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SHAPE GROUPS FOR C*-ALGEBRAS

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Abstract: We shall describe in this paper shape groups for C^*-algebras and prove some of their basic properties.

1. Introduction

In the paper [5] the author has defined homotopy groups for C^*-algebras. More precisely, we have described how to associate with every pair (A, B) of C^*-algebras and every integer $n \geq 0$ pointed sets $\pi_n(A;\, B)$ for $n = 0$ and groups $\pi_n(A;\, B)$ for $n \geq 1$ which are commutative for $n \geq 2$, which depend only on homotopy types of A and B, and which have other properties similar to the properties of absolute homotopy groups of spaces. We called $\pi_n(A;\, B)$ for $n \geq 1$ the n-th (absolute) homotopy group of B over A. The pointed set $\pi_0(A;\, B)$ is the pointed set of all homotopy classes of $*$-homomorphisms from A into B (see [5]). There is a corresponding relative theory in which we associate to C^*-algebras A and B and a C^*-subalgebra B' of B pointed sets $\pi_n(A;\, (B,\, B'))$ for $n = 1$ and groups $\pi_n(A;\, (B,\, B'))$ for $n \geq 2$ which are commutative for $n \geq 3$, which depend only on homotopy types of A and (B, B'), and which have other properties similar to properties of the relative homotopy groups of spaces. We called

$\pi_n(A;\,(B,\,B'))$ for $n \geq 2$ the n-th (relative) homotopy group of B over A modulo B'.

On the other hand, in the paper [6], we have described shape theory for arbitrary C^*-algebras following the original Borsuk's method based on the notion of a fundamental sequence in [2] and [3]. More precisely, we constructed the $*$-shape category with objects C^*-algebras and with morphisms $*$-homotopy classes of fundamental $*$-sequences. Our prime objective was to improve homotopy theory of C^*-algebras by relaxing the requirement that one must use $*$-homomorphisms. Instead, we have utilised so called $*^{\varepsilon}$-homomorphisms, i.e., nonexpansive functions which satisfy conditions for $*$-homomorphisms only approximately. An analogous approach to strong shape theory of separable C^*-algebras based on the notion of an asymptotic homomorphism was earlier considered by A. Connes and N. Higson [7].

The goal in this paper is to study shape groups of C^*-algebras that correspond to Borsuk's shape groups of spaces. The main results are that these groups which we call $*$-shape groups are $*$-shape invariants and that they can be put into a long weakly exact sequence. For the above homotopy groups of C^*-algebras or $*$-homotopy groups this sequence is exact and the whole theory resembles even more results on homotopy groups of spaces. However, in both homotopy theory and it's generalisation shape theory in the case of C^*-algebras there are no problems with base points. Moreover, we get bifunctors instead of functors which is not surprising bearing in mind the KK-theory [12].

The organisation of this paper is briefly as follows. The §2 explains our notation and recalls some standard conventions in our exposition. In §3 we recall the definition of the $*$-shape category from [6]. This requires to define first $*^{\varepsilon}$-homomorphisms and the relation of $*^{\varepsilon}$-homotopy for them. Next we introduce fundamental $*$-sequences and the notion of the $*$-homotopy for them. The most demanding is the description of the composition of $*$-homotopy classes of fundamental $*$-sequences.

After these preliminary sections in §§4–8 we develop basic definitions and results of shape groups for C^*-algebras. The §4 deals with absolute groups while §5 is concerned with relative groups. The boundary operators and induced transformations are considered in the §§6 and 7, while the final §8 establishes weak exactness of the long shape groups sequence.

2. Preliminaries and notation

In this paper by a C^*-*algebra* we mean a complete normed algebra A over the field $\mathbb{C}$ of complex numbers with an involution $*$ such that

(1) $x^{**} = x$,
(2) $(\lambda x + \mu y)^* = \overline{\lambda} x^* + \overline{\mu} y^*$,
(3) $(xy)^* = y^* x^*$,
(4) $(||x_A)^2|| = ||x^* x_A||$,

for all $x, y \in A$ and all $\lambda, \mu \in \mathbb{C}$, where $\overline{\lambda}$ is complex conjugate of λ and $|| \, ||_A$ denotes the norm on A. Any algebraic $*$-homomorphism (i.e., respecting the involution) between two C^*-algebras is norm-decreasing thus uniformly continuous and every $*$-isomorphism between two C^*-algebras is isometric. When speaking of homomorphisms between C^*-algebras we shall always assume that they are $*$-homomorphisms. We recommend the books [4], [8], [10], and [11], as general references for the theory of C^*-algebras.

The symbol 0_A denotes the zero element of the C^*-algebra A.

For a C^*-algebra B and a compact topological space X, let $C(X; B)$ denote the C^*-algebra of all continuous functions from X into B. The norm $|| \, ||_{C(X;B)}$ on $C(X; B)$ is given by

$$||f||_{C(X;B)} = sup\{||f(t)||_B \mid t \in X\}.$$

Let I denote the unit closed segment $[0, 1]$ of real numbers. For every t in I, there is a natural evaluation $*$-homomorphism $e_t^B : C(I; B) \to$ $\to B$ defined by $e_t^B(f) = f(t)$ for every f in $C(I; B)$.

Our shape theory is an improvement of the $*$-homotopy theory for C^*-algebras which studies the equivalence relation of $*$-homotopy on $*$-homomorphisms. Recall that $*$-homomorphisms f and g between C^*-algebras A and B are $*$-*homotopic* and we write $f \simeq_* g$ provided there is a $*$-homomorphism $h : A \to C(I; B)$ such that $h_0 = e_0^B \circ$ $\circ\, h = f$ and $h_1 = e_1^B \circ h = g$. The $*$-homomorphism h is said to be a $*$-*homotopy* that joins f and g or which realises the relation $f \simeq_* g$. For an efficient introduction to some aspects of $*$-homotopy theory the reader should consult P. Kohn's thesis [9], J. Rosenberg's excellent expository article [12], and the author's paper [5].

3. Description of the $*$-shape category

This section includes an efficient description of the $*$-shape category from [6]. We recall the basic definitions and constructions necessary for our main results in §§4–8.

We begin with the definition of a $*^{\varepsilon}$-homomorphism that resembles asymptotic homomorphisms from [7]. Let A and B be C^*-algebras. Let ε be a positive real number. A function $f : A \to B$ is a $*^{\varepsilon}$-*homomorphism* provided

(1) f takes the zero element 0_A of A into the zero element 0_B of B,
(2) f is nonexpansive, i.e., the relation $||f(x) - f(y)||_B \leq ||x - y||_A$ holds for all x, $y \in A$, and
(3) $||f(x + y) - f(x) - f(y)||_B < \varepsilon(||x||_A + ||y||_A)$ for all x, $y \in A$.
(4) $||f(bx) - bf(x)||_B < \varepsilon||x||_A$ for each $x \in A$ and each $b \in \mathbb{C}$.
(5) $||f(xy) - f(x)f(y)||_B < \varepsilon||x||_A\,||y||_A$ for all x, $y \in A$.
(6) $||f(x^*) - f(x)^*||_B < \varepsilon||x||_A$ for each $x \in A$.

Observe that a $*^{\varepsilon}$-homomorphism is a uniformly continuous function. Moreover, for every real number ε between 0 and 1 and every C^*-algebra A the function f from A into itself which takes an $x \in A$ into the product of ε and x is an example of a $*^{\varepsilon}$-homomorphism which is not a $*$-homomorphism.

Another basic notion is that of the $*^{\varepsilon}$-*homotopy* for nonexpansive functions of C^*-algebras. Let $\varepsilon > 0$. Nonexpansive functions f and g between C^*-algebras A and B are $*^{\varepsilon}$-*homotopic* and we write $f \overset{\varepsilon}{\simeq}_* g$ provided there is an $*^{\varepsilon}$-homomorphism $h : A \to C(I;\, B)$ with $h_0 = e_0^B \circ$ $\circ\, h = f$ and $h_1 = e_1^B \circ = g$. We shall also say that h is a $*^{\varepsilon}$-homotopy which *joins* f and g or that it *realises* the relation or $*^{\varepsilon}$-homotopy $f \overset{\varepsilon}{\simeq}_* g$.

We can now introduce fundamental $*$-sequences and define the relation of $*$-homotopy for them. These definitions correspond to Borsuk's definitions in [2] and [3] of a fundamental sequence and a homotopy for fundamental sequences. Let A and B be C^*-algebras. A family $\varphi =$ $= \{f_i\}_{i=1}^{\infty}$ of nonexpansive functions $f_i : A \to B$ is a *fundamental $*$-sequence* from A into B provided for every $\varepsilon > 0$ there is an $i \in \mathbb{N}$ such that $f_j \overset{\varepsilon}{\simeq}_* f_i$ for every $j \geq i$.

We use functional notation $\varphi : A \to B$ to indicate that φ is a fundamental $*$-sequence from A into B. Let $F_*(A,\, B)$ denote all fundamental $*$-sequences $\varphi : X \to Y$. Two families $\varphi = \{f_i\}_{i=1}^{\infty}$ and $\psi = \{g_i\}_{i=1}^{\infty}$

of nonexpansive functions $f_i,\ g_i : A \to B$ are $*$-*homotopic* and we write $\varphi \simeq_* \psi$ provided for every $\varepsilon > 0$ there is an $i \in \mathbb{N}$ such that $f_j \overset{\varepsilon}{\simeq}_* g_j$ for every $j \geq i$.

The relation of $*$-homotopy is an equivalence relation on the set $F_*(A,\ B)$. The $*$-homotopy class of a fundamental $*$-sequence φ is denoted by $[\varphi]_*$ and the set of all $*$-homotopy classes by $\mathcal{S}h_*(A,\ B)$.

In order to organise C^*-algebras and $*$-homotopy classes of fundamental $*$-sequences into a $*$-shape category $\mathcal{S}h_*$, we must define a composition for $*$-homotopy classes of fundamental $*$-sequences and establish it's associativity. The definition of the composition is the only tricky part in setting up the category $\mathcal{S}h_*$. Our idea is to associate to every fundamental $*$-sequence $\varphi : A \to B$ two increasing functions $\varphi : \mathbb{N} \to \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$. The first function associates to an index $i \in \mathbb{N}$ of the sequence $\varphi = \{f_i\}$ a much larger index $\varphi(i)$ in $\mathbb{N}$ such that f_j and f_k are joined by a $*^{1/i}$-homotopy whenever $j,\ k \geq \varphi(i)$. The second function associates to an $i \in \mathbb{N}$ an element $f(i)$ of $\mathbb{N}$ such that the reciprocal value $1/f(i)$ of $f(i)$ is sufficiently small. This is a rough description of these functions and now we proceed with the details.

Let us agree that an *increasing* function $f : P \to P$ of a partially ordered set $(P, <)$ into itself is a function which satisfies $x < f(x)$ for every $x \in P$ and $x < y$ in P implies $f(x) < f(y)$. In the case when the domain and the codomain of a function f are different, the first requirement is dropped. When defining increasing functions that connect indexing sets we shall repeatedly use the following simple lemma.

Lemma 3.1. *Let $\{f_1, \ldots, f_n\}$ be functions from a cofinite directed set $(M,\ <)$ into a directed set $(L,\ <)$. Then there is an increasing function $g : M \to L$ such that $g(x) \geq f_1(x), \ldots, f_n(x)$ for every $x \in M$.*

For a positive real number ε and a natural number n, let

$$\varepsilon^{\langle n \rangle} = \{i \in \mathbb{N} \mid i > n/\varepsilon\}.$$

The sets $\varepsilon^{\langle 1 \rangle}$ and $\varepsilon^{\langle 2 \rangle}$ are denoted by ε^* and ε^{**}, respectively.

Let $\varphi = \{f_i\} : A \to B$ be a fundamental $*$-sequence between C^*-algebras. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be an increasing function such that for every $i \in \mathbb{N}$ the relation $j,\ k \geq \varphi(i)$ implies the relation $f_j \overset{1/i}{\simeq}_* f_k$. The multiple use of notation here can not lead to confusion provided one keeps in mind that fundamental $*$-sequences act only between C^*-algebras and that they can not be evaluated in an index (which is a natural number).

Let $\mathcal{L}_\varphi\{(i,\, j,\, k)|\; i,\, j,\, k \in \mathbb{N},\; j,\, k \geq \varphi(i)\}$. Then $\mathcal{L}_\varphi$ is a subset of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ that becomes a cofinite directed set when we define that $(i,\, j,\, k) \geq (m,\, n,\, p)$ if and only if $i \geq m$, $j \geq n$, and $k \geq p$.

We shall use the same notation φ for an increasing function $\varphi : \mathcal{L}_\varphi \to \mathbb{N}$ such that $\varphi(i, j, k) \geq \varphi(i)$ whenever $(i,\, j,\, k) \in \mathcal{L}_\varphi$.

It was observed [6, Claim (5.1)] that there is an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that

(1) $f(i) \geq \varphi(i,\, \varphi(i),\, \varphi(i))$ for every $i \in \mathbb{N}$, and
(2) f is cofinal in φ, i.e., for every $(i,\, j,\, k) \in \mathcal{L}_\varphi$ there is an $m \in \mathbb{N}$ with $f(m) \geq \varphi(i,\, j,\, k)$.

The above discussion shows that every fundamental $*$-sequence $\varphi : A \to B$ determines two increasing functions $\varphi : \mathbb{N} \to \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$. With the help of these functions we shall define the *composition* of $*$-homotopy classes of fundamental $*$-sequences as follows. Let $\varphi = \{f_i\}_{i \in \mathbb{N}} : A \to B$ and $\psi = \{g_i\}_{i \in \mathbb{N}} : B \to C$ be fundamental $*$-sequences. Let $\psi \circ \varphi$ denote the collection $\chi = \{h_i\}_{i \in \mathbb{N}}$, where we define $h_i = g_{\psi(i)} \circ f_{\varphi(g(i))}$ for every $i \in \mathbb{N}$. Observe that each h_i is a nonexpansive function because it is the composition of two nonexpansive functions. Since the collection $\psi \circ \varphi$ is a fundamental $*$-sequence from A into C, we can now define an *associative composition* of $*$-homotopy classes of fundamental $*$-sequences by the rule $[\psi]_* \circ [\varphi]_* = [\psi \circ \varphi]_*$ (see [6, Claims (5.2), (5.3) and (5.4)]).

Finally, it remains to observe that for every fundamental $*$-sequence $\varphi : A \to B$, the following relations hold.

$$[\varphi]_* \circ [\iota^A]_* = [\varphi]_* = [\iota^B]_* \circ [\varphi]_*,$$

where for a C^*-algebra A, we let $\iota^A = \{I_i\} : A \to A$ be the identity fundamental $*$-sequence defined by $I_i = \mathrm{id}_A$ for every $i \in \mathbb{N}$.

We can summarise the above with the following main theorem from [6].

Theorem 3.2. *The C^*-algebras as objects together with the $*$-homotopy classes of fundamental $*$-sequences as morphisms, the $*$-homotopy classes $[\iota^A]_*$ as identity morphisms, and the above composition of $*$-homotopy classes form the $*$-shape category $\mathcal{S}h_*$.*

4. Absolute $*$-shape groups

In this section we shall introduce absolute $*$-shape groups which correspond in $*$-shape theory to absolute $*$-homotopy groups considered in [5].

Let A and B be C^*-algebras. Let $n \geq 0$ be an integer. Let $FS^n = FS^n(A;\, B)$ denote the set of all fundamental $*$-sequences from A into the C^*-algebra $C_\partial(I^n;\, B)$ of all continuous functions from the n-dimensional cube I^n into B which map the boundary ∂I^n of I^n into the zero element 0_B of the algebra B. These fundamental $*$-sequences are divided into $*$-homotopy classes. We shall denote by $\varrho_n(A;\, B)$ the totality of these $*$-homotopy classes. We shall also denote by $[\varphi]_*$ the $*$-homotopy class which contains the fundamental $*$-sequence φ and by 0 the $*$-homotopy class which contains the trivial fundamental $*$-sequence $\zeta : A \to C_\partial(I^n;\, B)$.

We may define an *addition* (usually non-commutative) in FS^n as follows. For any two fundamental $*$-sequences φ and ψ in FS^n, their *sum* $\varphi + \psi$ is the fundamental $*$-sequence $\chi = \{h_i\}_{i=1}^{\infty}$ from A into $C_\partial(I^n;\, B)$ defined by

$$h_i(a)(t) = \begin{cases} f_i(a)(2t_1,\, t_2, \ldots, t_n), & 0 \leq t_1 \leq \frac{1}{2}, \\ g_i(a)(2t_1 - 1,\, t_2, \ldots,\, t_n), & \frac{1}{2} \leq t_1 \leq 1, \end{cases}$$

for every $a \in A$, every point $t = (t_1,\, \ldots,\, t_n)$ in I^n, and every $i \in \mathbb{N}$.

Our first claim shows that χ is indeed a fundamental $*$-sequence.

Claim 4.1. *The collection χ is a fundamental $*$-sequence from A into $C_\partial(I^n;\, B)$.*

Proof. We must see (1) that each function h_i is nonexpansive and (2) that for every $\varepsilon > 0$ there is an $i \in \mathbb{N}$ such that $h_j \overset{\varepsilon}{\simeq}_* h_i$ for every $j \geq i$.

Add (1). Since functions f_i and g_i are nonexpansive, it is easy to see that the function h_i will have the same property.

Add (2). Since φ and ψ are fundamental $*$-sequences, there is an $i \in \mathbb{N}$ such that $f_j \overset{\varepsilon}{\simeq}_* f_i$ and $g_j \overset{\varepsilon}{\simeq}_* g_i$ for every $j \geq i$. Let $m^{ji}\, n^{ji} : A \to C(I;\, C_\partial(I^n;\, B))$ and be $*^\varepsilon$-homotopies which realize these relations.

Define $k^{ji} : A \to C(I;\, C_\partial(I^n;\, B))$ by the rule

$$k^{ji}(a)(s)(t) = \begin{cases} m^{ji}(a)(s)(2t_1,\, t_2, \ldots, t_n), & 0 \leq t_1 \leq \frac{1}{2}, \\ n^{ji}(a)(s)(2t_1 - 1,\, t_2, \ldots, t_n), & \frac{1}{2} \leq t_1 \leq 1, \end{cases}$$

for every $a \in A$, every $s \in I$, and every point $t = (t_1,\, \ldots,\, t_n)$ in I^n. Then k^{ji} is a $*^\varepsilon$-homotopy joining h_j and h_i. $\diamondsuit$

In the following claim we shall prove that the operation of sum can be consistently introduces for $*$-homotopy classes of members of FS^n.

Claim 4.2. *Let φ, ψ, χ, $\kappa \in FS^n$. If $\varphi \simeq_* \chi$ and $\psi \simeq_* \kappa$, then $\varphi + \psi \simeq_* \chi + \kappa$.*

Proof. Let $\pi = \varphi + \psi$ and $\varrho = \chi + \kappa$, where $\pi = \{p_i\}$ and $\varrho = \{r_i\}$. In order to show that π and ϱ are $*$-homotopic, we must see that for every $\varepsilon > 0$ there is an $i \in \mathbb{N}$ with $p_j \overset{\varepsilon}{\simeq}_* r_j$ for every $j \geq i$. Let an $\varepsilon \geq 0$ be given. Since $\varphi \simeq_* \chi$ and $\psi \simeq_* \kappa$, there is an $i \in \mathbb{N}$ such that $f_j \overset{\varepsilon}{\simeq}_* h_j$ and $g_j \overset{\varepsilon}{\simeq}_* k_j$ for every $j \geq i$. Let these last two $*^\varepsilon$-homotopies be realized by $*^\varepsilon$-homomorphisms u^j and v^j from A into $C(I;\, C_\partial(I^n;\, B))$, respectively. Define a function $m^j : A \to C(I;\, C_\partial(I^n;\, B))$ by

$$m^j(a)(s)(t) = \begin{cases} u^j(a)(s)(2t_1, t_2, \ldots, t^n), & 0 \leq t_1 \leq \frac{1}{2}, \\ v^j(a)(s)(2t_1 - 1, t_2, \ldots, t_n), & \frac{1}{2} \leq t_1 \leq 1, \end{cases}$$

for every $a \in A$, every $s \in I$, and every point $t = (t_1, \ldots, t_n)$ in I^n. Then m^i is a $*^\varepsilon$-homotopy joining p_j and r_j. $\Diamond$

Now we can define a binary operation of addition on the set $\varrho_n(A;\, B)$ by the rule $[\varphi]_* + [\psi]_* = [\varphi + \psi]_*$.

Claim 4.3. *The operation of addition of $*$-homotopy classes of fundamental $*$-sequences is associative.*

Proof. Let $\varphi,\, \psi,\, \chi \in FS^n$. Let $\pi = \varphi + \psi$, $\varrho = \pi + \chi$, $\alpha = \psi + \chi$, and $\beta = \varphi + \sigma$. We must prove that ϱ and β are $*$-homotopic fundamental $*$-sequences. In other words, that for every $\varepsilon \geq 0$ there is an $i \in \mathbb{N}$ with $r_j \overset{\varepsilon}{\simeq}_* b_j$ for every $j \geq i$. Let an $\varepsilon > 0$ be given. Since φ, ψ, and χ are fundamental $*$-sequences, there is an $i \in \mathbb{N}$ such that functions f_j, g_j, and h_j are $*^\varepsilon$-homomorphisms for every $j \geq i$. Let $j \geq i$. Define the function $d^j : A \to C(I;\, C_\partial(I^n;\, B))$ by the rule

$$d^j(a)(s)(t) = \begin{cases} f_j(a)\left(\frac{4t_1}{s+1},\, t_2, \ldots, t_n\right), & 0 \leq t_1 \leq \frac{s+1}{4}, \\ g_j(a)(4t_1 - s - 1,\, t_2, \ldots, t_n), & \frac{s+1}{4} \leq t_1 \leq \frac{s+2}{4}, \\ h_j(a)\left(\frac{4t_1 - 2 - s}{2 - s},\, t_2, \ldots, t_n\right), & \frac{s+2}{4} \leq t_1 \leq 1, \end{cases}$$

for every $a \in A$, every $s \in I$, and every point $t = (t_1, \ldots, t_n)$ in I^n. Then d^j is a $*^\varepsilon$-homotopy joining r_j and b_j. $\Diamond$

Claim 4.4. *The $*$-homotopy class $[\zeta]$ of the trivial fundamental $*$-sequence ζ from A into $C_\partial(I^n;\, B)$ is an identity for the addition operation.*

Proof. Let $\varphi \in FS^n$. We must show that $[\varphi]_* + [\zeta]_* = [\varphi]_* = [\zeta]_* + [\varphi]_*$. Let $\psi = \varphi + \zeta$. The left hand side equality of the above extended equality will follow provided we show that the fundamental $*$-sequences ψ and φ are $*$-homotopic. Let $\varepsilon > 0$. Since φ is a fundamental $*$-sequence, there is an $i \in \mathbb{N}$ such that f_j is a $*^\varepsilon$-homomorphism for every $j \geq i$. For such an index j, define a function $h^j : A \to C(I;\, C_\partial(I^n;\, B))$ by the rule

$$h^j(a)(s)(t) = \begin{cases} f_j(a)(\frac{2t_1}{s+1},\, t_2, \ldots, t_n), & 0 \le t_1 \le \frac{s+1}{2}, \\ z_j(a)(t), & \frac{s+1}{2} \le t_1 \le 1, \end{cases}$$

for every $a \in A$, every $s \in I$, and every point $t = (t_1, \ldots, t_n)$ in I^n, where $\zeta = \{z_i\}$ and z_i is the trivial $*$-homomorphism from A into $C_\partial(I^n;\, B)$. Then h^j is a $*^\varepsilon$-homotopy joining g_j and f_j. The argument for the other equality is similar. ♢

Claim 4.5. *With respect to the addition every element has an inverse, i.e., for every $\varphi = FS^n$ there is a $\psi = FS^n$ such that $[\varphi]_* + [\psi]_* = [\zeta]_*$ $* = [\psi]_* + [\varphi]_*$.*

Proof. For every $i \in \mathbb{N}$, define a function $g_i : A \to C_\partial(I^n;\, B)$ by the rule

$$g_i(a)(t) = f_i(a)(1 - t_1,\, t_2, \ldots,\, t_n),$$

for every $a \in A$ and every point $t = (t_1, \ldots, t_n)$ in I^n. The collection $\psi = \{g_i\}$ is a fundamental $*$-sequence from A into $C_\partial(I^n;\, B)$. Let $\chi =$ $= \varphi + \psi$. We must show that the fundamental $*$-sequences χ and ζ are $*$-homotopic. Let $\varepsilon > 0$. Since φ is a fundamental $*$-sequence, there is an $i \in \mathbb{N}$ such that f_j is a $*^\varepsilon$-homomorphism for every $j \ge i$. For all such indices j the function

$$k^j(a)(s)(t) = \begin{cases} z_j(a)(t), & 0 \le t_1 \le \frac{s}{2}, \\ f_j(a)(2t_1 - s,\, t_2, \ldots, t_n), & \frac{s}{2} \le t_1 \le \frac{1}{2}, \\ f_j(a)(2 - 2t_1 - s,\, t_2, \ldots, t_n), & \frac{1}{2} \le t_1 \le \frac{1-s}{2}, \\ z_j(a)(t), & \frac{1-s}{2} \le t_1 \le 1 \end{cases}$$

is a $*^\varepsilon$-homotopy joining h_j and z_j. The argument for the other equality is analogous. ♢

The group $\varrho_n(A;\, B)$ will be called the n-th $*$-*shape group* of $(A,\, B)$ or the n-th (*absolute*) $*$-*shape group* of B *over* A. For $n = 1$, we call $\varrho_1(A;\, B)$ the $*$-*fundamental group* of the pair $(A,\, B)$. Just as in the ordinary homotopy theory, for $n > 1$, these groups are abelian.

Theorem 4.6. *For $n > 1$, the group $\varrho_n(A;\, B)$ is commutative.*

Proof. Let $\varphi,\, \psi \in FS^n$. We must show that $\pi \simeq_* \varrho$, where $\pi = \varphi +$ $+ \psi$ and $\varrho = \psi + \varphi$. For a given $\varepsilon > 0$, select an $i \in \mathbb{N}$ such that f_j and g_j are $*^\varepsilon$-homomorphisms for every $f \ge i$. Let $j \ge i$. Define $*^\varepsilon$-homomorphisms h^j, k^j, $m^j A \to C(I;\, C_\partial(I^n;\, B))$ by the rules

$$h^j(a)(s)(t) =$$

$$= \begin{cases} f_j(a)(2t_1,\, (1+s)t_2,\, t_3, \ldots, t_n), & 0 \le t_1 \le \frac{1}{2}, \quad 0 \le t_2 \le \frac{1}{1+s}, \\ g_j(a)(2t_1 - 1,\, (1+s)t_2 - t,\, t_3, \ldots, t_n), & \frac{1}{2} \le t_1 \le 1, \quad \frac{s}{1+s} \le t_2 \le 1, \\ z_j(a)(t), & \text{otherwise}, \end{cases}$$

$$k^j(a)(s)(t) = \\ = \begin{cases} f_j(a)(2t_1,\, 2t_2,\, t_3, \ldots, t_n), & \frac{s}{2} \le t_1 \le \frac{s+1}{2}, \quad 0 \le t_2 \le \frac{1}{2}, \\ g_j(a)(2t_1 + s - 1,\, 2t_2 - 1,\, t_3, \ldots,\, t_n), & \frac{1-s}{2} \le t_1 \le \frac{2-s}{2}, \quad \frac{1}{2} \le t_2 \le 1, \\ z_j(a)(t), & \text{otherwise}, \end{cases}$$

$$m^j(a)(s)(t) = \\ = \begin{cases} f_j(a)(2t_1 - 1,\, (2-s)t_2,\, t_3, \ldots, t_n), & \frac{1}{2} \le t_1 \le 1, \quad 0 \le t_2 \le \frac{1}{2-s}, \\ g_j(a)(2t_1,\, (2-s)t_2 - t,\, t_3, \ldots,\, t_n), & 0 \le t_1 \le \frac{1}{2}, \quad \frac{1-s}{2-s} \le t_2 \le 1, \\ z_j(a)(t), & \text{otherwise}, \end{cases}$$

for every $a \in A$, every $s \in I$, and every point $t = (t_1, \ldots, t_n)$ in I^n. Observe that $h_0^j = p_j$, $h_1^j = k_0^j$, $k_1^j = m_0^j$, and $m_1^j = q_j$. Hence, π and ϱ are $*$-homotopic. $\Diamond$

If the boundary ∂I^n of I^n is identified to a point, we get a quotient space which is topologically equivalent to an n-sphere S^n with a given basic point s_0 in S^n. It follows that one might equally well define an element of $\varrho_n(A;\, B)$ as a $*$-homotopy class of a fundamental $*$-sequence of A into the C^*-algebra $C_{s_0}(S^n;\, B)$ of all continuous functions from S^n into B which map the point s_0 into the zero element 0_B of B. Since the two halves of I^n, defined by the conditions $t_1 \le \frac{1}{2}$ and $t_1 \ge \frac{1}{2}$ respectively, correspond to two hemispheres of S^n, it is clear how to define group operation of $\varrho_n(A;\, B)$ from this point of view. Since, when $n > 1$, there exists a rotation of S^n which leaves s_0 fixed and interchanges the two hemispheres, we get an alternative proof of Th. (4.6).

As a consequence of the following result, it follows that every $*$-shape group of a pair $(A,\, B)$ of C^*-algebras can be expressed as the $*$-fundamental group of some other pair.

Theorem 4.7. *Let p be any positive integer less than n and let $q = n - p$. Then the groups $\varrho_n(A;\, B)$ and $\varrho_p(A;\, C_\partial(I^q;\, B))$ are isomorphic.*

Proof. Let $\varphi \in FS^n$. For every $i \in \mathbb{N}$ define a function $g_i : A \to C_\partial(I^p;\, C_\partial(I^q;\, B))$ by the formula

$$g_i(a)(s_1,\, \ldots,\, s_p)(t_1,\, \ldots,\, t_q) = f_i(a)(s_1,\, \ldots,\, s_p,\, t_1,\, \ldots,\, t_q),$$

for every $a \in A$, every point $(s_1, \ldots, s_p)$ in I^p, and every point $(t_1, \ldots, t_q)$ in I^q.

One can easily show that $\psi = \{g_i\}$ is a fundamental $*$-sequence from A into $C_\partial(I^p;\, C_\partial(I^q;\, B))$ and that the function ξ defined by $\xi(\varphi) = \psi$ is a bijection between FS^n and $FS^p(A;\, C_\partial(I^q;\, B))$ which respects addition and $*$-homotopy relation and thus induces the required isomorphism. $\Diamond$

5. Relative $*$-shape groups

The objective of the present section is to generalise the notion of $*$-shape groups by defining the relative $*$-shape groups $\varrho_n(A;\ (B,\ B'))$ for C^*-algebras A and B and a C^*-subalgebra B' of B.

Let $n > 0$ be an integer and define the n-th *relative $*$-shape set* $\varrho_n(A;\ (B,\ B'))$ as follows. Consider again the n-cube I^n. The *initial* $(n-1)$-face of I^n defined by $t_n = 0$ will be denoted by J_n and identified with I^{n-1} hereafter. The union of all remaining $(n-1)$-faces of I^n is denoted by K_n. Then we have

$$\partial I^n = J_n \cup K_n, \quad \text{and} \quad \partial I^{n-1} = J_n \cap K_n.$$

When $n = 1$, we drop 1 from our notation and talk about $I = [0,\ 1]$ and two of its subsets $J = \{0\}$ and $K = \{1\}$. Let $C_{K_n}(I^n,\ J_n;\ B,\ B')$ denote the C^*-algebra of all continuous functions from I^n into B which take points of J_n into B' and all of K_n to the zero element 0_B of B. We denote by $FS^n = FS^n(A;\ (B,\ B'))$ the set of all fundamental $*$-sequences from A into $C_{K_n}(I^n,\ J_n;\ B,\ B')$. These fundamental $*$-sequences are divided into $*$-homotopy classes. We shall denote by $\varrho_n(A;\ (B,\ B'))$ the totality of these $*$-homotopy classes. We shall also denote by $[\varphi]_*$ the $*$-homotopy class which contains the fundamental $*$-sequence φ and by 0 the $*$-homotopy class which contains the trivial fundamental $*$-sequence ζ from A into $C_{K_n}(I^n,\ J_n;\ B,\ B')$.

If $n \geq 2$, we may define an *addition* (possibly non-commutative) in FS^n. For any two fundamental $*$-sequences φ and ψ in FS^n, their sum $\varphi + \psi \in FS^n$ is defined by the formula given in §4 for the absolute case. The $*$-homotopy class $[\varphi + \psi]_*$ depends only on the $*$-homotopy classes $[\varphi]_*$ and $[\psi]_*$ and hence we may define an addition in $\varrho_n(A;\ (B,\ B'))$ by taking $[\varphi]_* + [\psi]_* = [\varphi + \psi]_*$. As in the §4, one can verify that this addition makes $\varrho_n(A;\ (B,\ B'))$ for $n > 1$ into a group which will be called the *n-th relative $*$-shape group of $(A,\ B)$ modulo B'*. The class 0 is the group-theoretic neutral element of $\varrho_n(A;\ (B,\ B'))$, and the inverse element of $[\varphi]_*$ is the $*$-homotopy class $[\psi]_*$, where the i-th function g_i of the fundamental $*$-sequence $\psi : A \to C_{K_n}(I^n,\ J_n;\ B,\ B')$ is defined by

$$g_i(a)(t_1,\ t_2, \ldots,\ t_n) = f_i(a)(1 - t_1,\ t_2, \ldots,\ t_n),$$

for every $a \in A$ and every $(t_1, \ldots,\ t_n) \in I^n$.

If B' is a trivial C^*-subalgebra of B consisting only of the zero element 0_B, then we have

$$FS^n(A;\,(B,\,B')) = FS^n(A;\,B).$$

Hence, in this case, $\varrho_n(A;\,(B,\,B'))$ reduces to the absolute $*$-shape group $\varrho_n(A;\,B)$ defined in §4.

If K_n is pinched to a point s_0, then (I^n, J_n, K_n) becomes a configuration topologically equivalent to the triplet (D^n, S^{n-1}, s_0) consisting of the unit disc D^n, its boundary $(n-1)$-sphere S^{n-1}, and a reference point $s_0 \in S^{n-1}$. It follows that one might equally well define an element of $\varrho_n(A;\,(B,\,B'))$ as a $*$-homotopy class of fundamental $*$-sequences from A into the C^*-algebra $C_{s_0}(D^n,\,S^{n-1};\,B,\,B')$ of all continuous functions from D^n into B which take points of S^{n-1} to B' and s_0 to 0_B. Since, when $n \geq 3$, there exists a rotation of D^n which leaves s_0 fixed and interchanges the two halves of D^n, we see that $\varrho_n(A;\,(B,\,B'))$ is abelian for every $n \geq 3$. This commutativity property can be also proved by a method similar to the proof of Th. (4.7).

Let us call the C^*-algebra $B'' = C_K(I,\,J;\,B,\,B')$ the *derived C^*-algebra* of the pair $(B,\,B')$.

Proposition 5.1. *For every $n > 0$, the groups $\varrho_n(A;\,(B,\,B'))$ and $\varrho_{n-1}(A;\,B'')$ are isomorphic.*

Proof. Let $\varphi : A \to C_{K_n}(I^n,\,J_n;\,B,\,B')$ be a fundamental $*$-sequence. Define a fundamental $*$-sequence $\psi = \xi(\varphi)$ from A into $C_\partial(I^{n-1};\,C_K(I,\,J;\,B,\,B'))$, where $\psi = \{g_i\}$ and the function g_i is given by the rule

$$g_i(a)(s_1,\,\ldots,\,s_{n-1})(t) = f_i(a)(s_1,\,\ldots,\,s_{n-1},\,t),$$

for every $a \in A$, every point $(s_1,\,\ldots,\,s_{n-1})$ in I^{n-1}, and every point t in I. The transformation ξ has properties that $\xi(\varphi + \psi) = \xi(\varphi) + \xi(\psi)$ and $\varphi \simeq_* \psi$ implies $\xi \simeq_* (\varphi)(\psi)$. Hence, ξ induces the required isomorphism because it has an inverse θ defined by the formula $\theta(\psi) = \varphi$, where $\psi = \{g_i\}$ is a fundamental $*$-sequence from A into $C_\partial(I^{n-1};\,C_K(I,\,J;\,B,\,B'))$, $\varphi = \{f_i\}$ is a fundamental $*$-sequence from A into $C_{K_n}(I^n,\,J_n;\,B,\,B')$, and for each $i \in \mathbb{N}$ the function f_i is given by

$$f_i(a)(s_1,\,\ldots,\,s_n) = g_i(a)(s_1,\,\ldots,\,s_{n-1})(s_n),$$

for every $a \in A$ and every point $(s_1,\,\ldots,\,s_n)$ in I^n. $\diamondsuit$

On the other hand, let $\underline{B}$ denote the C^*-algebra $C_\partial(I;\,B)$ of all continuous functions of the unit segment I into the C^*-algebra B which take ends of I into the zero element of B. Let us call the C^*-algebra pair $(\underline{B},\,\underline{B}')$ the *derived C^*-algebra pair* of the pair $(B,\,B')$.

Proposition 5.2. *For every integer $n > 2$, the $*$-shape groups $\varrho_n(A;\,(B,\,B'))$ and $\varrho_{n-1}(A;\,(\underline{B},\,\underline{B}'))$ are isomorphic.*

Proof. Let $\varphi : A \to C_{K_n}(I^n,\, J_n;\, B,\, B')$ be a fundamental $*$-sequence. Define a fundamental $*$-sequence $\psi = \xi(\varphi)$ from A into $C_{K_{n-1}}(I^{n-1},\, J_{n-1};\, \underline{B},\, \underline{B'})$ by the rule $\psi = \{g_i\}$, where for every $i \in \mathbb{N}$ the function g_i is given by

$$g_i(a)(s_1,\, \ldots,\, s_{n-1})(t) = f_i(a)(s_1,\, t,\, s_2,\, \ldots,\, s_{n-1}),$$

for every $a \in A$, every point $(s_1,\, \ldots,\, s_{n-1})$ in I^{n-1}, and every point t in I. The transformation ξ has properties that $\xi(\varphi + \psi) = \xi(\varphi) + \xi(\psi)$ and $\varphi \simeq_* \psi$ implies $\xi(\varphi) \simeq_* \xi(\psi)$. Hence, ξ induces the required isomorphism because it has an inverse θ defined by the formula $\theta(\psi) = \varphi$, where $\psi = \{g_i\}$ is a fundamental $*$-sequence from A into $C_{K_{n-1}}(I^{n-1},\, J_{n-1};\, \underline{B},\, \underline{B'})$, $\varphi = \{f_i\}$ is a fundamental $*$-sequence from A into $C_{K_n}(I^n,\, J_n;\, B,\, B')$, and for each $i \in \mathbb{N}$ the function f_i is given by

$$f_i(a)(s_1,\, \ldots,\, s_n) = g_i(a)(s_1,\, s_3,\, \ldots,\, s_n)(s_2),$$

for every $a \in A$, and every point $(s_1,\, \ldots,\, s_n)$ in I^n. $\Diamond$

The following result will be needed later in the §8.

Proposition 5.3. *If an element α of $\varrho_n(A;\, (B,\, B'))$ is represented by a fundamental $*$-sequence $\varphi = \{f_i\} \in FS^n(A;\, (B,\, B'))$ such that $f_i(a)(t_1,\, \ldots,\, t_n) \in B'$ for every $i \in \mathbb{N}$, every $a \in A$, and every $(t_1,\, \ldots$ $\ldots,\, t_n) \in I^n$, then $\alpha = 0$.*

Proof. It suffices to show that the fundamental $*$-sequence φ is $*$-homotopic to the trivial fundamental $*$-sequence $\zeta = \{z_i\}$, i.e., that for every $\varepsilon > 0$ there is an $i \in \mathbb{N}$ such that $j \geq i$ implies the existence of a $*^{\varepsilon}$-homomorphism $h : A \to C(I;\, C_{K_n}(I^n,\, J_n;\, B,\, B'))$ with $h_0 = f_j$ and $h_1 = z_i$. Let an $\varepsilon > 0$ be given. Since φ is a fundamental $*$-sequence, there is an $i \in \mathbb{N}$ such that f_j is a $*^{\varepsilon}$-homomorphism for every $j \geq i$. For such indices we can define the required function h by the formula

$$h(a)(s)(t_1, \ldots,\, t_n) = f_j(a)(t_1, \ldots,\, t_{n-1},\, s + t_n - st_n)$$

for $a \in A$, $s \in I$, and $(t_1, \ldots,\, t_n) \in I^n$. $\Diamond$

6. The boundary operator

Let A and B be C^*-algebras and let B' be a C^*-subalgebra of B. For every $n \geq 0$, we shall define a transformation

$$\partial_n : \varrho_{n+1}(A;\, (B,\, B')) \longrightarrow \varrho_n(A;\, B').$$

Let ξ be any element of $\varrho_{n+1}(A;\, (B,\, B'))$. By definition, ξ is in fact a $*$-homotopy class represented by a fundamental $*$-sequence $\varphi : A \to$

$\to C_{K_n}(I^n,\, J_n;\, B,\, B')$. Define a fundamental $*$-sequence $\varphi^{\partial_n} : A \to$ $\to C_{\partial}(I^n;\, B')$ by the formula $\varphi^{\partial_n} : \{f_i^{\partial_n}\}$, where the functions $f_i^{\partial_n}$ are given by

$$f_i^{\partial_n}(a)(t_1, \ldots, t_n) = f_i(a)(t_1, \ldots, t_n, 0),$$

for every $a \in A$ and every $(t_1, \ldots, t_n) \in I^n$. Let

$$\partial_n(\xi) = \partial_n([\varphi]_*) = [\varphi\partial_n]_*.$$

It could be proved easily that this definition is correct, i.e., that it is independent from the choice of the fundamental $*$-sequence φ above. Hereafter, ∂_n will be called the *boundary operator*.

The following two properties of ∂_n are obvious from the definition.

Proposition 6.1. *The boundary operator ∂_n sends the neutral element of the set $\varrho_{n+1}(A;\, (B,\, B'))$ into that of $\varrho_n(A;\, B')$.*

Proposition 6.2. *If $n > 0$, then the boundary operator ∂_n is a homomorphism.*

7. Induced transformations

In this section we shall show that ϱ_n is a bifunctor from the product category $\mathcal{S}h_*^{\mathrm{op}} \times \mathcal{S}h_*^2$ into the category of pointed sets $\mathcal{S}_*$ $(n = 0)$ and into the category of groups $\mathcal{G}$ $(n \geq 1)$, where $\mathcal{S}h_*^{\mathrm{op}}$ is the opposite category of the $*$-shape category $\mathcal{S}h_*$ of C^*-algebras while $\mathcal{S}h_*^2$ is the $*$-shape category of C^*-algebra pairs consisting of a C^*-algebra and its C^*-subalgebra. The morphisms in the category $\mathcal{S}h_*^2$ between objects $(A,\, A')$ and $(B,\, B')$ are $*$-homotopy classes rel$(A',\, B')$ of fundamental $*$-sequences $\varphi : (A,\, A') \to (B,\, B')$. Here we require that $\varphi = \{f_i\}$, where each $f_i : A \to B$ is a nonexpansive function which takes 0_A into 0_B and A' into B' and for every $\varepsilon > 0$ there is an $i \in \mathbb{N}$ such that $f_j \overset{\varepsilon}{\simeq}_* f_i$ rel$(A',\, B')$ whenever $j \geq i$. This last relation means that there is a $*^{\varepsilon}$-homomorphism $h : A \to C(I;\, B)$ such that $h(A') \subset C(I;\, B')$, $h_0 = f_j$, and $h_1 = f_i$.

Let A and B be C^*-algebras and let $(C,\, C')$ and $(D,\, D')$ be C^*-algebra pairs. Let $\varphi : B \to A$ and $\psi : (C,\, C') \to (D,\, D')$ be fundamental $*$-sequences. The pair $\alpha = ([\varphi]_*,\, [\psi]_*)$ is a morphism from $(A,\, (C,\, C'))$ into $(B,\, (D,\, D'))$ in the category $\mathcal{S}h_*^{\mathrm{op}} \times \mathcal{S}h_*^2$. For every fundamental $*$-sequence $\chi : A \to C_{K_n}(I^n,\, J_n;\, C,\, C')$, we can define a fundamental $*$-sequence $\kappa : B \to C_{K_n}(I^n,\, J_n;\, D,\, D')$ by the formula $\kappa : \tilde{\psi} \circ \chi \circ \varphi$, where $\tilde{\psi} = \{\tilde{g}_i\}$ and the function

$$\tilde{g}_i : C_{K_n}(I^n,\, J_n;\, C,\, C') \to C_{K_n}(I^n,\, J_n;\, D,\, D')$$

is defined by the formula $\tilde{g}_i(\lambda)(t) = g_i(\lambda(t))$, for every $\lambda \in C_{K_n}(I^n,\, J_n;\, C,\, C')$ and every $t \in I^n$. One can show that $\tilde{\psi}$ is indeed a fundamental $*$-sequence so that the above definition is correct. Moreover, the $*$-homotopy class of $\tilde{\psi}$ depends only on the $*$-homotopy class of ψ. Let us write $\kappa = \alpha_\sharp(\varphi,\, \psi)(\chi)$. One can show that $\varphi_1 \simeq_* \varphi_2$, $\psi_1 \simeq_* \psi_2$, and $\chi_1 \simeq_* \chi_2$ implies $\kappa_1 \simeq_* \kappa_2$, where $\kappa_1 = \alpha_\sharp(\varphi_1,\, \psi_1)(\chi_1)$ and $\kappa_2 = \alpha_\sharp(\varphi_2,\, \psi_2)(\chi_2)$ for fundamental $*$-sequences $\varphi_1,\, \varphi_2 : B \to A$, $\psi_1,\, \psi_2 : (C,\, C') \to (D,\, D')$, and $\chi_1,\, \chi_2 : A \to C_{K_n}(I^n,\, J_n;\, C,\, C')$. Moreover, the functions $\alpha_\sharp(\varphi,\, \psi)$ send the trivial fundamental $*$-sequence into the trivial fundamental $*$-sequence and they respect addition, i.e.,

$$\alpha_\sharp(\varphi,\, \psi)(\chi_1 + \chi_2) = \alpha_\sharp(\varphi,\, \psi)(\chi_1) + \alpha_\sharp(\varphi,\, \psi)(\chi_2),$$

so that these functions induce morphisms α_{*n} of pointed sets when $n = 0$, $C' = \{0_C\}$, and $D' = \{0_D\}$ and when $n = 1$ and either $C' \neq \{0_C\}$ or $D' \neq \{0_D\}$ while they induce the homomorphism α_{*n} of groups when $n = 1$, $C' = \{0_C\}$, and $D' = \{0_D\}$ and when $n \geq 1$. Thus we have established the following two properties of α_{*n}.

Proposition 7.1. *If $n = 0$, $C' = \{0_C\}$, and $D' = \{0_D\}$, or if $n > 0$, then the induced transformation α_{*n} sends the neutral element of $\varrho_n(A;\, (C,\, C'))$ into that of $\varrho_n(B;\, (D,\, D'))$*

Proposition 7.2. *If $n = 1$, $C' = \{0_C\}$, and $D' = \{0_D\}$, or if $n > 1$, then the induced transformation α_{*n} is a homomorphism.*

In the case of (7.2) we shall call α_{*n} the *induced homomorphism.*

Proposition 7.3. *The induced transformation has the following functorial properties:*

(1) *It sends the neutral element of $\varrho_n(A;\, (C,\, C'))$ into that of $\varrho_n(B;\, (D,\, D'))$,*
(2) $([\mathrm{id}_C]_*,\, [\mathrm{id}_{(C,\, C')}]_*)_* = \mathrm{id}_{\varrho_n}(A;\, (C,\, C'))$,
(3) $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ *for morphisms $\varphi : (A,\, (C,\, C')) \to (B,\, (D,\, D'))$ and $\psi : (B,\, (D,\, D')) \to (E,\, (G,\, G'))$ in the category $\mathcal{S}h_*^{\mathrm{op}} \times \mathcal{S}h_*^2$.*

Proof. The relations (1) and (2) are obvious, while the relation (3) is an immediate consequence of associativity of composition of $*$-homotopy classes of fundamental $*$-sequences and the fact that the transformation $\psi \longmapsto \tilde{\psi}$ satisfies $\widetilde{\nu \circ \psi} = \tilde{\nu} \circ \tilde{\psi}$. $\Diamond$

The Prop. (7.3) has the following consequence.

Corollary 7.4. *Let $n > 1$. If C^*-algebras A and B are $*$-shape equivalent and if C^*-algebra pairs (C, C') and (D, D') are $*$-shape equivalent,* *then the groups $\varrho_n(A;\,(C,\,C'))$ and $\varrho_n(B;\,(D,\,D))$ are isomorphic. The $*$-shape groups are invariants of $*$-shape type. The $*$-shape equivalences induce isomorphisms of $*$-shape groups.*

Proposition 7.5. *For every morphism $\psi : (A, (C, C')) \to (B, (D, D'))$ of the category $\mathcal{S}h_*^{\mathrm{op}} \times \mathcal{S}h_*^2$ and every $n > 0$ the following rectangle commutes.*

$$\begin{array}{ccc} \varrho_{n+1}(A;\,(C,\,C')) & \xrightarrow{\partial_n} & \varrho_n(A;\,C') \\ {\scriptstyle \psi_{*n+1}}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{*n}} \\ \varrho_{n+1}(B;\,(D,\,D')) & \xrightarrow[\partial_n]{} & \varrho_n(A;\,D'). \end{array}$$

8. The weak exactness property

The main result of this section is that $*$-shape groups of C^*-algebras can be put into a long weakly exact sequence.

Let A and B be C^*-algebras, and let B' be a C^*-subalgebra of B. Let j and i denote the inclusions of C^*-algebra pairs $(B,\,\{0_B\})$ and $(B',\,\{0_B\})$ into C^*-algebra pairs $(B,\,B')$ and $(B,\,\{0_B\})$, respectively. Let η and ι denote simple fundamental $*$-sequences generated by $*$-homomorphisms j and i. The pairs $([\iota^A]_*, [\eta]_*)$ and $([\iota^A]_*, [\iota]_*)$ are denoted also as j and i and are morphisms of the category $\mathcal{S}h_*^{\mathrm{op}} \times \mathcal{S}h_*^2$. Hence, for every $n > 0$ the morphism j induces the transformation j_{*n} : $: \varrho_n(A;\,B) \to \varrho_n(A;\,(B,\,B'))$, while for every $n \geq 0$ the morphism i induces the transformation $i_{*n} : \varrho_n(A;\,B') \to \varrho_n(A;\,B)$. Together with the boundary operators ∂_n, they form an endless sequence

$$\varrho_0(A;\,B) \xleftarrow{i_{*0}} \varrho_0(A;\,B') \xleftarrow{\partial_0} \varrho_1(A;\,(B,\,B')) \xleftarrow{j_{*1}}$$
$$\longleftarrow \varrho_1(A;\,B') \xleftarrow{i_{*1}} \varrho_1(A;\,B) \xleftarrow{\partial_1} \dots$$

which will be called the *$*$-shape groups sequence* of the pair $(A,\,(B,\,B'))$ and will be denoted by $\varrho(A;\,B,\,B')$.

Every set in $\varrho(A;\,B,\,B')$ has a specified element called its neutral element and every transformation in $\varrho(A;\,B,\,B')$ carries the neutral element into the neutral element. The *kernel* of a transformation in $\varrho(A;\,B,\,B')$ is defined to be the inverse image of the neutral element. Such a sequence is *exact* if the kernel of each transformation coincides exactly with the image of the preceding transformation. It is highly unlikely that $*$-shape groups sequence is exact because the long shape groups sequence is not exact. However, we shall now prove that it has the weaker property called weak exactness.

The main purpose of this section is to prove the following theorem. In the statements below, the symbol 0 denotes either the neutral element of the set involved or the transformation which sends every element into the neutral element.

Theorem 8.1. *The $*$-shape groups sequence $\varrho(A, B, B')$ of a pair $(A, (B, B'))$ has the following properties.*

(1) $j_{*n} \circ i_{*n} = 0$,
(2) $\partial_n \circ j_{*n+1} = 0$,
(3) $i_{*n} \circ \partial_n = 0$, *and*
(4) *The $*$-shape groups sequence is weakly exact, i.e., if $\varrho_n(A;\, B) = 0$ for all $n \geq 0$, then ∂_n sends $\varrho_{n+1}(A;\, (B,\, B'))$ onto $\varrho_n(A;\, B')$ in a one-to-one fashion for every $n > 0$.*

Proof. (1): Let $\alpha = \varrho_n(A;\, B')$ and choose a fundamental $*$-sequence $\varphi : A \to C_\partial(I^n;\, B')$ which represents α. Then the element $(j_{*n} \circ i_{*n})(\alpha)$ in $\varrho_n(A;\, (B,\, B'))$ is represented by the composition $\psi = i_\partial^K \circ \widehat{i_{B'}^B} \circ \varphi$, where

$$i_\partial^K : C_\partial(I^n,\, B) \to C_{K_n}(I^n;\, J_n;\, B,\, B')$$

is the inclusion $*$-homomorphism and

$$\widehat{i_{B'}^B} : C_\partial(I^n;\, B') \to C_\partial(I^n;\, B)$$

is a $*$-homomorphism induced by the inclusion $i_{B'}^B : B' \to B$ by the rule

$$\widehat{i_{B'}^B}(h)(t) = i_{B'}^B(h(t)),$$

for every $h \in C_\partial(I^n;\, B')$ and every $t \in I^n$. Since obviously $g_i(a)(t) \in B'$ for every $i \in \mathbb{N}$, every $a \in A$, and every $t \in I^n$, it follows from Prop. (5.3) that $j_{*n} \circ i_{*n}(\alpha) = 0$. Since α is arbitrary, this implies $j_{*n} \circ i_{*n} = 0$.

(2): Let α be an element of $\varrho_{n+1}(A;\, B)$ and choose a fundamental $*$-sequence $\varphi : A \to C_\partial(I^{n+1};\, B)$ which represents the element α. Then the element $(\partial_n \circ j_{*n+1})(\alpha)$ in $\varrho_n(A;\, B')$ is represented by the fundamental $*$-sequence $\psi : A \to C_\partial(I^n;\, B')$, where $\psi = \{g_i\}$ and g_i is $(i_\partial^K \circ \varphi)_i^{\partial_n}$. Now, for every $i \in \mathbb{N}$, every $a \in A$, and every $t = (t_1,\, \ldots,\, t_n)$ in I^n, we have

$$\begin{aligned} g_i(a)(t) &= (i_\partial^K \circ f_{\varphi(i)})(a)(t_1,\, \ldots,\, t_n,\, 0) = \\ &= i_\partial^K(f_{\varphi(i)}(a))(t_1,\, \ldots,\, t_n,\, 0) = 0_B, \end{aligned}$$

because $(t_1,\, \ldots,\, t_n,\, 0)$ is a point of ∂I^{n+1}. Hence, $g_i(a)$ is a zero element of $C_\partial(I^n;\, B')$ for every $a \in A$ so that g_i is a trivial function. It follows that the element $\partial_n \circ j_{*n+1}(\alpha)$ is equal to 0. Hence, $\partial_n \circ j_{*n+1} = 0$.

(3): Let $\alpha \in \varrho_{n+1}(A;\, (B,\, B'))$ and choose a fundamental $*$-sequence φ from A into $C_{K_{n+1}}(I^{n+1},\, J_{n+1};\, B,\, B')$ which represents α. Then the element $(i_{*n} \circ \partial_n)(\alpha)$ in $\varrho_n(A;\, B')$ is represented by the fundamental $*$-sequence $\psi : A \to C_\partial(I^n;\, B)$, where ψ is $\widehat{i^B_{B'}} \circ \varphi^{\partial_n}$. We claim that ψ is $*$-homotopic to the trivial fundamental $*$-sequence $\zeta = \{z_i\}$ from A into $C_\partial(I^n;\, B)$. Indeed, let an $\varepsilon > 0$ be given. Since φ is a fundamental $*$-sequence, there is an $i \in \mathbb{N}$ such that f_j is an $*^\varepsilon$-homomorphism for every $j \geq i$. For each such index j we can define a function $h^j : A \to C(I;\, C_\partial(I^n;\, B))$ by the rule

$$h^j(a)(s)(t_1,\, \ldots,\, t_n) = f_j(a)(t_1,\, \ldots,\, t_n,\, s),$$

for every $a \in A$, every point $(t_1,\, \ldots,\, t_n)$ of I^n, and every $s \in I$. It could be easily checked that h^j is a $*^\varepsilon$-homotopy which joins the function g_j with the trivial $*$-homomorphism. We obtain that the element $i_{*n} \circ \partial_n(\alpha)$ is equal to 0. Hence, $i_{*n} \circ \partial_n = 0$.

(4): In order to show that ∂_n is onto, let $\alpha \in \varrho_n(A;\, B')$ and choose a fundamental $*$-sequence $\varphi \in FS^n(A;\, B')$ which represents α. Let $\psi = \iota \circ \varphi$, where ι denotes the simple fundamental $*$-sequence generated by the $*$-homomorphism $\widehat{i^B_{B'}} : C_\partial(I^n;\, B') \to C_\partial(I^n;\, B)$ defined by $\widehat{i^B_{B'}}(\lambda) = i^B_{B'}(\lambda(t))$ for every $\lambda \in C_\partial(I^n;\, B')$ and every $t \in I^n$ from the inclusion $i^B_{B'}$ of B' into B. Then $\psi \in FS^n(A;\, B)$ so that ψ is $*$-homotopic to the trivial fundamental $*$-sequence. Hence, by [6, Prop. (4.3)], there is a fundamental $*$-sequence $\chi = \{h_i\}_{i \in \mathbb{N}}$ from A into $C(I;\, C_\partial(I^n;\, B))$ and an increasing function $\lambda : \mathbb{N} \to \mathbb{N}$ such that $h^i_0 = g_{\lambda(i)}$ and $h^i_1 = z_{\lambda(i)}$ for every $i \in \mathbb{N}$. For every natural number i we can now define a function $k_i : C_{K_{n+1}}(I^{n+1},\, J_{n+1};\, B,\, B')$ by

$$k_i(a)(t_1,\, \ldots,\, t_{n+1}) = h_i(a)(t_{n+1})(t_1,\, \ldots,\, t_n),$$

for every $a \in A$ and every point $(t_1,\, \ldots,\, t_{n+1})$ in I^{n+1}. The family $\kappa = \{k_i\}$ is a fundamental $*$-sequence such that κ^{∂_n} is $*$-homotopic to φ. Hence, $\alpha = \partial_n(\beta)$ for $\beta = [\kappa]_*$.

Finally, it remains to check that ∂_n is one-to-one. Assume that α and β from the group $\varrho_{n+1}(A;\, (B,\, B'))$ are $*$-homotopy classes such that $\partial_n(\alpha) = \partial_n(\beta)$. Let fundamental $*$-sequences φ and ψ from A into $C_{K_{n+1}}(I^{n+1},\, J_{n+1};\, B,\, B')$ be representatives of α and β, respectively. For every $i \in \mathbb{N}$ define functions

$$c_i,\, d_i : A \to C_\partial(I^n;\, B')$$

by

$$c_i(a)(t_1,\, \ldots,\, t_n) = f_i(a)(t_1,\, \ldots,\, t_n,\, 0)$$

and

$$d_i(a)(t_1,\, \ldots,\, t_n) = g_i(a)(t_1,\, \ldots,\, t_n,\, 0),$$

for every $a \in A$ and every point $(t_1,\, \ldots,\, t_n)$ of I^n. Since φ and ψ are fundamental $*$-sequences, one can show that the families $\gamma = \{c_i\}$ and $\delta = \{d_i\}$ are also fundamental $*$-sequences. By assumption γ and δ are $*$-homotopic. It follows that there is an increasing function $\pi : \mathbb{N} \to \mathbb{N}$ and a fundamental $*$-sequence $\chi = \{h_i\}_{i \in \mathbb{N}}$ from A into $C(I;\, C_\partial(I^n;\, B'))$ such that $h_0^i = c_{\pi(i)}$ and $h_1^i = d_{\pi(i)}$ for every $i \in \mathbb{N}$.

Let $W = I^n \times \{0\} \times I$, $S_1 = I^n \times I \times \{0\}$, $S_2 = \partial I^n \times I \times I$, $S_3 = I^n \times I \times \{1\}$, and let V denote the union $W \cup S_1 \cup S_2 \cup S_3$. Let $q : I^n \times I \times I \to V$ be a retraction. For each $i \in \mathbb{N}$, define a function $\ell_i : A \to C(I^{n+2};\, B)$ by the rule

$$\ell_i(a)(t) = \begin{cases} h^i(a)(s)(x_1,\, \ldots,\, x_n), & \text{if } \ q(t) = (x_1,\, \ldots,\, x_n,\, 0,\, s) \in W, \\ f_{\pi(i)}(a)(y_1,\, \ldots,\, y_n,\, s), & \text{if } \ q(t) = (y_1,\, \ldots,\, y_n,\, s,\, 0) \in S_1, \\ 0_B, & \text{if } \ q(t) \in S_2, \\ g_{\pi(i)}(a)(z_1,\, \ldots,\, z_n,\, s), & \text{if } \ q(t) = (z_1,\, \ldots, z_n,\, s,\, 1) \in S_3, \end{cases}$$

for every $a \in A$ and every point $t \in I^{n+2}$. The collection $\lambda = \{\ell_i\}$ is a fundamental $*$-sequence from A into $C_\partial(I^{n+2};\, B)$. For every $i \in \mathbb{N}$ define a function

$$m_i : A \to C_\partial(I^{n+1};\, B)$$

by the rule

$$m_i(a)(t) = \ell_i(a)(t_1,\, \ldots,\, t_n,\, 1,\, t_{n+1}),$$

for every $a \in A$ and every $t = (t_1,\, \ldots,\, t_{n+1}) \in I^{n+1}$. The collection $\mu = \{m_i\}_{i \in \mathbb{N}}$ is a fundamental $*$-sequence from A into $C_\partial(I^{n+1};\, B)$. Since $\varrho_{n+1}(A;\, B) = 0$, the fundamental $*$-sequence μ is $*$-homotopic to the trivial fundamental $*$-sequence. Hence, there is an increasing function $\omega : \mathbb{N} \to \mathbb{N}$ and a fundamental $*$-sequence $\nu = \{n^i\}_{i \in \mathbb{N}}$ from A into $C([1,\, 2];\, C_\partial(I^{n+1};\, B))$ with $n_1^i = m_{\omega(i)}$ and $n_2^i = z_{\omega(i)}$ for every $i \in \mathbb{N}$.

Let $P = [0,\, 2] \times [0,\, 1]$ be the product of closed segments $[0,\, 2]$ and $I = [0,\, 1]$. Let $r : I^2 \to P$ be a map which extends a homeomorphism of the boundary of I^2 onto the boundary of P such that $r(0,\, t) = (0,\, t)$, $r(s,\, 0) = (s,\, 0)$, and $r(s,\, 1) = (s,\, 1)$, for all $s,\, t \in I$. Let $r_1(s,\, t)$ and $r_2(s,\, t)$ denote the first and the second coordinate of the point $r(s,\, t)$ for every $(s,\, t) \in I^2$.

For every $i \in \mathbb{N}$ define a function $k^i : A \to C(I; C_{K_{n+1}}(I^{n+1}, J_{n+1}; B, B'))$ by the rule

$$k^i(a)(s)(t) = \\ = \begin{cases} \ell_{\omega(i)}(a)(t_1, \ldots, t_n, r_1(t_{n+1}, s), r_2(t_{n+1}, s)), & 0 \leq r_1(t_{n+1}, s) \leq 1, \\ n^i(a)(r_1(t_{n+1}, s))(t_1, \ldots, t_n, r_2(t_{n+1}, s)), & 1 \leq r_1(t_{n+1}, s) \leq 2, \end{cases}$$

for every $a \in A$, every $s \in I$, and every point $t = (t_1, \ldots, t_{n+1})$ of I^{n+1}. The collection $\kappa = \{k^i\}_{i \in \mathbb{N}}$ is a $*$-homotopy between fundamental $*$-sequences φ' and ψ', where $f_i' = f_{\pi(\chi(i))}$ and $g_i' = g_{\pi(\chi(i))}$ for every $i \in \mathbb{N}$. It follows that $\varphi \simeq_* \varphi'$ and $\psi \simeq_* \psi'$. Hence, $\alpha = \beta$ and ∂_n is indeed one-to-one. $\diamondsuit$

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