,C,-C)$ yield by (3) that $K=M$.

Turning to the general case, define

$$K_0=\frac{1}{|G|}\sum_{g\in G} g(K).$$

Then K_0 also satisfies $V(K_0,C,-C)=3V(K_0,C,C)$, and hence $K_0 = M$ by the considerations above. As the M has at least as many vertices as K, K must be an affine octahedron. On the other hand, any face-normal of K is a face-normal of M, and we conclude that $K=M$. $\Diamond$

Remark. If $d=4$ then $V(K,C,C,-C)=4V(K,C,C,C)$ yields that C is a 4-simplex but there are various choices for K. For example, removing a pair of opposite vertices of $C \cap -C$, one can take as K the convex hull of the remaining vertices.

For $d \geq 5$, the following statements support Conj. 6.1:

Proposition 6.3. *Let C and K be convex bodies with $K = -K$.*

(i) $V(C-C,K;1) \leq \sqrt{d}\binom{2(d-1)}{d-1}V(C,K;1) < 4^{d-1}V(C,K;1)$.

(ii) *Conj. 6.1 holds if C is a d-simplex.*

Proof. John's theorem (see [5]) provides an ellipsoid E such that $E \subset \subset K \subset \sqrt{d}E$. As E is a zonoid, i) follows by Th. 5.1.

Now let C be a regular d-simplex whose centroid is the origin and denote by G the symmetry group of C. By the linearity of the mixed volumes, it sufficient to verify the conjecture for

$$K_0 = \frac{1}{|G|}\sum_{g \in G} g(K).$$

We may assume that the supporting hyperplanes of C containing its facets are also supporting hyperplanes of K_0. Then $K_0 \subset C \cap -C$, and the relations $V(C,K_0;1) = V(C,C\cap -C;1)$ and $V(C-C,K_0;1) \leq \leq V(C-C,C\cap -C;1)$ yield the theorem. ◊

7. Parallelbodies

A possible generalization of the inequality of Rogers and Shepard is the following: Find the smallest ρ such that for any compact, convex set K and convex body C, we have the inquality

$$V(K+\rho C) \geq V(K+C-C). \tag{15}$$

If K is a point then (15) reduces to the original difference body problem, and hence the inequality holds for $\rho = \binom{2d}{d}^{1/d}$. As $\binom{2d}{d}^{1/d} < 4$, one may hope to find a ρ independent of the dimension d. In general, it is not possible (see Cor. 7.2), and so we assume in Section 8 that K has lower dimension.

Theorem 7.1. *Let C be a convex body. Then for $\rho \geq q(C)+1$ and for any $K \in \mathcal{K}^d$, the inequality*

$$V(K+\rho C) \geq V(K+C-C)$$

holds, and the lower bound for ρ is the best possible.

Proof. The inequality follows right away by (4). Now assume that (15) holds for any convex body K. Replacing K by λK for large λ, the formulae

$$\begin{aligned} V(\lambda K+\rho C) =& \lambda^d V(K)+d\lambda^{d-1}\rho V(K,C;1)+O(\lambda^{d-2}) \quad \text{and} \\ V(\lambda K+C-C) =& \lambda^d V(K)+d\lambda^{d-1}(V(K,C;1)+ \\ &+V(K,-C;1))+O(\lambda^{d-2}) \end{aligned}$$

yield that

$$\rho \geq 1+\frac{V(K,-C;1)}{V(K,C;1)}. \tag{16}$$

We deduce by (6) the existence of $u_0,\dots,u_k \in \mathbf{S}^{d-1}$ such that the family $u_0,\dots,u_k$ is minimal with respect to the property $o \in \in \operatorname{conv}\{u_0,\dots,u_k\}$ and $h_{-C}(u_i) = q(C)\,h_C(u_i)$. Define T be a k-simplex whose outer $(k-1)$-face-normals in aff T are $u_0,\dots,u_k$ and let P be some $(d-k)$-polytope orthogonal to T. Since for large λ and for $K = T+\lambda P$, (3) yields that

$$\frac{V(K,-C;1)}{V(K,C;1)} = \frac{\lambda^{d-k}\sum_{i=0}^{k} h_{-C}(u_i)v(F_{T+P}(u_i)) \;+\; O(\lambda^{d-k-1})}{\lambda^{d-k}\sum_{i=0}^{k} h_{C}(u_i)v(F_{T+P}(u_i)) \;+\; O(\lambda^{d-k-1})},$$

the inequality $\rho \geq q(C)+1$ follows by (6) and (16). Note that in the case $k=d$, the use of λ and P is unnecessary. $\Diamond$

Taking C to be a d-simplex in Th. 7.1 yields

Corollary 7.2. *Let $\rho \geq d+1$. Then for any $K \in \mathcal{K}^d$ and for any convex body C, the inequality*

$$V(K+\rho C) \geq V(K+C-C)$$

holds, and the lower bound for ρ is the best possible.

8. The low dimensional case

In this section, we consider lower bounds for ρ such that (15) holds in the case when the dimension of K is small. The next proposition is the main ingredient of the proof of Th. 8.3.

Proposition 8.1. *For any $\rho > 0$, $K \in \mathcal{K}^d$ and for any convex body C,*

$$V(K+\rho C) \geq \sum_{i=0}^{i=d} \binom{d}{i} \rho^i \binom{d+i}{i}^{-1} V(K, C-C; i).$$

The **proof** follows the line of the classical paper [9] of Rogers and Shepard. Consider the double integral

$$I = \iint \chi_C(y-x)\chi_{K+C}(y)\,dy\,dx.$$

Changing the order of integration yields that

$$I = \int \chi_{K+C}(y) \int \chi_C(y-x)\,dx\,dy = V(K+C)\cdot V(C). \tag{17}$$

On the other hand, set $D = C - C$ and define $\lambda(x) = \min\{\lambda \geq 0 \,|\, x \in \in K + \lambda D\}$. We claim that if $\lambda(x) \leq 1$ then

$$\int \chi_C(y-x)\chi_{K+C}(y)\,dy \geq (1-\lambda(x))^d\, V(C). \tag{18}$$

First assume that $\lambda(x) > 0$, and let $a \in K$ and $b, c \in C$ so that $x = a + + \lambda(b-c)$ for $\lambda = \lambda(x)$. Note that $\lambda b + (1-\lambda)C$ and $\lambda c + (1-\lambda)C$ are contained in C, which in turn yields that

$$a + \lambda b + (1-\lambda)C \subset K + C,$$
$$x + \lambda c + (1-\lambda)C \subset x + C,$$

where $a + \lambda b = x + \lambda c$. This proves the claim for $\lambda > 0$. The case $\lambda = 0$ readily holds.

Combining (17) and (18) yields that

$$V(K+C) \geq \int_{V(K+D)} (1-\lambda(x))^d\,dx. \tag{19}$$

Lemma 8.2.

$$\int_{V(K+D)} (1-\lambda(x))^d\,dx = V(K) + d\int_0^1 (1-\mu)^d V(K+\mu D, D; 1)\,d\mu.$$

Proof. Define for $0 \leq \mu \leq 1$,

$$f(\mu) = \int_{V(K+\mu D)} (1-\lambda(x))^d\,dx.$$

Note that $\lambda(x)$ is continuous and

$$\begin{aligned}\frac{d}{d\mu} V(K+\mu D) &= \frac{d}{d\mu}\sum_{i=0}^{i=d}\binom{d}{i} V(K,D;i)\,\mu^i \\ &= d\sum_{i=1}^{i=d}\binom{d-1}{i-1} V(K,D;i)\,\mu^{i-1} \\ &= d\,V(K+\mu D, D; 1).\end{aligned}$$

It follows that $f'(\mu) = (1-\mu)^d d\,V(K+\mu D, D; 1)$, which in turn yields the lemma as $f(0) = V(K)$. $\Diamond$

We deduce by replacing C by ρC in Lemma 8.2 and (19) that

(20)
$$V(K+\rho C)\geq V(K)+d\sum_{i=1}^{i=d}\binom{d-1}{i-1}\rho^i V(K,D;i)\int_0^1(1-\mu)^d\mu^{i-1}\,d\mu.$$
Finally, the simple identity
$$\int_0^1(1-\mu)^d\mu^{i-1}\,d\mu=\frac{1}{i}\binom{d+i}{i}^{-1}$$
yields the inequality
$$V(K+\rho C)\geq V(K)+\sum_{i=1}^{i=d}\binom{d}{i}\rho^i\binom{d+i}{i}^{-1}V(K,D;i).\quad\Diamond$$
Remark. The proof shows that if K and C are homothetic d simplices then equality holds in Prop. 8.1.

Next we prove that if $\dim K\leq m<d$ then the optimal ρ in (15) approximately is of order $d/(d-m)$.

Theorem 8.3. (i) *For any $K\in\mathcal{K}^d$ with $\dim K\leq m<d$ and for any convex body C,*
$$V(K+\rho C)\geq V(K+C-C)\quad\textit{for}\quad\rho\geq e\left(1+\frac{d}{d-m}\right).$$
(ii) *Assume that $d\geq 25$ and $m\geq\frac{24}{25}d$. Then there exists a $K\in\mathcal{K}^d$ with $\dim K\leq m<d$ and some convex body C such that*
$$V(K+\rho C)<V(K+C-C)\quad\textit{for}\quad\rho\leq\frac{1}{2\ln\frac{d}{d-m}}\cdot\frac{d}{d-m}.$$

Proof. Note that $V(K+D)=\sum_{i=0}^{i=d}\binom{d}{i}V(K,D;i)$ for $D=C-C$ and $V(K,D;i)=0$ for $i<d-m$. Thus by Prop. 8.1, it is sufficient to check the condition
$$\rho\geq\binom{d+i}{i}^{1/i}$$
for $i=d-m,\dots,d$. Using the Stirling formula
$$\frac{i^i}{e^i}\sqrt{2\pi\, i}<i!<\frac{i^i}{e^i}\sqrt{2\pi\,(i+1)},\tag{21}$$
we conclude the estimate
$$\binom{d+i}{i}^{1/i}<\left(1+\frac{i}{d}\right)^{d/i}\left(1+\frac{d}{i}\right)\left(\frac{d+i+1}{2\pi di}\right)^{1/i}<e\left(1+\frac{d}{d-m}\right).$$
On the other hand, assume that $k=d/(d-m)\geq 25$ for some $m<d$. We need some constants in the course of the proof; namely,

$$\alpha = \left\lceil 1.9(d-m)\ln\frac{d}{d-m}\right\rceil \quad \text{and} \quad \omega = \frac{\alpha^2}{m-\alpha^2}.$$

Let T be an m-simplex and let Q be some centrally symmetric $(d-m)$-body whose affine hull is orthogonal to T, and define $K = T$ and $C = \omega T + Q$.

First observe that by (2), we have the formula

$$V_m(T + (\omega T - \omega T)) = \sum_{j=0}^{m} \binom{m}{m-j}\binom{m}{j}(1+\omega)^{m-j}\omega^j\, V_m(T) = \sum_{i=0}^{m} \binom{m}{i}\binom{m+i}{i}\omega^i\, V_m(T).$$

Thus we investigate the quotient

$$\frac{V(K+\rho C)}{V(K+C-C)} = \frac{(1+\rho\omega)^m \rho^{d-m}}{\left(\sum_{i=0}^{m}\binom{m}{i}\binom{m+i}{i}\omega^i\right) 2^{d-m}}. \tag{22}$$

Applying the Stirling formula for $i = \alpha$ (where $\alpha \geq 6$) results in the estimate

$$\binom{m}{\alpha}\binom{m+\alpha}{\alpha}\omega^{\alpha} \geq \left(\frac{m+\alpha}{m-\alpha}\right)^m \sqrt{\frac{m+\alpha}{2\pi(m-\alpha+1)(\alpha+1)^2}} \geq \left(\frac{m+\alpha}{m-\alpha}\right)^m \frac{1}{3\alpha}.$$

We claim that for $\rho = \frac{1}{2}k/\ln k$,

$$\frac{\left(\frac{m+\alpha}{m-\alpha}\right)^m \frac{1}{3\alpha}}{(1+\rho\omega)^m} > \frac{\rho^{d-m}}{2^{d-m}}. \tag{23}$$

Observe that this claim yields (ii) in the light of (22).

As $\rho < m/\alpha$, it is sufficient to verify

$$\left(1+\frac{\alpha}{m}\right)^{m/(d-m)} > \frac{(3\alpha)^{1/(d-m)}}{2}\,\frac{m}{\alpha}.$$

On the left hand side, we deduce by $d/(d-m) \geq 25$, $\alpha/m < 0.5$ and $\alpha > 1.6(d-m)\ln d/(d-m)$ that

$$\frac{m}{d-m}\ln\left(1+\frac{\alpha}{m}\right) > 1.6\cdot\frac{\ln(1+0.5)}{0.5}\,\ln\frac{d}{d-m} > \ln 1.5 + \ln\frac{d}{d-m}.$$

As $\alpha > d-m$ on the right hand side and $\frac{1}{2}(3(d-m))^{1/(d-m)} < 1.5$, the claim (23) now follows, which in turn yields the theorem. ◊

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