Mathematica Pannonica
7/1 (1996), 147 – 162

ONE-SIDED k-IDEALS AND h-IDEALS IN SEMIRINGS

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Received May 1995

MSC 1991: 16 Y 60, 20 M 99

Keywords: Semirings; left and two-sided ideals, k-ideals, h-ideals; maximal ideals of these kinds; congruence class semirings.

Abstract: Since they are closer related to the ring-theoretical concept of ideals than arbitrary semiring ideals, two-sided k-ideals and h-ideals occur in several statements on semirings. In this paper we investigate those ideals and also left-sided ones in arbitrary semirings S with commutative addition. For instance, we prove (for the first time without any restrictions) that an ideal A of S is a maximal (left) k-ideal or h-ideal of S iff the congruence class semiring S/κ_A of S determined by A has only two (trivial) ideals of this kind. Concerning other results we refer to a survey in the second part of the Introduction.

1. Introduction

A semiring as considered in this paper is an algebra $S = (S, +, \cdot)$

such that $(S,+)$ is a commutative semigroup, $(S,\cdot)$ an arbitrary one, both connected by ring-like distributivity. A semiring S may have an identity e [a zero o], defined by $es = se = s$ $[o + s = s]$ for all $s \in$ $\in S$. An element $a \in S$ is called multiplicatively (briefly: mult.) left absorbing if $as = a$ holds for all $s \in S$. This yields $a + a = a$, i. e. a is additively (briefly: add.) idempotent. There is at most one mult. absorbing element O [add. absorbing element ∞] in S, defined by $Os =$ $= sO = O$ $[\infty + s = \infty]$ for all $s \in S$. Note that O and ∞ as well as O and o may coincide, and that a zero o [an add. absorbing element ∞] of a semiring need not even satisfy $oo = o$ $[\infty\infty = \infty]$.

Let $(S,+)$ be a commutative semigroup, also called a *semimodule* henceforth. For each subsemimodule A of $(S,+)$ the *k-closure* $\overline{A}$ of A is defined by

$$\overline{A} = \{\overline{a} \in S \mid \overline{a} + a_1 = a_2 \text{ for some } a_i \in A\}$$

and the *h-closure* $\widehat{A}$ of A by

$$\widehat{A} = \{\widehat{a} \in S \mid \widehat{a} + a_1 + u = a_2 + u \text{ for some } a_i \in A, u \in S\}.$$

These closures are again subsemimodules of $(S,+)$ and satisfy $A \subseteq \overline{A} \subseteq$ $\subseteq \widehat{A}$ as well as $\overline{\overline{A}} = \overline{A}$ and $\widehat{\widehat{A}} = \widehat{A}$. The *zeroid* Z of $(S,+)$, defined by

$$Z = \{z \in S \mid z + u = u \text{ for some } u \in S\},$$

is either empty or a subsemimodule. In the latter case one has $\widehat{Z} =$ $= Z$ and $Z \subseteq \widehat{A}$ for each subsemimodule A of $(S,+)$. Note also that each idempotent element a of $(S,+)$ is contained in Z and that Z may coincide with S.

A *left ideal* [*ideal*] A of a semiring $S = (S,+,\cdot)$ is a subsemimodule of $(S,+)$ satisfying $sa \in A$ [$sa \in A$ and $as \in A$] for all $a \in A$ and $s \in S$. If A is a left ideal or an ideal of S, the same holds for $\overline{A}$ and $\widehat{A}$. Our main interest is with k-closed and h-closed left ideals or ideals $A = \overline{A}$ and $A = \widehat{A}$ of S, simply called (*left*) *k-ideals* and (*left*) *h-ideals*. In particular, if the zeroid Z of $(S,+)$ is not empty, $Z = \widehat{Z}$ is an h-ideal of S and the intersection of all left h-ideals $A = \widehat{A}$ of S. Each ideal A of a semiring S defines a *congruence* κ_A on $(S,+,\cdot)$ by

$$s\kappa_A s' \iff s + a_1 = s' + a_2 \text{ for some } a_i \in A.$$

The corresponding congruence class semiring S/κ_A, consisting of the classes $[s]_{\kappa_A}$, contains the k-closure $\overline{A}$ of A as one of its classes, and

$\overline{A}$ is the mult. absorbing zero of S/κ_A, regardless of whether S has a zero o or a mult. absorbing element O (which yields $\overline{A} = [o]_{\kappa_A}$ or $\overline{A} = [O]_{\kappa_A}$, respectively). Moreover, one has $\kappa_A = \kappa_{\overline{A}}$, whereas $\kappa_{\overline{A}} = \kappa_{\overline{B}}$ implies $\overline{A} = \overline{B}$. Similarly, each ideal A of S defines a congruence η_A by

$$s\eta_A s' \iff s + a_1 + u = a_2 + u \text{ for some } a_i \in A, u \in S.$$

Now S/η_A contains the h-closure $\widehat{A}$ of A as its mult. absorbing zero, and one has $\kappa_A \subseteq \eta_A = \eta_{\widehat{A}}$ and $\eta_{\widehat{A}} = \eta_{\widehat{B}} \Longrightarrow \widehat{A} = \widehat{B}$. Moreover, S/η_A is add. cancellative. Finally,

$$s\delta s' \iff s + u = s' + u \text{ for some } u \in S$$

defines the smallest congruence on $(S, +, \cdot)$ such that $(S/\delta, +)$ is cancellative, which yields $\delta \subseteq \eta_A$ for each ideal A of S. We note that the congruences κ_A, η_A and δ have appeared at first (for semirings with a mult. absorbing zero) in [1], [4] and [2], and we also refer to [5] and [3], I.7 and I.8 in this context.

In Section 2 we consider at first left ideals and ideals which consist of a single element and investigate the k-closure and the h-closure of those ideals. Then we introduce a condition (C^*) and obtain that a semiring S satisfies (C^*) iff S has no left k-ideals except S and, possibly, one more consisting of a single element (cf. Th. 2.6 a)). This improves results of [7], where a stronger condition (C) was shown to be sufficient for a semiring S to contain only S and, possibly, $\{O\}$ as k-ideals. However, the latter implies (C) only with supplementary assumptions, e. g. for mult. commutative semirings with an identity. Similarly, we characterize in Th. 2.6 b) those semirings S which have no left h-ideals except S and, possibly, the zeroid Z of S.

A *maximal k-ideal* A of a semiring S is defined as a proper (i. e. $A \subset S$) k-ideal which is maximal in the set of all proper k-ideals of S. *Maximal left k-ideals* and *maximal (left) h-ideals* of S are defined correspondingly. We deal with those ideals in Section 3 and give in Th. 3.1 sufficient conditions on S such that each proper (left) k-ideal or each proper (left) h-ideal is contained in a maximal one. In all four cases, our condition is implied by the ascending chain condition for the corresponding ideals, but less weaker and in many cases easily verified (cf. Examples 3.2 and 3.3).

In the next part of Section 3 we deal with semiring-theoretical generalizations of the fact that a proper ideal A of a ring R is a maximal ideal of R iff the congruence class ring R/A has only the trivial ideals

$\{O\}$ and R/A. Now it is well known that R/A, if one considers R as a semiring, coincides with R/κ_A. However, a proper ideal A of a semiring S need not be a maximal ideal of S if S/κ_A has only the trivial ideals $\{O\}$ and S/κ_A (cf. Prop. 3.8). Inspite of this we obtain (together with results including also left ideals as well as S/η_A) in Theorems 3.5 and 3.6 b): A proper k-ideal $A = \overline{A}$ of S is a maximal k-ideal of S iff S/κ_A has only $\{O\}$ and S/κ_A as k-ideals and a proper h-ideal $A = \widehat{A}$ of S is a maximal h-ideal of S iff S/κ_A has only $\{O\}$ and S/κ_A as h-ideals. We remark that these both statements, to our best knowledge, have so far only been published with considerably restrictions on the considered semirings, namely the first one in [7, Th. 2.9] for mult. commutative semirings with an identity, and the second one in [5, Th. 3.11] for semirings with a mult. absorbing zero, satisfying a certain condition (H). This may be caused by the disadvantage that k-ideals and h-ideals are not preserved in general under semiring homomorphisms. However, to prove the above statements one only needs that, for each ideal A of S, the natural homomorphism φ of S onto S/κ_A maps certain k-ideals [h-ideals] of S onto k-ideals [h-ideals] of S/κ_A. We show this in Lemma 3.4, the key result in this context, whereas condition (H) in [5] restricts the considered semirings by the assumption that, for each homomorphism φ as above, $\varphi(B)$ is h-closed for all h-ideals B of S.

We also mention the well known fact that a maximal (left) k-ideal (or h-ideal) of a semiring S need not be a maximal (left) ideal of S, and that the same holds for subsemimodules of a semimodule $(S,+)$. (A semiring proving all these statements occurs in the context of Ex. 3.3.) Similarly, a maximal (left) h-ideal of a semiring S need not be a maximal (left) k-ideal of S as we show in Ex. 3.9. In this case, however, the situation changes if one only deals with subsemimodules of a semimodule $(S,+)$. According to Th. 3.10, each h-closed proper subsemimodule of $(S,+)$ is a maximal h-closed subsemimodule of $(S,+)$ iff it is a maximal k-closed one.

In the following we denote by $|A|$ the cardinal number of any set A, and we assume $|S| \geq 2$ for general statements on a semiring S in order to avoid trivial exceptions. We further write $\mathbb{N}_0 = (N_0, +, \cdot)$ for the semiring of non-negative integers with the usual operations and $\mathbb{N}$ for $\mathbb{N}_0 \setminus \{0\}$. Finally, for each semiring S, we introduce the notion S' by $S' = S \setminus \{O\}$ if S has a mult. absorbing element O, and $S' = S$ otherwise.

2. Semirings with at most two k-ideals or h-ideals

We start with the following observations about (left) ideals consisting of a single element:

Lemma 2.1. a) *Let S be a semiring and $a \in S$. Then $A = \{a\}$ is a left ideal of S iff a is mult. right absorbing. This yields $a + a = a$ and hence $A = \{a\} \subseteq Z$ for the zeroid Z of S.*

b) *A semiring S contains exactly one left ideal L satisfying $|L| = 1$ iff S contains an ideal A satisfying $|A| = 1$, which in turn holds iff S has a mult. absorbing element O. Clearly, one has $L = A = \{O\} \subseteq Z$ in this case.*

c) *Let $A = \{a\}$ be a left ideal of S. Then $\widehat{A} = A$ implies $a = O$ and $A = Z$.*

d) *If S is an add. cancellative semiring, then either $Z = \emptyset$ or $Z = \{O\}$ holds, where O is the mult. absorbing zero of S in the latter case.*

Proof. a) If $A = \{a\}$ is a left ideal of S, clearly $sa = a$ holds for all $s \in S$. Conversely, the latter yields $a + a = a$ by multiplying $s_1 + s_2 = s_3$ by a.

b) If $\{a\}$ is a left ideal of S, the same holds by a) for $\{at\}$ and each $t \in S$. Hence $\{a\} = \{at\}$ yields that $a = O$ is the mult. absorbing element of S. If such an element O exists, $\{O\}$ is an ideal of S, again by a). In this case, $\{O\}$ coincides which each left ideal $L = \{a\}$ of S because of $a = Oa = O$.

c) From $\{a\} = A = \widehat{A} \subseteq Z$ by a) we get $A = Z$ and, since Z is an ideal of S, also $a = O$ by b).

d) Let S be add. cancellative. Then, for each $z \in Z$, from $z + u = u$ it follows that z is the zero o of S, which is well known to be mult. absorbing if S is add. cancellative. $\Diamond$

Contrasting statement c) above, very less can be said in general about (left) k-ideals consisting of one element. We illustrate this by the following examples:

Example 2.2. a) Recall that each commutative and idempotent semigroup $(S, +)$ corresponds to an upper semilattice $(S, \leq)$ by $s \leq r \iff s + r = r$ and $s + r = \sup\{s, r\}$ for all $s, r \in S$. Defining $s \cdot r = r$ for all $s, r \in S$, we obtain a semiring $(S, +, \cdot)$ for which each $a \in S$ determines a left ideal $A = \{a\}$, and one has $\overline{A} = \{s \in S \mid s \leq a\}$. Hence $A = \overline{A}$ holds iff a is minimal in $(S, \leq)$ and $A \subset \overline{A}$ otherwise. By a suitable choice of $(S, \leq)$, the semiring $(S, +, \cdot)$ contains an arbitrary cardinal

number of left ideals $A = \{a\}$ satisfying $A = \overline{A}$ as well as $A \subset \overline{A}$. An extreme case corresponds to the addition $s + s = s$ and $s + r = \infty$ for all $s \neq r$ of S and one element $\infty \in S$ picked out arbitrarily. The latter is then add. absorbing, the left ideals $A = \{a\}$ satisfy $A = \overline{A}$ for each $a \neq \infty$, whereas $\overline{A} = S$ holds for each other left ideal A of S. Note also that, for any semiring S as considered in this example, $\widehat{A} = S$ holds for each left ideal $A = \{a\}$, due to $s + a + (s + a) = a + (s + a)$.

b) Let S be a semiring with a mult. absorbing element O and hence $A = \{O\}$ the unique (left) ideal of S consisting of one element. Then, as one would expect, there are various examples such that $A = \overline{A}$ or $A \subset \overline{A} = S$ or $A \subset \overline{A} \subset S$ is satisfied. For instance, the semigroup $(S, \cdot) = (\mathbb{Z}, \cdot)$ of integers established with an addition $s \oplus r = \max\{s, r\}$ (or $s \oplus r = \min\{s, r\}$) provides a semiring S of the latter kind (cf. also Ex. 2.7).

For each semiring S, a sufficient condition (C) was introduced in [7] such that S contains at most two k-ideals, where S satisfies (C) iff, for all $a \in S'$ and $s \in S$,

$$s + s_1 a = s_2 a \text{ holds for suitable } s_i \in S.$$

Clearly, this condition corresponds to the elements $s_i a$ of the left ideal $Sa = \{sa \mid s \in S\}$ of S. We need a condition (C^*) corresponding to the elements of the left ideal A generated by $a \in S$, which are sa, na and $sa + na$ for all $s \in S$ and $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ where na is defined by $\sum_{i=1}^{n} a$. To simplify our notation, we introduce an operator domain T for S consisting of the elements $s \in S, n \in \mathbb{N}$ and their sums $s + n$, such that $A = Ta = \{ta \mid t \in T\}$ holds for the left ideal A generated by $a \in S$. We only remark that T may be considered as a semiring $(T, +, \cdot)$ containing S as a subsemiring. If S has a mult. absorbing zero o, T is the Dorroh semiring extension $T = \mathbb{N}_0 e + S$ with $e + o = e$ as the identity and $0 + o = o$ as the zero of T (cf. [3], I.3). Otherwise, one has to adjoin a mult. absorbing element, say w, to S and obtains T as $\mathbb{N}_0 e + (S \cup \{w\}) \setminus \setminus \{w\}$.

Definition 2.3. A semiring S satisfies *condition* (C^*) iff for all $a \in S'$ and $s \in S$

$$s + t_1 a = t_2 a \quad \text{holds for suitable} \quad t_i \in T, \tag{2.1}$$

and S satisfies *condition* (D^*) iff for all $a \in S \setminus Z$ and $s \in S$

(2.2) $\quad s + t_1 a + u = t_2 a + u$ holds for suitable $t_i \in T$ and $u \in S$.

The conditions (C) and (D) for S are obtained replacing $t_i \in T$ in (2.1) and (2.2) by $s_i \in S$.

Remark 2.4. a) If S has a right identity e_r or at least elements e_1, e_2 satisfying $x + xe_1 = xe_2$ for all $x \in S$ (a pair of this kind was called a right identity pair in [6]), these four conditions can be simplified: It is enough to check them for $s = e_r$ or for $s = e_1$ and $s = = e_2$.

b) One clearly has $(C) \Longrightarrow (C^*)$ and $(D) \Longrightarrow (D^*)$. The converse implications hold if S has a left identity or at least a left identity pair.

c) Also $(C) \Longrightarrow (D)$ and $(C^*) \Longrightarrow (D^*)$ are obvious, and the converse implications are true if S is add. cancellative. To see the latter, observe that such a semiring satisfies $Z = \emptyset \Longrightarrow S = S'$ and $Z \neq \emptyset \Longrightarrow Z = \{o\}$ as well as $o = O$.

Now we come to statements depending on one of these conditions. We leave it to the reader to take notice of consequences resulting from the following statements together with Remark 2.4 or with supplementary assumptions on the considered semirings as e.g. mult. commutativity.

Lemma 2.5. *A semiring S satisfying condition (C^*) contains at most one left ideal A consisting of a single element.*

Proof. By way of contradiction, assume that $\{a\} \neq \{b\}$ are left ideals of S. Then S has no mult. absorbing element by Lemma 2.1 b), and we can apply (C^*) to a and to b. The former yields $s + a = a$ for all $s \in S$ and hence $b + a = a$, and the latter $a + b = b$ in the same way. So we get $a = b$ contradicting $\{a\} \neq \{b\}$. $\Diamond$

Theorem 2.6. a) *If a semiring S satisfies condition (C^*), it contains at most two left k-ideals, which are in fact two-sided ones, namely S and, possibly, the ideal $\{O\}$ consisting of the mult. absorbing element O of S. Conversely, if S contains no other left k-ideals, S satisfies (C^*). However, if such a semiring has a mult. absorbing element O, the k-ideals $\overline{\{O\}}$ and S may coincide, where $\overline{\{O\}} = S$ holds iff O is also add. absorbing, and $\overline{\{O\}} = \{O\}$ otherwise.*

b) *If a semiring S satisfies (D^*), it contains at most two left h-ideals, which are in fact two-sided ones, namely S and, possibly, the zeroid $Z = Z(S)$ of S. Conversely, if S contains no other left h-ideals, S satisfies (D^*). However, if such a semiring has a zeroid $Z \neq \emptyset$, both*

cases $Z \subset S$ and $Z = S$ are possible.

Proof. a) Assume at first (C^*) for S and let $A = \overline{A}$ be a left k-ideal of S. If $|A| = 1$ holds, we get $A = \{O\}$ by Lemma 2.5 and Lemma 2.1 b). Otherwise A contains an element $a \in S'$, and $s + t_1 a = t_2 a$ for each $s \in S$ by (C^*) implies $\overline{A} = S$. Conversely, let S and, possibly, $\{O\}$ be all left k-ideals of S. Then each $a \in S'$ generates a left ideal $A = Ta$ of S satisfying $\overline{A} = S$. Thus, for each $s \in S$, there are $t_1, t_2 \in T$ such that $s + t_1 a = t_2 a$ holds, which proves (C^*) for S. Now assume (C^*) for a semiring S with a mult. absorbing element O. Then the k-ideal $\overline{\{O\}}$ coincides either with $\{O\}$ or with S as proved above, and we show in Ex. 2.7 that both cases may appear. Clearly, $\overline{\{O\}} = S$ is equivalent with $s + O = O$ for all $s \in$ $\in S$.

b) Again we start assuming (D^*) for S. Since $Z \subseteq A$ holds for each left h-ideal $A = \widehat{A}$ (regardless whether $Z \neq \emptyset$ holds or not), we may assume that A contains an element $a \in S \setminus Z$. Then $s + t_1 a +$ $+ u = t_2 a + u$ by (D^*) implies $A = \widehat{A} = S$. Conversely, let S and, possibly, Z be all left h-ideals of S. Then each $a \in S \setminus Z$ generates a left ideal $A = Ta$ of S satisfying $\widehat{A} = S$, which proves (D^*) for S. Examples for semirings according to the last statement of b) follow below.

Example 2.7. a) It is well known that each semiring H can be extended to a semiring $S = H \cup \{O\}$ by adjoining an element $O \notin H$ once as a mult. absorbing zero $O = o$ and once as a twofold absorbing element $O = \infty$. In the first case S satisfies $\overline{\{O\}} = \{O\}$ and $\widehat{\{O\}} =$ $= Z(S) = \{O\} \cup Z(H)$, in particular $Z(S) = \{O\}$ or $Z(S) = S$ by choosing $Z(H) = \emptyset$ or $Z(H) = H$, respectively. In the latter case one has $\overline{\{O\}} = S$ and hence $\widehat{\{O\}} = S$.

b) A simple way to obtain semirings S as above satisfying (C) and hence $(C^*), (D)$ and (D^*) is to choose $(H, +, \cdot)$ as a semifield without a zero (i. e. a semiring such that $(H, \cdot)$ is a group). In particular, there are semifields H which satisfy $Z(H) = \emptyset$ (e. g. the add. cancellative semifield $(\mathbb{H}, +, \cdot)$ of positive rational numbers with the usual operations) or $Z(H) = H$ (e. g. each add. idempotent semifield as $(\mathbb{H}, \oplus, \cdot)$ where $a \oplus b = \max\{a, b\}$ replaces the usual addition).

c) There are semirings S (in particular mult. commutative ones) with a mult. absorbing element O such that

$$\{O\} \subset \overline{\{O\}} = \widehat{\{O\}} = Z(S) \subset S$$

holds, where $|Z(S)| \geq 2$ can be chosen arbitrarily. Such a semiring S can be obtained as an inflation of any semiring U satisfying $\widehat{\{O\}} = \{O\} \subset U$ for the mult. absorbing element O of U by adjoining at least one shadow s of O (i. e. an element $s \notin U$ which behaves like O in all sums and products, cf. [3], I.2). If $\widehat{\{O\}}$ and U are all left h-ideals of U, then $\widehat{\{O\}}$ and S are all left h-ideals of S. In this case, according to Th. 2.6, S satisfies (D^*) but not (C^*).

For a semiring S with a mult. absorbing zero $o = O$ satisfying (C) it was shown in [7], that $ab = O$ implies $a = O$ or $b = O$. Due to Th. 2.6 a), the same holds for each mult. absorbing element O of S which is not add. absorbing. The proof is similar to that of the following statement:

Proposition 2.8. *Assume $Z \neq \emptyset$ and (D) for a semiring S. Then $ab \in Z$ for $a, b \in S$ implies $a \in Z$ or $b \in Z$.*

Proof. By way of contradiction, we assume $ab \in Z$ for $a \notin Z$ and $b \notin Z$. Then $s + s_1 a + u = s_2 a + u$ according to (D) yields $sb + s_1 ab + ub = s_2 ab + ub$, i.e. $sb \in \widehat{Z} = Z$ for all $s \in S$. Applying again (D), we obtain $r + s_1 b + u = s_2 b + u$, i.e. $r \in \widehat{Z} = Z$ for each $r \in S$, contradicting $a \notin Z$. $\Diamond$

3. Maximal k-ideals and h-ideals

According to the Introduction, a (left) k-ideal $A = \overline{A}$ of a semiring S is called a maximal (left) k-ideal of S if, for each (left) k-ideal $B = \overline{B}$ of S, $A \subset B \subseteq S$ implies $B = S$. Maximal (left) h-ideals $A = \widehat{A}$ of S are defined in the same way. Note that a k-ideal A of S may be a maximal k-ideal or a maximal left k-ideal of S. Clearly, the latter implies the former, and the corresponding implications holds for h-ideals. The semiring S of all 2×2-matrices over the semifield $\mathbb{H}_0$ of all non-negative rational numbers disproves both converse implications: S has only two ideals, both h-closed (and hence k-closed), but an infinite number of h-closed left-ideals. Similarly, a (left) h-ideal A of S may even be a maximal (left) k-ideal, which yields that A is a maximal (left) h-ideal of S. The converse implication is disproved in Ex. 3.9.

Theorem 3.1. a) *Let S be a semiring. Then each proper k-ideal [h-*

ideal] of S is contained in a maximal k-ideal [h-ideal] of S if there exists a finitely generated ideal F of S satisfying $\overline{F} = S$ [$\widehat{F} = S$].

b) *The same statements hold for left ideals of S.*

Proof. From the four statements, we only show that on k-ideals and that on left k-ideals simultaneously. So we assume $\overline{F} = S$ for a (left) ideal F of S generated by $\{a_1, \ldots, a_n\} \subseteq S$. Let $A = \overline{A}$ be a proper (left) k-ideal of S and $\mathfrak{B}$ the set of all (left) k-ideals $B = \overline{B}$ of S satisfying $A \subseteq B \subset S$, partially ordered by inclusion. Consider a chain $\{B_i \mid i \in I\}$ in $\mathfrak{B}$. One easily checks that $C = \bigcup_{i \in I} B_i$ is a (left) k-ideal of S. We claim $C = \overline{C} \subset S$. Otherwise, $C = S$ would imply $\{a_1, \ldots, a_n\} \subseteq C$ and hence $\{a_1, \ldots, a_n\} \subseteq B_j$ for a suitable $j \in I$. But the latter implies $F \subseteq B_j$ and thus $S = \overline{F} \subseteq \overline{B_j} = B_j$, contradicting $B_j \subset S$. So we have $A \subseteq C = \overline{C} \subset S$, i. e. $C \in \mathfrak{B}$, and Zorn's Lemma yields that $\mathfrak{B}$ has a maximal element as we were to show. $\Diamond$

In Th. 3.1 we have dealt with four types of ideals of a semiring S, namely left or two-sided k- or h-ideals of S, and we could have also included two more, namely all left and all two-sided ideals. The used conditions are that S, considered as an ideal of one of these four (or six) types, is finitely generated. In particular, each of these conditions holds if S has a (right) identity or is a finitely generated semiring. Moreover, each of these conditions is implied by the ascending chain condition for ideals of the corresponding type, since this is equivalent with the statement, that all ideals of S of that type are finitely generated. The following Ex. 3.2 presents a semiring which satisfies our condition for all six types of ideals, but none of the ascending chain conditions. However, also our conditions are not necessary for the corresponding statements in Th. 3.1 as we show in Ex. 3.3.

Example 3.2. Let $S = \mathbb{N}_0[x_1, x_2, \ldots]$ be the polynomial semiring in the set $\{x_i \mid i \in \mathbb{N}\}$ of indeterminates over $\mathbb{N}_0$. Then S is mult. commutative and add. cancellative, such that left and two-sided ideals coincide as well as k-closed and h-closed ones. Since S has an identity, the ideal $S = \overline{S} = \widehat{S}$ is finitely generated. For each $n \in \mathbb{N}$, the ideal A_n of S generated by $x_1, \ldots, x_n$ consists of the elements $\sum_{i=1}^{n} f_i x_i$ for arbitrary $f_i \in S$, from which $A_n = \overline{A}_n = \widehat{A}_n$ is obvious. Thus the chain $A_1 \subset \subset A_2 \subset A_3 \subset \ldots$ disproves the ascending chain condition for all ideals under discussion.

Example 3.3. Let S be the homomorphic image of the semiring

$\mathbb{N}_0[x_1, x_2, \dots] \setminus \mathbb{N}$ defined by the relations $x_i x_j = 0$ and $x_i^2 = x_i$ for all $i \neq j$. Then S consists of the elements

$$f = \sum_{i \in \mathbb{N}} n_i x_i, \text{ almost all} n_i \in \mathbb{N}_0 \text{ equal } 0,$$

i.e. S is the direct sum of its subsemirings $S_i = \mathbb{N}_0 x_i = \{nx_i \mid n \in \in \mathbb{N}_0\}$, which are isomorphic to $(\mathbb{N}_0, +, \cdot)$ and have x_i as their identity. As in the above example, S is mult. commutative and add. cancellative.

Now let A be an ideal of S. Then $f \in A$ for $f = \sum_{i \in \mathbb{N}} n_i x_i$ implies $f x_i = n_i x_i \in A$, such that A is the direct sum of ideals A_i of S_i. Obviously, one has $A = \overline{A}$ and hence $A = \widehat{A}$ iff $A_i = \overline{A}_i$ holds for all these ideals A_i of S_i. It is well known that the k-ideals of $\mathbb{N}_0$ are the ideals $m\mathbb{N}_0 = \{mn \mid n \in \mathbb{N}_0\}$ for all $m \in \mathbb{N}_0$, and that $m\mathbb{N}_0$ is a maximal k-ideal iff m is a prime number p. (Note in this context that each maximal k-ideal $p\mathbb{N}_0$ of $\mathbb{N}_0$ is properly contained in the unique maximal ideal $\mathbb{N}_0 \setminus \{1\}$ of $\mathbb{N}_0$.) Consequently, each k-ideal $A = \overline{A}$ of S is characterized by a sequence $(m_i)_{i \in \mathbb{N}}$ of integers $m_i \in \mathbb{N}_0$, where A is the direct sum of the k-ideals $A_i = m_i \mathbb{N}_0 x_i$ of S_i. Moreover, if $A = \overline{A}$ corresponds to $(m_i)_{i \in \mathbb{N}}$ and $B = \overline{B}$ to $(k_i)_{i \in \mathbb{N}}$, then $A \subseteq B$ is equivalent to $k_i | m_i$ for all $i \in \mathbb{N}$. Hence $B = \overline{B}$ is a maximal k-ideal of S iff $(k_i)_{i \in \mathbb{N}}$ consists of one prime number, whereas all other k_i equal 1, and each proper k-ideal $A = \overline{A}$ of S is contained in a maximal one. (Similarly one obtains that each proper ideal of S is contained in a maximal one.)

On the other hand, let F be a finitely generated ideal of S. Then only a finite number of elements x_i occur in the summands nx_i of the elements of F. This yields $\overline{F} = \widehat{F} \subset S$.

Our next point is to characterize maximal k-ideals [h-ideals] A of S by corresponding properties of the semirings S/κ_A and S/η_A, which needs some preparations. Let $\varphi : S \to \varphi(S)$ be a surjective homomorphism of a semiring S. Recall that each (left) ideal B of S is mapped onto the (left) ideal $\varphi(B)$ of $\varphi(S)$, whereas $\varphi^{-1}(C) = \{s \in S \mid \varphi(s) \in \in C\}$ is a (left) ideal of S for each (left) ideal C of $\varphi(S)$. Moreover, it is well known that $\varphi^{-1}(C)$ is a (left) k-ideal or a (left) h-ideal of S if C is such an ideal of $\varphi(S)$. However, for a (left) k-ideal or a (left) h-ideal B of S, the (left) ideal $\varphi(B)$ of $\varphi(S)$ need not be k-closed or h-closed, respectively. But the latter is true with supplementary assumptions.

Lemma 3.4. a) *Let A be a proper ideal of S and consider the natural homomorphism $\varphi : S \to \varphi(S) = S/\kappa_A$ given by $\varphi(s) = [s]_{\kappa_A}$. Assume $A \subseteq B$ for a (left) k-ideal or a (left) h-ideal B of S. Then $\varphi(B)$ is a (left) k-ideal or a (left) h-ideal of S/κ_A, respectively.*

b) *Consider in the same way the homomorphism $\varphi : S \to \varphi(S) = S/\eta_A$ and assume $A \subseteq B$ for a (left) h-ideal B of S. Then $\varphi(B)$ is a (left) h-ideal of S/η_A.*

Proof. We only show a) for a (left) h-ideal B of S satisfying $A \subseteq B$, since the other statements follow in the same pattern. We write $[s]$ for $[s]_{\kappa_A}$ and suppose $[s] + [b_1] + [u] = [b_2] + [u]$ for some $s, u \in S$ and $b_i \in B$. This yields $(s + b_1 + u)\kappa_A(b_2 + u)$, i. e. $s + b_1 + a_1 + u = b_2 + a_2 + u$ for some $a_i \in A$. This and $b_i + a_i \in B$ imply $s \in \widehat{B} = B$ and thus $[s] \in \varphi(B)$. Hence $\varphi(B)$ is a (left) h-ideal of S/κ_A. $\Diamond$

Theorem 3.5. *A proper k-ideal $A = \overline{A}$ of a semiring S is a maximal (left) k-ideal of S iff S/κ_A has only trivial (left) k-ideals, which means that the ideals $\{A\}$ and S/κ_A are all (left) k-ideals of S/κ_A.*

Proof. Assume that the ideal A is a maximal (left) k-ideal of S and $\{A\} \subseteq C = \overline{C} \subseteq S/\kappa_A$ for a (left) k-ideal C of S/κ_A. (Recall that $A = \overline{A}$ is the mult. absorbing zero and hence $\{A\}$ a k-ideal of S/κ_A.) Let $\varphi : S \to \varphi(S) = S/\kappa_A$ be the homomorphism defined by $\varphi(s) = [s]_{\kappa_A}$. Then, as remarked above, $B = \varphi^{-1}(C)$ is a (left) k-ideal of S, and one has $A \subset B$. This yields $B = S$ by the maximal property of A and hence $C = \varphi(B) = \varphi(S) = S/\kappa_A$. Conversely, assume that $\{A\} \subset C = \overline{C} \subseteq S/\kappa_A$ implies $C = S/\kappa_A$ for each (left) k-ideal C of S/κ_A and consider a (left) k-ideal $B = \overline{B}$ of S such that $A \subset B \subseteq S$. Then, by Lemma 3.4 a), $C = \varphi(B)$ is a (left) k-ideal of S/κ_A, clearly satisfying $\{A\} \subset C$. So we get $C = S/\kappa_A$, i.e. each $[s]_{\kappa_A} \in S/\kappa_A$ equals some $[b]_{\kappa_A} \in \varphi(B)$. Now $s + a_1 = b + a_2$ for some $a_i \in A \subseteq B$ implies $s \in \overline{B} = B$ for each $s \in S$, which proves $B = S$. $\Diamond$

Theorem 3.6. a) *A proper h-ideal $A = \widehat{A}$ of a semiring S is a maximal (left) k-ideal of S iff S/κ_A has only trivial (left) k-ideals. This implies that S/η_A has only trivial (left) k-ideals, but not conversely.*

b) *A proper h-ideal $A = \widehat{A}$ of S is a maximal (left) h-ideal of S iff S/κ_A has only trivial (left) h-ideals, which in turn holds iff S/η_A has only trivial (left) h-ideals.*

Proof. a) Since $A = \widehat{A}$ implies $A = \overline{A}$, the equivalence is a special case of Th. 3.5. Using that S/η_A is a homomorphic image of S/κ_A, we obtain the implication directly from the remark before Lemma 3.4. We show in Ex. 3.9 b) that the converse implication does not hold.

b) These are two statements corresponding to Th. 3.5, and the proof of the latter can be transferred to both with near at hand modifications. $\Diamond$

Remark 3.7. Recall that a (one- or two-sided) semiring ideal of a ring R need not be a ring ideal of R in the usual meaning, which is the case iff A is k-closed and hence h-closed. Moreover, for such an ideal $A = \overline{A} = \widehat{A}$ of R, both semirings R/κ_A and R/η_A coincide with the ring R/A. Hence we obtain from Th. 3.5 (and also from Th. 3.6) that a proper ring ideal A of a ring R is a maximal (left) ideal of R iff R/A has no (left) ideals except its trivial ones. As already mentioned in the Introduction, the direct translation of this ring-theoretical result to semirings fails to be true, and we state in this context:

Proposition 3.8. *Let A be a proper ideal of a semiring S. If A is a maximal (left) ideal of S, then S/κ_A and S/η_A have no (left) ideals except their trivial ideals. However, if the latter holds, A need not be a maximal (left) ideal of S.*

Proof. Let the ideal A be a maximal (left) ideal of S. Recall that the k-closure $\overline{A}$ is one κ_A-class and the mult. absorbing zero of S/κ_A. If $\overline{A} = S$ holds, S/κ_A consists of a single element and there is nothing to prove. Otherwise, let C be a (left) ideal of S/κ_A satisfying $\{\overline{A}\} \subset C$ and $\varphi : S \to \varphi(S) = S/\kappa_A$ the natural homomorphism. Then $B = \varphi^{-1}(C)$ is a (left) ideal of S satisfying $A \subseteq \overline{A} \subset B$. By the assumption on A, this yields $S = B$ and hence $C = S/\kappa_A$. In the same way one obtains that S/η_A has no (left) ideals except $\{\widehat{A}\}$ and S/η_A.

Concerning the converse, consider the semiring $S = \mathbb{N}_0$ with the usual operations. For each prime number p, the ideal $A = p\mathbb{N}_0$ is not a maximal ideal of $\mathbb{N}_0$ as already stated in Ex. 3.3. However, the semirings $\mathbb{N}_0/\kappa_A$ and $\mathbb{N}_0/\eta_A$ coincide and are isomorphic to the field $\mathbb{Z}/(p)$ and so without non-trivial ideals. $\Diamond$

Example 3.9. a) We give an example of a mult. commutative semiring S which has only two h-ideals, namely its zeroid Z and S, but an infinite

chain of k-ideals B_i satisfying

$$A = Z \subset B_1 \subset B_2 \subset \ldots \subset S.$$

In particular, *$A = Z$ is a maximal h-ideal of S, but not a maximal k-ideal.* For this purpose, let $(\mathbb{H}, +, \cdot)$ be the semifield of positive rational numbers with the usual operations and $(\mathbb{N}, +, \cdot)$ the semiring defined on $(\mathbb{N}, \leq)$ by $s + r = \max\{s, r\}$ and $s \cdot r = \min\{s, r\}$. Clearly, the direct product of these semirings, defined on $\mathbb{H} \times \mathbb{N} = \{(\sigma, s) \mid \sigma \in \in \mathbb{H}, s \in \mathbb{N}\}$ by component-wise operations, is again a semiring. We consider the semiring $S = (\mathbb{H} \times \mathbb{N}) \cup \{O\}$, obtained from $\mathbb{H} \times \mathbb{N}$ by adjoining a mult. absorbing zero $o = O$. Now $(\sigma, s) + O \neq O$ and $(\sigma, s) + (\varrho, r) \neq (\varrho, r)$ for all $(\sigma, s), (\varrho, r) \in \mathbb{H} \times \mathbb{N}$ shows that the ideal $\{O\}$ is the zeroid Z of S. Next let B be any ideal of S which contains at least one element $(\beta, b) \neq O$. Then $(\beta, b)(\beta^{-1}\sigma, c) \in B$ shows that $(\beta, b) \in B$ implies $(\sigma, c) \in B$ for all $\sigma \in \mathbb{H}$ and all $c \in \mathbb{N}$ such that $c \leq b$. Moreover, $\widehat{B} = S$ holds since each $(\sigma, s) \in \mathbb{H} \times \mathbb{N}$ for some $s > b$ satisfies $(\sigma, s) + (\sigma_1, b) + (\mu, s) = (\sigma_2, b) + (\mu, s)$ if one chooses $\sigma_i \in \mathbb{H}$ according to $\sigma + \sigma_1 = \sigma_2$. This proves that $A = Z = = \widehat{Z}$ is a maximal h-ideal of S. Now we define $B_n = \{(\sigma, c) \in \mathbb{H} \times \times \mathbb{N} \mid c \leq n\} \cup \{O\}$ for each $n \in \mathbb{N}$. Clearly, B_n is an ideal of S, and $\overline{B}_n = B_n$ holds since $(\varrho, r) + (\sigma_1, c_1) = (\sigma_2, c_2)$ implies $r \leq n$. Thus $A = Z$ is contained in the chain $B_1 \subset B_2 \subset \ldots$ of k-ideals of S.

b) We use this example to complete the proof of Th. 3.6 a). For $A = Z$, the congruence κ_A is obviously the identical relation on S and hence $S/\kappa_A \cong S$. On the other hand, $(\sigma, s)\eta_A(\varrho, r)$ holds for elements of $\mathbb{H} \times \mathbb{N}$ iff $\sigma = \varrho$, since $(\sigma, s) + O + (\mu, u) = (\varrho, r) + O + (\mu, u)$ implies $\sigma = \varrho$, whereas $s + u = r + u$ is satisfied e. g. for $u = s + r$. Hence S/η_A is isomorphic to the semifield $(\mathbb{H}_0, +, \cdot)$. Therefore, S/η_A has no (left) ideals except $\{A\}$ and S/η_A, whereas $S/\kappa_A \cong S$ has an infinite number of k-ideals.

c) We denote one of the mult. non-commutative semirings of Ex. 2.2 a) by $(S_1, +, \cdot)$ and consider the semiring $S = (\mathbb{H}_0 \times S_1) \cup \cup \{O\}$ in the same way as above. Then $A = Z = \{O\}$ and S are the only ideals of S and *the maximal h-ideal $A = Z$ is now also a maximal k-ideal of S.* Moreover, one checks that *$A = Z$ is also a maximal left h-ideal, but not a maximal left k-ideal of S.* The latter follows since $B_a = \{(\sigma, s) \in \mathbb{H} \times S_1 \mid s \leq a\} \cup \{O\}$ is a left k-ideal of S for each $a \in S_1$. Corresponding to part b), one has again

$S/\kappa_A \cong S$ and $S/\eta_A \cong \mathbb{H}_0$ and thus another example in the context of Th. 3.6 a).

As a contrast to the fact that a maximal (left) h-ideal of a semiring need not be a maximal (left) k-ideal illustrated by the above examples we show for subsemimodules of a semimodule:

Theorem 3.10. *Let $S = (S, +)$ be a semimodule and $A = \widehat{A} \subset S$ a maximal h-closed subsemimodule of S. Then A is also k-closed in S, i. e. $A \subset B = \overline{B} \subseteq S$ implies $B = S$ for each k-closed subsemimodule B of S.*

Proof. We proceed by showing two auxiliary statements for a maximal h-closed subsemimodule A of S.

i) For each $r \in S \setminus A$ and each $s \in S$ there are $a_i \in A$, $n_i \in \in \mathbb{N}$ and $u \in S$ satisfying $s + a_1 + n_1 r + u = a_2 + n_2 r + u$. To prove this, let C be the subsemimodule of S generated by A and r. From $A \subset C$ and the assumption on A we have $\widehat{C} = S$, which yields $s + + c_1 + v = c_2 + v$ for some $c_i \in C$ and $v \in S$. Since C consists of the elements $a \in A$, $nr \in \mathbb{N}r$ and all sums $a + nr$ of those elements, we can add any of these sums $a + nr$ to $s + c_1 + v = c_2 + v$ and obtain i).

ii) For each $s \in S$ there is some $m \in \mathbb{N}$ such that $ms \in A$ holds. We go by contradiction and assume $ms \notin A$ for all $m \in \mathbb{N}$ and some $s \in S$. Applying i) to $r = 2s \in S \setminus A$ and s we get

$$a_1 + (2n_1 + 1)s + u = a_2 + 2n_2 s + u$$

for suitable $a_i \in A, n_i \in \mathbb{N}$ and $u \in S$. Because of $2n_1 + 1 \neq 2n_2$ we may assume $2n_1 + 1 > 2n_2$ without loss of generality, i. e. $2n_1 + 1 = 2n_2 + d$ for some $d \in \mathbb{N}$. This yields

$$ds + a_1 + (2n_2 s + u) = a_2 + (2n_2 s + u),$$

which implies $ds \in \widehat{A} = A$, contradicting our assumption $ms \notin A$ for all $m \in \mathbb{N}$.

To complete our proof, we consider any k-closed subsemimodule $B = \overline{B}$ of S containing A properly and the subsemimodule C generated by A and some $r \in B \setminus A$. Then we have $A \subset C \subseteq B = \overline{B}$, and we obtain $B = S$ by showing $\overline{C} = S$. Indeed, for each $s \in S$ we have

$$s + a_1 + n_1 r + mu = a_2 + n_2 r + mu$$

according to i), where $mu \in A$ holds by ii) for at least one $m \in \mathbb{N}$. This states $s + c_1 = c_2$ for each $s \in S$ and suitable $c_i \in C$, i. e. $\overline{C} = S$.

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