Mathematica Pannonica
7/2 (1996), 223 – 232

BOUNDED SOLUTIONS OF SCHILLING'S PROBLEM

Janusz **Morawiec**

Instytut Matematyki, Uniwersytet Śląski, ul. Bankowa 14, PL-40-007 Katowice, Poland

Received: August 1995

MSC 1991: 39 B12, 39 B 22

Keywords: Schilling's problem, bounded and continuous functions.

Abstract: Let n be a positive integer, q_n be the unique $x \in (\frac{1}{3}, \frac{1}{2})$ with $x^{n+1} - 3x + 1 = 0$, and $q \in (0, q_n]$. We found a set A_q^n of reals with the following property (P): Every solution $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$f(qx) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)]$$

which vanishes outside of $[-\frac{q}{1-q}, \frac{q}{1-q}]$ and is bounded in a neighbourhood of a point of that set vanishes everywhere. It is also observed that for $q \in (0, \frac{1}{3}]$ the set $\bigcup_{n=1}^{\infty} A_q^n$, which equals then

$$\Big\{ \sum_{n=1}^{\infty} \varepsilon(n) q^n \quad : \quad \varepsilon \in \{-1, 0, 1\}^{\mathbb{N}} \Big\},$$

is the largest one with property (P).

Following R. Schilling [9] we consider solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$(1) \qquad f(qx) = \frac{1}{4q}\big[f(x-1) + f(x+1) + 2f(x)\big]$$

such that

$$(2) \qquad f(x) = 0 \quad \text{for} \quad |\, x \,| > Q$$

where q is a fixed number from the open interval $(0, 1)$ and

$$Q = \frac{q}{1-q}.$$

In what follows any solution $f : \mathbb{R} \to \mathbb{R}$ of (1) satisfying (2) will be called a *solution of Schilling's problem.*

If

$$3q \le 1 - \sqrt[3]{2} + \sqrt[3]{4} \tag{3}$$

then according to [7] the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of a point of the set

$$\Big\{\varepsilon \sum_{i=1}^{n} q^i : \quad n \in \mathbb{N} \cup \{0, +\infty\},\ \varepsilon \in \{-1, 1\}\Big\}. \tag{4}$$

This generalizes in particular [1; Th. 1]. It is the aim of the present paper to obtain such a result with the set (4) replaced by a larger one. However, we are not able to enlarge (4) for all q's satisfying (3) but, on the other hand, for $q \le \frac{1}{3}$ we succeeded in finding even the largest set to be put in the place of (4) (cf. Cor. 1).

Given a positive integer n and $q \in (0,1)$ consider the set A_q^n of all the real numbers of the form

$$\varepsilon \sum_{l=1}^{L} (-1)^l \sum_{k=1}^{K_l} q^{\sum_{m=k}^{K_l} \nu(l,m) + \sum_{j=l+1}^{L} \sum_{m=1}^{K_j} \nu(j,m) + M} + \varepsilon(-1)^L \sum_{m=1}^{M} q^m, \tag{5}$$

where $\varepsilon \in \{-1,1\}$, M, L are non-negative integers, $K_1, \dots, K_L \in \in \{1, \dots, n\}$, and $\nu : \{1, \dots, L\} \times \{1, \dots, n\} \to \mathbb{N}$. Evidently, the set (4) is a subset of $\operatorname{cl} A_q^n$. Let us observe also that for $l_1, l_2 \in \{1, \dots L\}$, $k_1 \in \{1, \dots, K_{l_1}\}$, $k_2 \in \{1, \dots, K_{l_2}\}$, if $(l_1, k_1) \ne (l_2, k_2)$ then

$$\sum_{m=k_1}^{K_{l_1}} \nu(l_1, m) + \sum_{j=l_1+1}^{L} \sum_{m=1}^{K_j} \nu(j,m) \ne \sum_{m=k_2}^{K_{l_2}} \nu(l_2, m) + \sum_{j=l_2+1}^{L} \sum_{m=1}^{K_j} \nu(j,m). \tag{6}$$

The proof of the following fact is left to the reader (cf. also [6; Th. 21(a), (d)]).

Remark 1. If $q \in (0, \frac{1}{3}]$ then

$$\operatorname{cl} \bigcup_{n=1}^{\infty} A_q^n = \Big\{ \sum_{n=1}^{\infty} \varepsilon(n) q^n : \quad \varepsilon \in \{-1, 0, 1\}^{\mathbb{N}} \Big\},$$

and if $q \in [\frac{1}{3}, 1)$ then

$$\Big\{\sum_{n=1}^{\infty}\varepsilon(n)q^n : \quad \varepsilon\in\{-1,0,1\}^{\mathbb{N}}\Big\}=[-Q,Q].$$

For every positive integer n let q_n denote the unique $x\in(\frac{1}{3},\frac{1}{2})$ with

$$x^{n+1}-3x+1=0, \tag{7}$$

and observe that if $q\in(0,\frac{1}{2})$ then

$$q\le q_n \quad \text{iff} \quad q^{n+1}-3q+1\ge 0.$$

Our main result reads.

Theorem 1. *If n is a positive integer and $q\in(0,q_n]$ then the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of a point of the set* $\operatorname{cl}A_q^n$.

The proof of this theorem is based on four lemmas. However, we start with the following simple remarks.

Remark 2. If f is a solution of Schilling's problem then so is the function $g:\mathbb{R}\to\mathbb{R}$ defined by the formula $g(x)=f(-x)$.

Remark 3. Assume f is a solution of Schilling's problem.

If $q\neq\frac{1}{4}$ then $f(-Q)=f(Q)=0$. If $q<\frac{1}{2}$ then $f(0)=0$.

Lemma 1. *Assume $q\in(0,\frac{1}{2})$. If f is a solution of Schilling's problem then*

$$f\Big(q^{N+M}x+\varepsilon\sum_{m=1}^{M}q^m\Big)=\Big(\frac{1}{2}\Big)^M\Big(\frac{1}{2q}\Big)^{N+M}f(x) \tag{8}$$

for all $x\in(Q-1,1-Q)$ (for all $x\in[Q-1,1-Q]$ if $q\neq\frac{1}{4}$), for all $\varepsilon\in\{-1,1\}$, and for all non-negative integers M and N.

For $x\in(Q-1,1-Q)$ this was proved in [7] as Lemma 2. In the case of the closed interval $[Q-1,1-Q]$ and $q\neq\frac{1}{4}$ we argue similarly as in the proof of [7; Lemma 2] using also [7; Remarks 1 and 2(i)].◊

Lemma 2. *Let $n\in\mathbb{N}$, $q\in(0,q_n]$ and*

$$y=q^{N+\sum_{l=1}^{L}\sum_{k=1}^{K_l}\nu(l,k)}x+$$
$$+\sum_{l=1}^{L}(-1)^l\sum_{k=1}^{K_l}q^{\sum_{m=k}^{K_l}\nu(l,m)+\sum_{j=l+1}^{L}\sum_{m=1}^{K_j}\nu(j,m)},$$

where N is a non-negative integer, L is a positive integer, $K_1,\dots,K_L\in\{1,\dots,n\}$, $\nu:\{1,\dots,L\}\times\{1,\dots,n\}\to\mathbb{N}$, and $x\in[0,1-Q)$ ($x\in[0,1-Q]$ if $q\neq\frac{1}{4}$).

If L is even then $y\in(0,1-Q)$.

If L is odd then $y \in [Q-1,0]$ ($y \in (Q-1,0]$ if $q<\frac{1}{3}$).

Proof. Since $q \le q_n < \frac{1}{2}$ we have

$$Q < 1. \tag{9}$$

Moreover, as q_n is a solution of (7),

$$\sum_{i=1}^{n} q^i \le \sum_{i=1}^{n} q_n^i = 1 - \frac{q_n}{1-q_n} \le 1 - \frac{q}{1-q} = 1 - Q, \tag{10}$$

and

$$\text{if} \quad q < \frac{1}{3} \quad \text{then} \quad \sum_{i=1}^{n} q^i < Q < 1 - Q. \tag{11}$$

Observe also that

$$\begin{aligned} y =& q^{\nu(L,K_L)} \Big(q^{N+\sum_{l=1}^{L}\sum_{k=1}^{K_l}\nu(l,k)-\nu(L,K_L)} x + \\ &+ \sum_{l=1}^{L-1}(-1)^l \sum_{k=1}^{K_l} q^{\sum_{m=k}^{K_l}\nu(l,m)+\sum_{j=l+1}^{L}\sum_{m=1}^{K_j}\nu(j,m)-\nu(L,K_L)} + \\ &+ (-1)^L \sum_{k=1}^{K_L-1} q^{\sum_{m=k}^{K_L-1}\nu(L,m)} + (-1)^L \Big), \end{aligned} \tag{12}$$

$$\begin{aligned} y =& q^{\sum_{l=1}^{L}\sum_{k=1}^{K_l}\nu(l,k)} \left(q^N x - 1\right) - \\ &- \sum_{k=2}^{K_1} q^{\sum_{m=k}^{K_1}\nu(1,m)+\sum_{j=2}^{L}\sum_{m=1}^{K_j}\nu(j,m)} + \\ &+ \sum_{l=2}^{L}(-1)^l \sum_{k=1}^{K_l} q^{\sum_{m=k}^{K_l}\nu(l,m)+\sum_{j=l+1}^{L}\sum_{m=1}^{K_j}\nu(j,m)}, \end{aligned} \tag{13}$$

$$q^{\sum_{l=1}^{L}\sum_{k=1}^{K_l}\nu(l,k)} \left(q^N x - 1\right) < 0 \tag{14}$$

and

$$\sum_{k=1}^{K_L} q^{\sum_{m=k}^{K_L}\nu(L,m)} \le \sum_{k=1}^{K_L} q^{\sum_{m=k}^{K_L} 1} \le \sum_{k=1}^{n} q^k. \tag{15}$$

Suppose first L is even. Applying (13), (14), (6), (15), (9) and (10) we obtain

$$y < \sum_{l=2}^{L}(-1)^l \sum_{k=1}^{K_l} q^{\sum_{m=k}^{K_l}\nu(l,m)+\sum_{j=l+1}^{L}\sum_{m=1}^{K_j}\nu(j,m)} \leq$$

$$\leq \sum_{l=2}^{L-2}\sum_{k=1}^{K_l} q^{\sum_{m=k}^{K_l}\nu(l,m)+\sum_{j=l+1}^{L}\sum_{m=1}^{K_j}\nu(j,m)} -$$

$$- \sum_{k=1}^{K_{L-1}} q^{\sum_{m=k}^{K_{L-1}}\nu(L-1,m)+\sum_{m=1}^{K_L}\nu(L,m)} + \sum_{k=1}^{K_L} q^{\sum_{m=k}^{K_L}\nu(L,m)} \leq$$

$$\leq \sum_{i=1}^{\infty} q^{\nu(L-1,K_{L-1})+\sum_{m=1}^{K_L}\nu(L,m)+i} -$$

$$- q^{\nu(L-1,K_{L-1})+\sum_{m=1}^{K_L}\nu(L,m)} + \sum_{k=1}^{n} q^k =$$

$$= q^{\nu(L-1,K_{L-1})+\sum_{m=1}^{K_L}\nu(L,m)}(Q-1) + \sum_{k=1}^{n} q^k < \sum_{k=1}^{n} q^k \leq 1 - Q,$$

whereas (12), (6) and (9) give

$$y \geq q^{\nu(L,K_L)}\Big(-\sum_{i=1}^{\infty} q^i + 1\Big) = q^{\nu(L,K_L)}(-Q+1) > 0.$$

Suppose now L is odd. If $L = 1$ then using the definition of y, (15) and (10) we see that

$$y \geq -\sum_{k=1}^{K_1} q^{\sum_{m=k}^{K_1}\nu(1,m)} \geq -\sum_{k=1}^{n} q^k \geq Q - 1,$$

with the last inequality being strict if $q < \frac{1}{3}$ (cf. (11)). If $L \geq 3$ then on account of the definition of y, (6), (15), (9) and (10) we have

$$y \geq -\sum_{l=1}^{L-2}\sum_{k=1}^{K_l} q^{\sum_{m=k}^{K_l}\nu(l,m)+\sum_{j=l+1}^{L}\sum_{m=1}^{K_j}\nu(j,m)} +$$

$$+ \sum_{k=1}^{K_{L-1}} q^{\sum_{m=k}^{K_{L-1}}\nu(L-1,m)+\sum_{m=1}^{K_L}\nu(L,m)} - \sum_{k=1}^{K_L} q^{\sum_{m=k}^{K_L}\nu(L,m)} \geq$$

$$\geq -\sum_{i=1}^{\infty} q^{\nu(L-1,K_{L-1})+\sum_{m=1}^{K_L}\nu(L,m)+i} +$$

$$+\, q^{\nu(L-1,K_{L-1})+\sum_{m=1}^{K_L}\nu(L,m)} - \sum_{k=1}^{K_L} q^k$$

$$> -\sum_{k=1}^{n} q^k \geq Q-1.$$

Finally, if L is odd then taking into account (12) and (6) we obtain

$$y \leq q^{\nu(L,K_L)}\Big(x + \sum_{i=1}^{\infty} q^i - 1\Big) \leq q^{\nu(L,K_L)}[(1-Q)+Q-1] = 0. \quad \diamond$$

Lemma 3. *Assume* $n \in \mathbb{N}$ *and* $q \in (0, q_n]$. *If* f *is a solution of Schilling's problem then for every* $x \in [0, 1-Q)$, *for every non-negative integers* M, L *and* N, *for every* $K_1, \dots, K_L \in \{1, \dots, n\}$, *and for every* $\nu : \{1, \dots, L\} \times \{1, \dots, n\} \to \mathbb{N}$ *we have*

$$\begin{aligned} &f(q^{N+\sum_{l=1}^{L}\sum_{k=1}^{K_l}\nu(l,k)+M}x+ \\ &+\sum_{l=1}^{L}(-1)^l \sum_{k=1}^{K_l} q^{\sum_{m=k}^{K_l}\nu(l,m)+\sum_{j=l+1}^{L}\sum_{m=1}^{K_j}\nu(j,m)+M} + \\ &+(-1)^L \sum_{m=1}^{M} q^m) = \\ &= \Big(\frac{1}{2}\Big)^{\sum_{l=1}^{L}K_l+M}\Big(\frac{1}{2q}\Big)^{N+\sum_{l=1}^{L}\sum_{k=1}^{K_l}\nu(l,k)+M} f(x). \end{aligned} \tag{16}$$

Proof. According to Lemma 1, (16) holds for $L = 0$. Assume L is a positive integer.

Consider first the case $M = 0$.

Let $L = 1$. Equality (16) takes then the form

$$\begin{aligned} &f(q^{N+\sum_{k=1}^{K_1}\nu(1,k)}x - \sum_{k=1}^{K_1} q^{\sum_{m=k}^{K_1}\nu(1,m)}) = \\ &= \Big(\frac{1}{2}\Big)^{K_1}\Big(\frac{1}{2q}\Big)^{N+\sum_{k=1}^{K_1}\nu(1,k)} f(x), \end{aligned} \tag{17}$$

and making use of Lemma 1 we see that if $K_1 = 0$ then (17) holds for all $x \in (Q-1, 1-Q)$ (for all $x \in [Q-1, 1-Q]$ if $q \neq \frac{1}{4}$) and for every non-negative integer N. Fix now a $K_1 \in \{0, \dots, n-1\}$ and suppose that (17) is satisfied for every non-negative integer N, for every

$\nu : \{1\} \times \{1, \ldots, n\} \to \mathbb{N}$, and for all $x \in [0, 1-Q)$ (for all $x \in [0, 1-Q]$ if $q \neq \frac{1}{4}$). Let $N \in \mathbb{N} \cup \{0\}$, $\nu : \{1\} \times \{1, \ldots, n\} \to \mathbb{N}$ and $x \in [0, 1-Q)$ ($x \in [0, 1-Q]$ if $q \neq \frac{1}{4}$). Putting

$$z = q^{N+\sum_{k=1}^{K_1} \nu(1,k)} x - \sum_{k=1}^{K_1} q^{\sum_{m=k}^{K_1} \nu(1,m)} - 1$$

we have

$$z \leq x - 1 \tag{18}$$

and, according to Lemma 2, $y := qz \in [Q-1, 0]$ (and $y \in (Q-1, 0]$ if $q < \frac{1}{3}$). This jointly with the definition of z, Lemma 1, (1), (17), (2), Remark 3 and (17) gives

$$\begin{aligned}
&f(q^{N+\sum_{k=1}^{K_1+1} \nu(1,k)} x - \sum_{k=1}^{K_1+1} q^{\sum_{m=k}^{K_1+1} \nu(1,m)}) = \\
&= f(q^{\nu(1,K_1+1)-1} y) == \Big(\frac{1}{2q}\Big)^{\nu(1,K_1+1)-1} f(y) = \\
&= \Big(\frac{1}{2q}\Big)^{\nu(1,K_1+1)-1} \frac{1}{4q} [f(z-1) + f(z+1) + 2f(z)] = \\
&= \Big(\frac{1}{2q}\Big)^{\nu(1,K_1+1)} \frac{1}{2} f(z+1) = \\
&= \Big(\frac{1}{2q}\Big)^{\nu(1,K_1+1)} \frac{1}{2} \Big(\frac{1}{2}\Big)^{K_1} \Big(\frac{1}{2q}\Big)^{N+\sum_{k=1}^{K_1} \nu(1,k)} f(x) = \\
&= \Big(\frac{1}{2}\Big)^{K_1+1} \Big(\frac{1}{2q}\Big)^{N+\sum_{k=1}^{K_1+1} \nu(1,k)} f(x).
\end{aligned}$$

Hence (17) holds for every $K_1 \in \{1, \ldots, n\}$, for every non-negative integer N, for every $\nu : \{1\} \times \{1, \ldots, n\} \to \mathbb{N}$, and for all $x \in [0, 1-Q)$ (for all $x \in [0, 1-Q]$ if $q \neq \frac{1}{4}$). Consequently, taking into account Remark 2 we have also

$$\begin{aligned}
&f(q^{N+\sum_{k=1}^{K_1} \nu(1,k)} x + \sum_{k=1}^{K_1} q^{\sum_{m=k}^{K_1} \nu(1,m)}) = \\
&= \Big(\frac{1}{2}\Big)^{K_1} \Big(\frac{1}{2q}\Big)^{N+\sum_{k=1}^{K_1} \nu(1,k)} f(x)
\end{aligned} \tag{19}$$

for every $K_1 \in \{1, \ldots, n\}$, for every non-negative integer N, $\nu : \{1\} \times \times \{1, \ldots, n\} \to \mathbb{N}$, and for all $x \in (Q-1, 0]$ (for all $x \in [Q-1, 0]$ if $q \neq \frac{1}{4}$).

Fix now a positive integer L and suppose that (16) holds with $M = 0$ for every $K_1, \dots, K_L \in \{1, \dots, n\}$, for every non-negative integer N, $\nu : \{1, \dots, L\} \times \{1, \dots, n\} \to \mathbb{N}$, and for all $x \in [0, 1 - Q)$ (for all $x \in [0, 1 - Q]$ if $q \neq \frac{1}{4}$). Defining y as in Lemma 2 and making use of Lemma 2, (17) and (19) with x replaced by y, and (16) with $M = 0$ we obtain

$$\begin{aligned}
&f(q^{N+\sum_{l=1}^{L+1}\sum_{k=1}^{K_l}\nu(l,k)}x+\\
&\quad+\sum_{l=1}^{L+1}(-1)^l\sum_{k=1}^{K_l}q^{\sum_{m=k}^{K_l}\nu(l,m)+\sum_{j=l+1}^{L+1}\sum_{m=1}^{K_j}\nu(j,m)}) =\\
&= f(q^{\sum_{k=1}^{K_{L+1}}\nu(L+1,k)}y + (-1)^{L+1}\sum_{k=1}^{K_{L+1}} q^{\sum_{m=k}^{K_{L+1}}\nu(L+1,m)}) =\\
&= \left(\frac{1}{2}\right)^{K_{L+1}}\left(\frac{1}{2q}\right)^{\sum_{k=1}^{K_{L+1}}\nu(L+1,k)} f(y) =\\
&= \left(\frac{1}{2}\right)^{K_{L+1}}\left(\frac{1}{2q}\right)^{\sum_{k=1}^{K_{L+1}}\nu(L+1,k)}\left(\frac{1}{2}\right)^{\sum_{l=1}^{L}K_l}\cdot\\
&\qquad\cdot\left(\frac{1}{2q}\right)^{N+\sum_{l=1}^{L}\sum_{k=1}^{K_l}\nu(l,k)} f(x) =\\
&= \left(\frac{1}{2}\right)^{\sum_{l=1}^{L+1}K_l}\left(\frac{1}{2q}\right)^{N+\sum_{l=1}^{L+1}\sum_{k=1}^{K_l}\nu(l,k)} f(x).
\end{aligned}$$

This ends the proof of (16) in the case where $M = 0$.

If M is a positive integer then defining once more y as in Lemma 2 and making use of this lemma, (8) with $N = 0$ and x replaced by y, and (16) with $M = 0$ we get

$$\begin{aligned}
&f(q^{N+\sum_{l=1}^{L}\sum_{k=1}^{K_l}\nu(l,k)+M}x+\\
&+\sum_{l=1}^{L}(-1)^l\sum_{k=1}^{K_l}q^{\sum_{m=k}^{K_l}\nu(l,m)+\sum_{j=l+1}^{L}\sum_{m=1}^{K_j}\nu(j,m)+M} + (-1)^L\sum_{m=1}^{M}q^m) =\\
&= f\Big(q^M y + (-1)^L\sum_{m=1}^{M}q^m\Big) = \left(\frac{1}{2}\right)^M\left(\frac{1}{2q}\right)^M f(y) =\\
&= \left(\frac{1}{2}\right)^{\sum_{l=1}^{L}K_l+M}\left(\frac{1}{2q}\right)^{N+\sum_{l=1}^{L}\sum_{k=1}^{K_l}\nu(l,k)+M} f(x). \quad \diamond
\end{aligned}$$

The fourth lemma is just [7; Lemma 1].

Lemma 4. *Assume* $q \in (0, \frac{1}{2})$. *If a solution of Schilling's problem vanishes either on the interval* $(-q, 0)$ *or on the interval* $(0, q)$ *then it vanishes everywhere.*

Proof of Theorem 1. Suppose f is a solution of Schilling's problem bounded in a neighbourhood of a point $x_0 \in \operatorname{cl} A_q^n$. We may (and we do) assume that x_0 is of the form (5), where $\varepsilon \in \{-1, 1\}$, M, L are non-negative integers, $K_1, \ldots, K_L \in \{1, \ldots, n\}$, and $\nu : \{1, \ldots, L\} \times \times \{1, \ldots, n\} \to \mathbb{N}$. Moreover, according to Remark 2, we may (and we do) assume $\varepsilon = 1$.

If $x \in [0, 1-Q)$ is fixed then the left-hand side of (16) is bounded with respect to N whereas the right-hand side is bounded iff $f(x) = 0$. This shows that f vanishes on $[0, 1-Q)$. Hence and from (10) it follows that f vanishes, in particular, on $[0, q)$ which jointly with Lemma 4 proves that f vanishes everywhere. ◊

To formulate a corollary accept the following definition.

Definition 1. Let $q \in (0, 1)$ and $x \in [-Q, Q]$. We say that $x \in B_q$ (resp. $x \in C_q$) if and only if the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of x (resp. continuous at x).

We will use also the following result of W. Förg–Rob; cf. [6; Theorems 20, 21, 23–26 and 28] and Remark 1.

If $q \in (0, 1)$ *and* f *is a solution of Schilling's problem then*

$$\operatorname{supp} f \subset \Big\{ \sum_{n=1}^{\infty} \varepsilon(n) q^n \quad : \quad \varepsilon \in \{-1, 0, 1\}^{\mathbb{N}} \Big\},$$

and for every $q \in (0, \frac{1}{3}]$ *the Schilling's problem has a nonzero solution.*

Corollary 1. *If* $q \in (0, \frac{1}{3}]$ *then*

$$B_q = C_q = \Big\{ \sum_{n=1}^{\infty} \varepsilon(n) q^n \quad : \quad \varepsilon \in \{-1, 0, 1\}^{\mathbb{N}} \Big\}.$$

Proof. Obviously $B_q \subset C_q$, whereas the above quoted result of W. Förg–Rob gives

$$C_q \subset \Big\{ \sum_{n=1}^{\infty} \varepsilon(n) q^n \quad : \quad \varepsilon \in \{-1, 0, 1\}^{\mathbb{N}} \Big\}.$$

Moreover, applying Remark 1 and Th. 1 we obtain that

$$\left\{\sum_{n=1}^{\infty}\varepsilon(n)q^n \quad : \quad \varepsilon \in \{-1,0,1\}^{\mathbb{N}}\right\} \subset B_q. \quad \diamondsuit$$

Applying Lemma 3 (formula (16) with $x = 0$ and Remark 2) and Remark 3 we obtain also the following result.

Theorem 2. *If n is a positive integer and $q \in (0, q_n]$ then any solution of Schilling's problem vanishes on the set A_q^n.*

The reader interested in further results on Schilling's problem is referred to [2] by K. Baron, A. Simon and P. Volkmann, [3] by K. Baron and P. Volkmann, [4] by J. M. Borwein and R. Girgensohn, [5] by G. Derfel and R. Schilling, [6] by W. Förg–Rob and [8].

Acknowledgement. This research was supported by the Silesian University Mathematics Department (Iterative Functional Equations program).

References

[1] BARON, K.: On a problem of R. Schilling, *Berichte der Mathematisch-statistischen Sektion in der Forschungsgesellschaft Joanneum–Graz, Bericht Nr.* **286** (1988).

[2] BARON, K., SIMON, A. et VOLKMANN, P.: Solutions d'une équation fonctionnelle dans l'espace des distributions tempérées, *Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématiques. (Paris)* **319** (1994), 1249–1252.

[3] BARON, K. et VOLKMANN, P.: Unicité pour une équation fonctionnelle, *Rocznik Naukowo–Dydaktyczny WSP w Krakowie. Prace Matematyczne* **13** (1993), 53–56.

[4] BORWEIN, J. M. and GIRGENSOHN, R.: Functional equations and distribution functions, *Results in Mathematics* **26** (1994), 229–237.

[5] DERFEL, G. and SCHILLING, R.: Spatially chaotic configurations and functional equations with rescaling, *Manuscript.*

[6] FÖRG-ROB, W.: On a problem of R. Schilling, *Mathematica Pannonica* **5**/1 (1994), I 29–65, II: 145–168.

[7] MORAWIEC, J.: On bounded solutions of a problem of R. Schilling, *Annales Mathematicae Silesianae* **8** (1994), 97–101.

[8] MORAWIEC, J.: On continuous solutions of a problem of R. Schilling, *Results in Mathematics* **27** (1995), 381-386.

[9] SCHILLING, R.: Spatially chaotic structures, *in*: H. Thomas (editor), Nonlinear Dynamics in Solids, Springer, Berlin, 1992, 213-241.