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THE AUTOMORPHISM GROUP OF UNARY ALGEBRAS

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Abstract: It was observed by several authors that the congruence lattice Con A and the automorphism group Aut A of a unary algebra are not independent [3]. This paper contains some results concerning this interdependence. For example, if A is a cyclic unary algebra, then S(Aut A), the subgroup lattice of its automorphism group, is isomorphic to an interval of Con A. If A is subalgebra-simple or subdirect irreducible, then S(Aut A) is isomorphic to a principal ideal of Con A. Other results are formulated in terms of fixpoints of automorphisms . For instance, we show that, if there exists an element of A which is not a fixpoint for any nontrival automorphism of A, then S(Aut A) can be embedded as a sublattice in Con A. Denote by Inv A the normal subgroup of all automorphisms of A under which all congruences of A are invariant. In general this is quite arbitrary, but we show that for a unary algebra Inv A is either a quaternion group or a subdirect irreducible Abelian group or it is a subdirect product of such groups. If A is cyclic, then Inv A is Hamiltonian. We also prove that if Con A is finite, then so is Aut A.

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1. Introduction

Let us denote, as usually by Aut A, Con A and by Sub A the automorphism group, the congruence lattice and the subalgebra lattice of an algebra (A, Γ). It is known that in the case of unary algebras Sub A and Con A are not independent (see Remark 1). It was established by several authors, that in the unary case between Aut A and Con A there exists also an interdependence. For example, W. Lex proved [3]: If (A, Γ) is a congruence-simple unary algebra (i.e. Con A is a two-element chain), then Aut A is either trivial or it is a cyclic group of prime order. *The aim of this paper* is to investigate the connection between Con A and Aut A. This connection is influenced by Sub A. A dependence between Aut A and Aut(Con A) — the automorphism group of the lattice Con A is also remarked.

Let (A, Γ) be a unary algebra. We note that the basic set Γ of operations can be substituted by the monoid $\langle\Gamma\rangle$ generated by it (using as a unit the identity mapping of A denoted by 1^A). It is easy to check that this substitution do not change Aut A, Con A and Sub A. (In view of this remark, in the rest of the paper we shall assume that Γ closed with respect to the composition of operations!) We denote by 0_A and 1_A the least and the greatest element of Con A, by 1^B the identity mapping of a subset $B \subseteq A$. $\theta(a, b)$ stands for the principal congruence of A corresponding to $a, b \in A$. The subgroup lattice of a group G is denoted by S(G), the cyclic subgroup generared by an element $g \in G$ by $\langle g\rangle$. For a group F, $F \leq G$ denotes that F is a subgroup of G. For $a \in A$, (a) denotes the subalgebra generated by $\{a\}$. A one element subalgebra is called a singleton. We note that $B \subseteq A$ is a subalgebra of A iff $(b) \subseteq B$ for all $b \in B$. (We denote it by $B \leq A$.) Sub A is a distributive lattice containing as 0 element $\emptyset$ and as unit element A. For an $a \in A$ and for a $\theta \in$ Con A the θ-congruence class of a is denoted by $\theta[a]$. For a lattice L and $a, b \in L$ $a \leq b$, the lattice interval $[a, b]$ denotes the set $\{x \in L | a \leq x \leq b\}$ and $(a]$ the principal (lattice) ideal belonging to a, i.e. the set $\{x \in L | x \leq a\}$.

Definition 1.1. For any subalgebra $B \leq A$ we define a *congruence*[1] ρ_B on A as follows:

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[1] The fact that ρ_B is a congruence on A is common knowledge.

$$(x, y) \in \rho_B \iff \text{ either } \quad x, y \in B \quad \text{or} \quad x = y.$$

Remark 1.2. (i) Since for any $B_1, B_2 \leq A$ we have $\rho_{B_1 \cap B_2} = \rho_{B_1} \wedge \rho_{B_2}$, the mapping $B \longmapsto \rho_B$, $(B \leq A)$ is a $\wedge$ (semilattice)-homomorphism of $\operatorname{Sub} A$ into $\operatorname{Con} A$. If B_1 (or B_2) has at least two elements, then $\rho_{B_1} = \rho_{B_2} \Rightarrow B_1 = B_2$. Notice, that in general $\rho_{B_1 \cup B_2} \neq \rho_{B_1} \vee \rho_{B_2}$ and $\rho_B = 0_A$ iff B has at most one element.

(ii) For any $B \subseteq A$, $\operatorname{Con} B$ is isomorphic to the principal ideal $(\rho_B]$ of $\operatorname{Con} A$ (see [1]).

Remark 1.3. (i) It is easy to check that for any $a \in A$ and any $f \in \operatorname{Aut} A$ we have: $f((a)) = (f(a))$. If $\{a\}$ is a singleton, then so does $\{f(a)\}$.

(ii) We say that the subalgebra $B \leq A$ is invariated by the subgroup $F \leq \operatorname{Aut} A$ if for any $f \in F$ holds $f(B) \subseteq B$. In fact this is equivalent to $f(B) = B$, $\forall f \in F$. (Indeed, since $f^{-1}(B) \subseteq B$, $\forall f \in F$, we can write $B = f(f^{-1}(B)) \subseteq f(B) \subseteq B$.)

Remark 1.4. For an arbitrary $f \in \operatorname{Aut} A$ and for any $F \leq \operatorname{Aut} A$ we denote by $\operatorname{Fix} f$ *the set of all fixpoints of* f and by $\operatorname{Fix} F$ *the set* $\cup\{ \operatorname{Fix} f | f \in F \setminus \{1^A\}\}$ respectively. $\operatorname{Fix} f$ and $\operatorname{Fix} F$ are subalgebras of A. If $f \neq 1^A$, then $\operatorname{Fix} f \neq A$. $\operatorname{Fix} F$ is invariated by F. Indeed, choose $x_0 \in \operatorname{Fix} f$; then $f(\gamma(x_0)) = \gamma(f(x_0)) = \gamma(x_0)$, i.e. $\gamma(x_0) \in \operatorname{Fix} f$, $\gamma \in \Gamma$. Evidentely $\operatorname{Fix} f = A$ implies $f = 1^A$. Take now $x \in \operatorname{Fix} F$ and $f \in F$; then we have $h(x) = x$ for a suitable $h \in F \setminus \{1^A\}$. Since $(f^{-1} \circ h \circ f)(f(x)) = f(x)$ and $f^{-1} \circ h \circ f \neq 1^A$, we obtain $f(x) \in \operatorname{Fix} F$.

Definition 1.5. (i) (A, Γ) is called *cyclic* if it is generated by a single element $a \in A$ (i.e. we have $(a) = A$).

(ii) (A, Γ) is called *subalgebra-simple* if it has no proper subalgebra (i.e. has no subalgebra different from itself and from the empty one).

Let us observe that (A, Γ) is subalgebra-simple iff $(x) = A$, $\forall x \in A$ — or equivalently: iff for any $a, b \in A$ there is a $\gamma \in \Gamma$ such that $\gamma(a) = b$.

2. Embedding the subgroup lattice of $\operatorname{Aut} A$ in $\operatorname{Con} A$

Proposition 2.1. *Any subgroup F of $\operatorname{Aut} A$ defines a congruence θ_F on A as follows:*

$$(a, b) \in \theta_F \iff \exists f \in F \quad \textit{such that} \quad f(a) = b.$$

Proof. It is clear that θ_F is an equivalence on A. Take now $(a,b)\in\theta_F$ and $\gamma\in\Gamma$ arbitrary; then by definition of θ_F there exists an $f\in F$ such that $f(a)=b$. We can write: $f(\gamma(a))=\gamma(f(a))=\gamma(b)$, thus $(\gamma(a),\gamma(b))\in\theta_F$ — i.e. θ_F is compatible. ◊

The congruence θ_F we shall call *the congruence induced by the group* $F\leq \operatorname{Aut} A$. We let $\theta_{\operatorname{Aut} A}$ stand for the congruence induced by the whole $\operatorname{Aut} A$.

Proposition 2.2. *For any unary algebra* (A,Γ) *the mapping* Υ : $: F\longmapsto\theta_F$, $F\leq\operatorname{Aut} A$ *is a complete join-homomorphism of* $\mathrm{S}(\operatorname{Aut} A)$ *into* $\operatorname{Con} A$ *with* $\operatorname{Ker}\Upsilon=\{1^A\}$.

Proof. We have to prove $\Upsilon(\bigvee\limits_{i\in I}F_i)=\bigvee\limits_{i\in I}\Upsilon(F_i)$, i.e. $\theta_{\bigvee_{i\in I}F_i}=\bigvee\limits_{i\in I}\theta_{F_i}$ for an arbitrary index set $I\neq\emptyset$ and $F_i\leq\operatorname{Aut} A$. Take $(x,y)\in\theta_{\bigvee_{i\in I}F_i}$; then $f(x)=y$, where $f=f_1\circ f_2\circ\ldots\circ f_n$ for some $f_1\in F_{i_1}$, $f_2\in F_{i_2},\ldots,f_n\in$ $\in F_{i_n}$ $(i_1,i_2,\ldots,i_n\in I,\ n\in\mathbb{N})$. Let us consider the elements $y_0=$ $=x,\ y_1=f_1(y_0),\ldots,y_k=f_k(y_{k-1}),\ldots,y=y_n=f_n(y_{n-1})$. Since $(y_{k-1},y_k)\in\theta_{F_{i_k}}$ holds for any $1\leq k\leq n$, we obtain $(x,y)\in\theta_{F_{i_1}}\circ$ $\circ\,\theta_{F_{i_2}}\circ\ldots\circ\theta_{F_{i_n}}\leq\bigvee\limits_{i\in I}\theta_{F_i}$. Conversely, take $(x,y)\in\bigvee\limits_{i\in I}\theta_{F_i}$; then there are $z_0,z_1,\ldots,z_m\in A$ and $i_1,\ldots,i_m\in I$, $(m\in\mathbb{N})$ not necessarily different, such that $z_0=x$, $z_m=y$ and such that $(z_{k-1},z_k)\in\theta_{F_{i_k}}$, for all $1\leq k\leq m$. Thus for each $1\leq k\leq m$ there exists an automorphism $f_k\in F_{i_k}$, such that $f_k(z_{k-1})=z_k$ holds. Take $f=f_1\circ f_2\circ\ldots\circ$ $\circ f_m\in\bigvee\limits_{i\in I}F_i$. Since $f(x)=y$, we obtain $(x,y)\in\theta_{\bigvee_{i\in I}F_i}$. Finally, take $F\in\operatorname{Ker}\Upsilon$, i.e. $F\leq\operatorname{Aut} A$ such that $\theta_F=0_A$. Then for all $f\in F$ and all $x\in A$ we have $f(x)=x$. Thus we obtain $F=\{1^A\}$. ◊

Remark 2.3. For any subgroup $F\leq\operatorname{Aut} A$ the restriction $\Upsilon_F\colon\mathrm{S}(F)\longrightarrow$ $\longrightarrow[0_A;\theta_F]$ of Υ is obviously a complete join-homomorphism. If the join-homomorphism $\Upsilon\colon\mathrm{S}(\operatorname{Aut} A)\longrightarrow[0_A,\theta_{\operatorname{Aut} A}]$ is a bijective mapping, then Υ is a lattice isomorphism. (The statement is true for any $h\colon L_1\longrightarrow$ $\longrightarrow L_2$ — where L_1,L_2 are arbitrary lattices and h is a semilattice homomorphism of $(L_1,\vee)$ into $(L_2,\vee)$.)

Proposition 2.4. *For any cyclic group* $\langle f\rangle\leq\operatorname{Aut} A$ *of finite order,* $\Upsilon_{\langle f\rangle}$ *is injective.*

Proof. Assume $\Upsilon(F_1)=\Upsilon(F_2)$ for $F_1,F_2\leq\langle f\rangle$. Since any subgroup of $\langle f\rangle$ is cyclic, there exists $f_1,f_2\in\langle f\rangle$ such that $F_1=\langle f_1\rangle$ and $F_2=$ $=\langle f_2\rangle$ — let us denote their order by n_1 and n_2 respectively. Since $\theta_{\langle f_1\rangle}=\theta_{\langle f_2\rangle}$, for any $x\in A$ there exists $k_x,\ l_x\in\mathbb{N}$, such that $f_1(x)=$

$= f_2^{l_x}(x)$ and $f_2(x) = f_1^{k_x}(x)$. We obtain: $f_1^{n_2}(x) = f_2^{n_2 \cdot l_x}(x) = x$ and $f_2^{n_1}(x) = f_1^{n_1 \cdot k_x}(x) = x$, $\forall x \in A$. Thus the order n_1 of f_1 divides n_2 and the order n_2 of f_2 divides n_1. We obtain $n_1 = n_2$. Since a cyclic group of finite order contains for any $n \in \mathbb{N}$ at most one subgroup of order n, we conclude $\langle f_1 \rangle = \langle f_2 \rangle$. $\Diamond$

Lemma 2.5. *For any $a \in A$ and for any $f \in$ Aut A with the property that $f(a) \in (a)$, we have $\theta(a, f(a)) = \theta_{\langle f \rangle} \wedge \rho_{(a)}$.*

Proof. First, note that $f(a) \in (a)$ implies

(1) $$f^k(x) \in (a), \quad \forall x \in (a), \quad \forall k \in \mathbb{N} \cup \{0\}.$$

Indeed, for $k = 0$, $f^0 = 1^A$ by definition and the statement holds trivially. Assume now that $f^k(x) \in (a)$, $\forall x \in (a)$ is true for a $k \in \mathbb{N} \cup \{0\}$. Then for each $x \in (a)$ there is a $\gamma_x \in \Gamma$ such that $\gamma_x(a) = f^k(x)$. We can write $f^{k+1}(x) = f(\gamma_x(a)) = \gamma_x(f(a)) \in (a)$. Thus we conclude by induction, that the statement holds for all $k \in \mathbb{N} \cup \{0\}$.

Let us prove now the required equality: Since $(a, f(a)) \in \rho_{(a)}$ holds by assumption and $(a, f(a)) \in \theta_{\langle f \rangle}$, we can write $\theta(a, f(a)) \leq \theta_{\langle f \rangle} \wedge \rho_{(a)}$. Take now $(x, y) \in \theta_{\langle f \rangle} \wedge \rho_{(a)}$. For $x = y$, $(x, y) \in \theta(a, f(a))$ holds trivially, so we can assume $x \neq y$; We have $x, y \in (a)$ and $f^k(x) = y$ for a suitable $k \in \mathbb{Z}$, $k \neq 0$. It is enough to consider the case $k > 0$, since in the case $k < 0$ we can write: $x = f^{(-k)}(y)$ with $-k > 0$ (and $y \in (a)$). Introducing the notations $z_i = f^i(x)$, $i \in \{0, \ldots, k\}$, we obtain $z_{i+1} = f(z_i)$, $z_0 = x$ and $z_k = y$. According to (1) we have $z_i \in (a)$. Thus there are $\gamma_i \in \Gamma$ such that

(2) $$\gamma_i(a) = z_i, \quad \forall i \in \{0, \ldots, k\}.$$

Now we can write:

(3) $$\gamma_i(f(a)) = f(\gamma_i(a)) = f(z_i) = z_{i+1}.$$

The relations (2) and (3) together imply $(z_i, z_{i+1}) \in \theta(a, f(a))$, $0 \leq i \leq k-1$, i.e.: $(x, y) = (z_0, z_k) \in \theta(a, f(a))$. Thus we conclude $\theta_{\langle f \rangle} \wedge \rho_{(a)} \leq \theta(a, f(a))$. $\Diamond$

Corollary 2.6. *For any $f \in$ Aut $A \setminus \{1^A\}$, Fix $f \cup (a) = A$ implies $\theta(a, f(a)) = \theta_{\langle f \rangle}$.*

Proof. Since $f(a) = a$ implies $(a) \subseteq \mathrm{Fix} f$, i.e. $\mathrm{Fix} f = A$, thus Remark 1.4. gives that $f(a) \neq a$. We must have $f(a) \in (a)$, otherwise $f(a) \in \mathrm{Fix} f$ would imply $f(f(a)) = f(a)$. Since for $x \notin (a)$ we have $f(x) = x$, now $(x, y) \in \theta_{\langle f \rangle}$ and $x \neq y$ imply $x, y \in (a)$, i.e. $\theta_{\langle f \rangle} \leq \rho_{(a)}$. Hence the statement follows by Lemma 2.5. $\Diamond$

Proposition 2.7. *Let F be a nontrivial subgroup of* Aut A. *Then the following statements hold:*

(i) *If* $\operatorname{Fix} F \neq A$, *then* $\Upsilon_F \colon \mathrm{S}(F) \longrightarrow \operatorname{Con} A$ *is an injective join-homomorphism.*

(ii) *If* $\operatorname{Fix} f_1 = \operatorname{Fix} f_2$, *for all* $f_1, f_2 \in F \setminus \{1^A\}$, *then* Υ_F *is a lattice embedding.*

(iii) *If for all* $f \in F$, $\operatorname{Fix} f$ *is not contained in any proper subalgebra different from it, then* $\Upsilon \colon \mathrm{S}(F) \longrightarrow [0_A, \theta_F]$ *is a lattice isomorphism.*

(iv) *If* $\Upsilon \colon \mathrm{S}(F) \longrightarrow [0_A, \theta_F]$ *is surjective and* $\operatorname{Fix} F \neq A$, *then* $\operatorname{Fix} f_1 = \operatorname{Fix} f_2$, $\forall f_1, f_2 \in F \setminus \{1^A\}$ *and for any subalgebra* $B \neq A$ *containing* $\operatorname{Fix} F$, *if* B *is invariated by an* $f \in F \setminus \{1^A\}$, *then we have* $B = \operatorname{Fix} F$.

Proof. (i) Assume $\Upsilon_F(F_1) = \Upsilon_F(F_2)$ for $F_1, F_2 \leq F$ and take an $x \notin \notin \operatorname{Fix} F$. By the definition of θ_{F_1} and θ_{F_2}, for any $f_1 \in F_1$, there exists an $f_2 \in F_2$ such that $f_1(x) = f_2(x)$, i.e.: $(f_1 \circ f_2^{-1})(x) = x$. Since $f_1 \circ f_2^{-1} \in F$ and $x \notin \operatorname{Fix} F$, we obtain $f_1 \circ f_2^{-1} = 1^A$, i.e $f_1 = f_2 \in F$. Thus we get $F_1 \subseteq F_2$. Symmetrically we can prove $F_2 \subseteq F_1$.

(ii) Since $\operatorname{Fix} F = \operatorname{Fix} f$, $\forall f \in F \setminus \{1^A\}$, we have $\operatorname{Fix} F \neq A$ (see Remark 1.4.) and Υ_F is an injective join-homomorphism, according to (i). Since Υ_F is isotone, we have $\Upsilon_F(F_1 \cap F_2) \leq \Upsilon_F(F_1) \wedge \Upsilon_F(F_2)$ for all $F_1, F_2 \subseteq F$. Thus it is enough to prove $\Upsilon_F(F_1) \wedge \Upsilon_F(F_2) \leq \leq \Upsilon_F(F_1 \cap F_2)$, i.e. $\theta_{F_1} \wedge \theta_{F_2} \leq \theta_{F_1 \cap F_2}$. Take now $(x, y) \in \theta_{F_1} \wedge \theta_{F_2}$, $x \neq y$. Then there exist $f_1 \in F_1 \setminus \{1^A\}$ and $f_2 \in F_2 \setminus \{1^A\}$ such that $f_1(x) = f_2(x) = y$. Since $x \neq y$ implies $x \notin \operatorname{Fix} F$, $(f_1 \circ f_2^{-1})(x) = = x$ implies $f_1 \circ f_2^{-1} = 1^A$, i.e.: $f_1 = f_2 \in F_1 \cap F_2$. Thus we obtain $(x, y) \in \theta_{F_1 \cap F_2}$.

(iii) Let us remark first, that the assumption of (iii) implies that for any $f \in F \setminus \{1^A\}$, $\operatorname{Fix} f$ is the same subalgebra. Indeed, let us assume $\operatorname{Fix} f_1 \neq \operatorname{Fix} f_2$, $f_1, f_2 \in F \setminus \{1^A\}$: Then we get $\operatorname{Fix} f_1 \cup \operatorname{Fix} f_2 = A$, according to the assumption of (iii). We can write:

$$A = (\operatorname{Fix} f_1 \setminus \operatorname{Fix} f_2) \cup (\operatorname{Fix} f_1 \cap \operatorname{Fix} f_2) \cup (\operatorname{Fix} f_2 \setminus \operatorname{Fix} f_1),$$

and by assumption of (iii) $\operatorname{Fix} f_1 \setminus \operatorname{Fix} f_2$ and $\operatorname{Fix} f_2 \setminus \operatorname{Fix} f_1$ are nonempty. It is not hard to see that $f = f_1 \circ f_2$ satisfies $\operatorname{Fix} f = \operatorname{Fix} f_1 \cap \operatorname{Fix} f_2$. We obtain $\operatorname{Fix} f \subseteq \operatorname{Fix} f_1 \neq A$, $\operatorname{Fix} f \neq \operatorname{Fix} f_1$ — contradicting the assumption of (iii). Applying the above (ii), we obtain that Υ_F is a lattice embedding. To prove (iii), it is enough to show, that for any $\theta \in [0_A, \theta_F]$, there is an $F_0 \leq F$ such that $\theta_{F_0} = \theta$. Since $\theta_{\langle 1^A \rangle} = 0_A$, we can assume $\theta \neq 0_A$. We can write: $\theta = \vee\{\theta(a, b) | (a, b) \in \theta,\ a \neq \neq b\}$. Take $(a, b) \in \theta$, $a \neq b$; then $\theta \leq \theta_F$ implies that there exists an $f_{ab} \in F$ such that $f_{ab}(a) = b$ and $a, b \notin \operatorname{Fix} f$. Since we have by assumption $\mathrm{Fix}\, f \cup (a) = A$, applying Corollary 2.6. we get $\theta(a,b) = \theta(a, f_{ab}(a)) = \theta_{\langle f_{ab}\rangle}$. Thus we can write: $\theta = \vee\{\theta_{\langle f_{ab}\rangle} | (a,b) \in \theta, a \neq b\} = \theta_{\vee\{\langle f_{ab}\rangle | (a,b)\in\theta,\ a\neq b\}} = \theta_{F_0}$ — where $F_0 = \vee\{\langle f_{ab}\rangle | (a,b) \in \theta, a \neq b\} \leq F$.

(iv) First we prove the second assertion. Let us consider a subalgebra $B \neq A$, $B \supseteq \mathrm{Fix}\, F$ and an $f \in F \setminus \{1^A\}$ arbitrary; since $\theta_{\langle f\rangle} \wedge \rho_B \leq \theta_F$, by assumption there is an $F_0 \leq F$ such that $\theta_{F_0} = \theta_{\langle f\rangle} \wedge \rho_B$. Take any $x \notin B$; we have $\theta_{F_0}[x] = \{x\}$, which means that $x \in \mathrm{Fix}\, F_0$. Since $x \notin \mathrm{Fix}\, F$, we must have $F_0 = \{1^A\}$, i.e.: $\theta_{\langle f\rangle} \wedge \rho_B = 0_A$, $f \in F \setminus \{1^A\}$. Assume now that B is invariated by an $f_0 \in F \setminus \{1^A\}$. Then $\theta_{\langle f_0\rangle} \wedge \rho_B = 0_A$ implies $B \subseteq \mathrm{Fix}\, f_0$. Now $\mathrm{Fix}\, F \subseteq B \subseteq \mathrm{Fix}\, f_0 \subseteq \mathrm{Fix}\, F$ implies $B = \mathrm{Fix}\, f_0 = \mathrm{Fix}\, F$. Concerning the first assertion, since $B = \mathrm{Fix}\, F$ is invariated by any $f \in F \setminus \{1^A\}$ (see Remark 1.4), we obtain $\mathrm{Fix}\, f_1 = \mathrm{Fix}\, f_2 = \mathrm{Fix}\, F$, $\forall f_1, f_2 \in F \setminus \{1^A\}$. ◊

Proposition 2.8. *If (A,Γ) is a subalgebra-simple unary algebra with a possible exception of a singleton, or it is the disjoint union of a singleton and of a subalgebra-simple unary algebra, then for any $F \leq \mathrm{Aut}\, A$, $\Upsilon_F\colon \mathrm{S}(F) \longrightarrow [0_A, \theta_F]$ is a lattice isomorphism.*

Proof. We can assume $F \neq \{1^A\}$. Now it is enough to prove that the condition of Prop. 2.7. (iii) holds in each of the above cases. If (A,Γ) is subalgebra-simple, then $\mathrm{Fix}\, f = \emptyset$, $\forall f \in F \setminus \{1^A\}$ — thus the condition of Prop. 2.7. (iii) holds. If (A,Γ) has only one singleton, let say $\{0\}$ (where $0 \in A$), then according to Remark 1.3. (i), we have $f(0) = 0$ — i.e. $0 \in \mathrm{Fix}\, f$, $\forall f \in F \setminus \{1^A\}$. If A is subalgebra-simple except $\{0\}$, we obtain $\mathrm{Fix}\, f = \{0\}$, $\forall f \in F \setminus \{1^A\}$. If we have $A = \{0\} \cup B$, $\{0\} \cap B = \emptyset$ and B is subalgebra-simple, then the relation $\mathrm{Fix}\, f \not\supseteq B$ implies $\mathrm{Fix}\, f \cap B = \emptyset$, i.e. we get again $\mathrm{Fix}\, f = \{0\}$, $\forall f \in F \setminus \{1^A\}$. Since in the both of the above cases $\{0\}$ is a maximal proper subalgebra of A, the condition (iii) holds in each of the cases. ◊

Remark 2.9. Υ in general is not an injective mapping! Indeed: Let us consider $A = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$ and $\Gamma = \{1^A\}$. Then $\mathrm{Aut}\, A = \Sigma_n$ — the symmetric group of order n, and $\mathrm{Con}\, A = \mathrm{Part}\, A$ — the partition lattice of the set A. Take for example $n = 3$. Then $\mathrm{Part}\, A$ contains 5 elements and Σ_3 contains 6 subgroups, thus the mapping $\Upsilon\colon \mathrm{S}(\Sigma_3) \longrightarrow \mathrm{Con}\, A$ can not be injective!

Lemma 2.10. *Let $B \leq A$ be a subalgebra invariated by a subgroup F of* $\mathrm{Aut}\, A$.

(i) *If $B \not\subseteq \mathrm{Fix}\, F$, then F can be embedded as a subgroup in the automorphism group of the subalgebra B.*

(ii) *If* $A \setminus B \not\subseteq \mathrm{Fix}\, F$, *then* F *can be embedded as a subgroup in the automorphism group of the factor algebra* $C = A/\rho_B$.

(iii)*If* $\mathrm{Fix}\, F \subseteq B \neq A$, *then* $\mathrm{S}(F)$ *can be embedded as a sublattice in the congruence lattice of the factor algebra* $C = A/\rho_B$.

Proof. We have $f(B) = B$, $\forall f \in F$, according to Remark 1.3. (ii).

(i) Since $B \not\subseteq \mathrm{Fix}\, F$, B is not empty. Let $\overline{f_B}$ denote the restriction of f to B. Since $\overline{f_B}(B) = B$, the injectivity of $\overline{f_B}$ implies $\overline{f_B} \in \mathrm{Aut}\, B$. Let us consider the mapping $\alpha \colon F \longrightarrow \mathrm{Aut}\, B$, $\alpha(f) = \overline{f_B}$. It is obvious that α is a group homomorphism. We have: $\mathrm{Ker}\, \alpha = \{f \in F | \overline{f_B} = 1^B\}$ — i.e. $B \subseteq \mathrm{Fix}\, f$ for all $f \in \mathrm{Ker}\, \alpha$. Since we have $\mathrm{Fix}\, f \subseteq \mathrm{Fix}\, F$, $\forall f \in$ $\in F \setminus \{1^A\}$ and since $\mathrm{Fix}\, F$ do not contain B, we conclude $\mathrm{Ker}\, \alpha = \{1^A\}$, i.e.: α is an embedding.

(ii) Since $A \setminus B$ is nonempty, C contains at least two elements. Let us denote the ρ_B congruence class of an element $x \in A$ by $\overline{x}$. Then we have $\overline{x} = \{x\}$ if $x \notin B$, and $\overline{x} = B$ if $x \in B$. For any $f \in F$ we define $f^* \in \mathrm{Aut}\, C$ as follows: $f^*(\overline{x}) = \overline{f(x_0)}$ — where $x_0 \in \overline{x}$. The function f^* is well-defined, since for any $x \in A \setminus B$ we have $\overline{x} = \{x\}$ and $f(x) \in$ $\in A \setminus B$, thus $f^*(\overline{x}) = \overline{f(x)} = \{f(x)\}$, and for any $x_0 \in B$, we have $f(x_0) \in B$, i.e. $f^*(B) = B$. Since $f(A \setminus B) = A \setminus B$ and the restriction of f to $A \setminus B$ is injective, follows that f^* is bijective. On the other hand for any $\gamma \in \Gamma$ we can write: $f^*(\gamma(\overline{x})) = f^*(\overline{\gamma(x_0)}) = \overline{f(\gamma(x_0))} =$ $= \gamma(\overline{f(x_0)}) = \gamma(f^*(\overline{x}))$ — where $x_0 \in \overline{x}$. Thus f^* is an automorphism of C. It is clear that $(f_1 \circ f_2)^* = f_1^* \circ f_2^*$, $\forall f_1, f_2 \in F$, i.e. the map $\beta \colon f \longmapsto f^*$ is a group homomorphism of F into $\mathrm{Aut}\, C$. β is injective: Indeed, let us assume $f_1^*(\overline{x}) = f_2^*(\overline{x})$, $\forall \overline{x} \in C$. Now for any $a \notin B$ we have $f_1(a) = f_2(a)$. Since by assumption there is an $a_0 \in A \setminus B$ such that $a_0 \notin \mathrm{Fix}\, F$, the equality $(f_1 \circ f_2^{-1})(a_0) = a_0$ implies $f_1 \circ f_2^{-1} = 1^A$ — i.e. $f_1 = f_2$.

(iii) Let us consider again the factor algebra $C = A/\rho_B$ and the mapping $\beta \colon f \longmapsto f^*$ from the proof of (ii): Since the assumption of (ii) holds, we have $F \cong \beta(F)$. For any $x \in A \setminus B$ and for any $f^* \in \beta(F)$, $f^* \neq 1^C$ the relation $f^*(\overline{x}) = \overline{x}$ implies $f(x) = x$, i.e. $x \in \mathrm{Fix}\, F$ — which contradicts the assumption of (iii). Thus we have $\mathrm{Fix}\, f^* = \{B\}$, for any $f^* \neq 1^C$. Applying now Prop. 2.7. (ii), we obtain that the subgroup lattice of $\beta(F)$ can be embedded as a sublattice in $\mathrm{Con}\, C$. The same is true for the subgroup lattice of F. $\Diamond$

Definition 2.11. Let (A, Γ) be a cyclic unary algebra. Then the union of all its proper subalgebras we denote by $\mathrm{K}(A)$ and we shall call it the *kernel of the unary algebra* A.

Remark 2.12. (i) For any unary algebra its kernel is its greatest proper subalgebra. For an $a \in A$, $\{a\}$ is a generator set of (A, Γ) iff $a \notin \mathrm{K}(A)$.

(ii) $\mathrm{K}(A)$ is invariated by all $f \in \mathrm{Aut}\, A$, i.e. for any $a \in A$ and $f \in \mathrm{Aut}\, A$: $f(a) \in \mathrm{K}(A) \iff a \in \mathrm{K}(A)$.

Proof. (i) It is trivial.

(ii) Since $f((a)) = (f(a))$, $a \in A$, we have $(a) = A \iff (f(a)) = = A$. Now (i) implies that $a \notin \mathrm{K}(A) \iff f(a) \notin \mathrm{K}(A)$ — which is equivalent to (ii). ◊

Theorem 2.13. (i) *If* (A, Γ) *is a cyclic unary algebra, then* Υ: $: \mathrm{S}(\mathrm{Aut}\, A) \longrightarrow [0_A, \theta_{\mathrm{Aut}\, A}]$ *is an injective join-homomorphism.*

(ii) Υ *is a lattice isomorphism iff the restriction of any* $f \in \mathrm{Aut}\, A$ *to* $\mathrm{K}(A)$ *is the identity mapping on* $\mathrm{K}(A)$.

(iii) $\mathrm{S}(\mathrm{Aut}\, A)$ *is isomorphic to an interval of* $\mathrm{Con}\, A$.

Proof. (i) Since $(\mathrm{K}(A))$ is the greatest proper subalgebra of A, we have $\mathrm{Fix}\, f \subseteq (\mathrm{K}(A))$, $f \in \mathrm{Aut}\, A \setminus \{1^A\}$. Thus $\mathrm{Fix}(\mathrm{Aut}\, A) \subseteq (\mathrm{K}(A)) \neq A$, and the assertion follows by Prop. 2.7. (i).

(ii) Evidently, we can assume $\mathrm{Aut}\, A \neq \{1^A\}$. If the restriction of any $f \in \mathrm{Aut}\, A$ to $\mathrm{K}(A)$ is $1^{\mathrm{K}(A)}$, then $\mathrm{Fix}\, f = \mathrm{Fix}(\mathrm{Aut}\, A) = \mathrm{K}(A)$, $\forall f \neq 1^A$, thus Υ is a lattice isomorphism according to Prop. 2.7. (iii). Conversely, assume that Υ is a surjective mapping; since $\mathrm{Fix}(\mathrm{Aut}\, A) \subseteq \subseteq \mathrm{K}(A) \neq A$ and since $\mathrm{K}(A)$ is invariated by all $f \in \mathrm{Aut}\, A$ (by Remark 2.12.), applying Prop. 2.7. (iv) we get $\mathrm{Fix}(\mathrm{Aut}\, A) = \mathrm{K}(A)$.

(iii) Let C denote the factor algebra $A/\rho_{\mathrm{K}(A)}$; then $\mathrm{K}(A) \subseteq A$ itself is a $\rho_{\mathrm{K}(A)}$ congruence class. Since C contains as proper subalgebra only the one element subalgebra $\mathrm{K}(A)$, it is subalgebra-simple except for a singleton. Let us observe, that the conditions of Lemma 2.10. (ii) hold for $\mathrm{K}(A) \leq A$ and $F = \mathrm{Aut}\, A$. Namely $\mathrm{K}(A)$ is invariated by the whole group $\mathrm{Aut}\, A$ (— according to Remark 2.12. (ii)) and $A \setminus \mathrm{K}(A) \nsubseteq \nsubseteq \mathrm{Fix}(\mathrm{Aut}\, A)$. Thus $\mathrm{Aut}\, A$ is isomorphic to a subgroup G of $\mathrm{Aut}\, C$. Since $\mathrm{Con}\, C$ is a principal filter in the lattice $\mathrm{Con}\, A$, applying Prop. 2.8 we obtain that $\mathrm{S}(G)$ is isomorphic to an interval of $\mathrm{Con}\, A$. Evidently, the same is true for $\mathrm{S}(\mathrm{Aut}\, A)$. ◊

Remark 2.14. If (A, Γ) is a cyclic unary algebra and $F \leq \mathrm{Aut}\, A$, then $\mathrm{S}(F)$ is isomorphic to an interval of $\mathrm{Con}\, A$.

Theorem 2.15. *Let* (A, Γ) *be a unary algebra and* $F \leq \mathrm{Aut}\, A$. *If* $\mathrm{Fix}\, F \neq A$, *then the following statements hold:*

(i) $\mathrm{S}(F)$ *can be embedded as a sublattice in* $\mathrm{Con}\, A$.

(ii) *If there exists an* $a \in A \setminus \mathrm{Fix}\, F$, *such that the cyclic subalgebra* (a) *is invariated by all* $f \in F$, *then* $\mathrm{S}(F)$ *is isomorphic to an interval of* $\mathrm{Con}\, A$.

Proof. (i) If $\mathrm{Fix}\, F = \emptyset$, then $\mathrm{Fix}\, f_1 = \mathrm{Fix}\, f_2 = \emptyset$, $\forall f_1, f_2 \in F \setminus \{1^A\}$ and now (i) follows from Prop. 2.7. (ii). If $\mathrm{Fix}\, F \neq \emptyset$, then the conditions of Lemma 2.10. (iii) hold for the subalgebra $B = \mathrm{Fix}\, F$, — thus $\mathrm{S}(F)$ can be embedded in $\mathrm{Con}(A/\rho_{\mathrm{Fix}\, F})$. Since the latter lattice is a principal filter in $\mathrm{Con}\, A$, the statement is true.

(ii) By Lemma 2.10. (i), F is isomorphic to a subgroup H of $\mathrm{Aut}(a)$. Applying Th. 2.13. (iii) to H (and (a)), we get that $\mathrm{S}(H)$ is isomorphic to an interval of the lattice $\mathrm{Con}(a)$. Since $\mathrm{Con}(a)$ is a principal ideal in $\mathrm{Con}\, A$ (see [2]), the proof is completed. ◇

Corollary 2.16. *If* (A, Γ) *is a unary algebra with* $\mathrm{Con}\, A$ *distributive, then any* $F \leq \mathrm{Aut}\, A$ *satisfying* $\mathrm{Fix}\, F \neq A$ *is a locally cyclic group.*

Proof. Since by Th. 2.15. (i) $\mathrm{S}(F)$ is isomorphic to a sublattice of $\mathrm{Con}\, A$, $\mathrm{S}(F)$ is distributive. Now the statement follows applying Ore's theorem. ◇

Lemma 2.17. *If* (A, Γ) *is a subdirectly irreducible unary algebra, then the following statements are true:*

(i) *Any* $f \in \mathrm{Aut}\, A$, $f \neq 1^A$, *has at most one fixpoint.*

(ii) A *has at most two singletons. If* A *has two singletons, then* $\mathrm{Aut}\, A$ *has at most two elements and* $f \in \mathrm{Aut}\, A$, $f \neq 1^A$, *has no fixpoints. If* A *has exactly one singleton, then it is the only fixpoint for any* $f \in \mathrm{Aut}\, A$, $f \neq 1^A$.

Proof. (i) For any $f \in \mathrm{Aut}\, A$, $f \neq 1^A$, let $\rho_{\mathrm{Fix}\, f}$ denote the congruence corresponding to the subalgebra $\mathrm{Fix} f$. We can write evidently $\theta_{\langle f \rangle} \wedge \wedge\, \rho_{\mathrm{Fix}\, f} = 0_A$. Since for $f \neq 1^A$, we have $\theta_{\langle f \rangle} \neq 0_A$, the subdirect irreducibility of (A, Γ) implies $\rho_{\mathrm{Fix}\, f} = 0_A$ — i.e. $\mathrm{Fix} f$ is either empty or it is a singleton.

(ii) If $\{a\}$, $\{b\}$, $\{c\}$ are different singletons in A, then $\{a, b\}$ and $\{a, c\}$ are also subalgebras in A. We have $\rho_{\{a,b\}} \neq 0_A$, $\rho_{\{a,c\}} \neq 0_A$, but $\rho_{\{a,b\}} \wedge \rho_{\{a,c\}} = 0_A$ — which is a contradiction, because of A is subdirectly irreducible. Assume now that the singletons of A are $\{a\}$ and $\{b\}$ only. $(a, b \in A,\ a \neq b)$. By Remark 1.3. (i), for any $x_0 \in \{a, b\}$, $f(x_0)$ is also a singleton, thus $f(x_0) \in \{a, b\}$. If $a \in \mathrm{Fix} f$ for an $f \in \in \mathrm{Aut}\, A \setminus \{1^A\}$, then the injectivity of f implies $f(b) = b$, i.e. $b \in \mathrm{Fix} f$ — contradicting (i). Thus in this case any $f \in \mathrm{Aut}\, A$, $f \neq 1^A$ is without fixpoints. Since now $\mathrm{Fix}(\mathrm{Aut}\, A) = \emptyset \subset \{a, b\}$, applying Lemma 2.10. (i) for $F = \mathrm{Aut}\, A$ and $B = \{a, b\}$, we obtain that $\mathrm{Aut}\, A$ can be embedded in the permutation group of the set B, thus it has at most two elements. If A has only one singleton, say $\{a\}$, then by Remark 1.3. (i) follows $f(a) = a$, $\forall f \in \operatorname{Aut} A$. Now (i) implies $\operatorname{Fix} f = \{a\}$, for any $f \in \operatorname{Aut} A \setminus \setminus \{1^A\}$. $\Diamond$

Corollary 2.18. *If* (A, Γ) *is subdirectly irreducible, then the subalgebra* $\operatorname{Fix}(\operatorname{Aut} A)$ *has at most one element. If* $\operatorname{Fix}(\operatorname{Aut} A)$ *is a singleton, then this is the only singleton in* A. *We have* $\operatorname{Fix} f_1 = \operatorname{Fix} f_2$ *for any* $f_1, f_2 \in \in \operatorname{Aut} A \setminus \{1^A\}$.

For a set B its cardinality will be denoted by $|B|$ in the rest of the paper.

Theorem 2.19. *If* (A, Γ) *is a subdirectly irreducible unary algebra, then the following assertions are true:*

(i) $\Upsilon\colon \mathrm{S}(\operatorname{Aut} A) \longrightarrow \operatorname{Con} A$ *is a lattice embedding.*

(ii) $\mathrm{S}(\operatorname{Aut} A)$ *is isomorphic to a principal ideal of* $\operatorname{Con} A$.

(iii) *If* $\operatorname{Aut} A$ *is nontrivial, then it is a subdirectly irreducible p-group.*

(iv) *If* $(a) \subseteq A$ *is a subalgebra with at least two elements and* (a) *is invariated by* $\operatorname{Aut} A$, *then* $\mathrm{S}(\operatorname{Aut} A)$ *is isomorphic to an interval of* $\operatorname{Con}(a)$.

(v) *If* $\operatorname{Aut} A \neq \{1^A\}$ *and* $\Upsilon\colon \mathrm{S}(\operatorname{Aut} A) \longrightarrow [0_A, \theta_{\operatorname{Aut} A}]$ *is a lattice isomorphism, then either* (A, Γ) *is subalgebra-simple with a possible exception of a singleton, or it is a disjoint union of a singleton and of a subalgebra-simple unary algebra.*

Proof. If $\operatorname{Aut} A$ is trivial, then (i), (ii), and (iv) hold trivially. Thus we may assume that $\operatorname{Aut} A \neq \{1^A\}$.

(i) It is an immediate consequence of Prop. 2.7. (ii) and Cor. 2.18.

(ii) Let θ_0 denote the least nonzero congruence (the monolith) of (A, Γ). Since for any subalgebra $B \leq A$ with at least two elements we have $\theta_0 \leq \rho_B$, we can write: $0_A < \theta_0 \leq \wedge\{\rho_B | B \leq A, |B| \geq 2\} = = \rho_{\cap\{B|B\leq A,|B|\geq 2\}}$. Thus $I = \cap\{B | B \leq A, |B| \geq 2\}$ is a subalgebra with at least two elements. It is not hard to see that I is invariated by the whole $\operatorname{Aut} A$. Since, by Cor. 2.18. $\operatorname{Fix}(\operatorname{Aut} A)$ has at most one element, the conditions of Lemma 2.10. (i) hold for $B = I$ and $F = \operatorname{Aut} A$, thus $\operatorname{Aut} A$ is isomorphic to a subgroup G of $\operatorname{Aut} I$. In the case when I has just two elements, then the assumption $\operatorname{Aut} A \neq \{1^A\}$ implies that $\operatorname{Aut} A$ and $\operatorname{Aut} I$ are both groups of order two, i.e. $\mathrm{S}(\operatorname{Aut} A)$ is a two-element chain. Since now $\operatorname{Con} I$ is also a chain with two elements and it is a principal ideal of $\operatorname{Con} A$, the assertion (ii) follows trivially. If I contains more than two elements, then for any $x \in I$ we have either $(x) = I$ or $(x) = \{x\}$, according to the construction of I. Since by Lemma 2.17. (ii) I contains at most two singletons, there exists an $a \in I$ such that $(a) = I$, i.e. I is a cyclic subalgebra. The kernel $\mathrm{K}(I)$ of I has at most one element, otherwise we get $I \subseteq \mathrm{K}(I)$ by construction of I — which is impossible. According to Remark 2.12. (i) this fact means, that I is subalgebra-simple with a possible exception of a singleton. Applying now Prop. 2.8 we obtain that $\mathrm{S}(G)$ is isomorphic to a principal ideal of the lattice $\mathrm{Con}\, I$. Since $\mathrm{Con}\, I$ is isomorphic to a principal ideal of $\mathrm{Con}\, A$ and since $\mathrm{S}(\mathrm{Aut}\, A) \cong \mathrm{S}(G)$, the proof of (ii) is completed.

(iii) Since (A, Γ) is subdirectly irreducible, any principal ideal of $\mathrm{Con}\, A$ contains the least nonzero element of $\mathrm{Con}\, A$. According to (ii), $\mathrm{S}(\mathrm{Aut}\, A)$ has also a least nonzero element, let say F_0, which implies that $\mathrm{Aut}\, A$ is subdirectly irreducible. Evidently F_0 is a cyclic group of prime order, let say of order $p \in \mathbb{N}$. Since $F_0 \subseteq \langle f \rangle$, $\forall f \in \mathrm{Aut}\, A \setminus \{1^A\}$, any $\langle f \rangle$ is a subdirectly irreducible cyclic group, thus its order is p^n for some $n \in \mathbb{N}$.

(iv) By Cor. 2.18. $\mathrm{Fix}(\mathrm{Aut}\, A)$ has at most one element. Since (a) has at least two elements, $a \notin \mathrm{Fix}(\mathrm{Aut}\, A)$. Applying now Th. 2.15. (ii) for $F = \mathrm{Aut}\, A$, we obtain (iv).

(v) By (i) it is clear that $\Upsilon \colon \mathrm{S}(\mathrm{Aut}\, A) \longrightarrow [0_A, \theta_{\mathrm{Aut}\, A}]$ is a lattice isomorphism iff Υ is surjective. We can assume $|A| \geq 2$. By Cor. 2.18 $\mathrm{Fix}(\mathrm{Aut}\, A)$ has at most two elements, hence $\mathrm{Fix}(\mathrm{Aut}\, A) \neq A$. Let us consider now the subalgebra I constructed in the proof of (ii) — which has at least two elements — and denote $B = \mathrm{Fix}(\mathrm{Aut}\, A) \cup I$. By Cor. 2.18, we have $\mathrm{Fix}(\mathrm{Aut}\, A) \neq B$. Even more B is invariated by the whole $\mathrm{Aut}\, A$ (see Remark 1.4 and the proof of (ii)). Assume that $B \neq A$. Since Υ is surjective, applying now Prop. 2.7. (iv), we obtain $B = \mathrm{Fix}(\mathrm{Aut}\, A)$ — contradiction. Thus we must have $B = A$. If $I = A$, then any proper subalgebra of A has exactly one element. But this is possible only if, either A is subalgebra-simple with a possible exception of a singleton, or A is a two-element algebra consisting of two singletons. If $I \neq A$, then according to Cor. 2.18, $\mathrm{Fix}(\mathrm{Aut}\, A)$ is a one element subalgebra $\{a\}$, $a \in A$ and $\{a\}$ is the unique singleton of A. Since now $\{a\} \cup I = A$, we have $a \notin I$. By the construction of I (see: (ii)), we get $(x) = I$, $\forall x \in I$, i.e.: $I = A \setminus \{a\}$ is a subalgebra-simple unary algebra. ◊

3. The invariant part of the automorphism group of a unary algebra

Let us consider first an arbitrary algebra (A, F).

Remark 3.1. (i) For any $f \in \operatorname{Aut} A$ and $\theta \in \operatorname{Con} A$, the relation $f(\theta) = \{(f(a), f(b)) \in A^2 | (a, b) \in \theta\}$ is a congruence of (A, F).

(ii) We have $(a, b) \in f(\theta) \iff (f^{-1}(a), f^{-1}(b)) \in \theta$. The equality $f(\theta) = \theta$ is equivalent to $(a, b) \in \theta \iff (f(a), f(b)) \in \theta$, $\forall (a, b) \in \theta$.

Proof. (i) It is clear that $f(\theta)$ is an equivalence on A. Since for any $f \in \operatorname{Aut} A$ the mapping $(f, f): A^2 \longrightarrow A^2$, $(f, f)(x, y) = (f(x), f(y))$ $((x, y) \in A^2)$ is an automorphism of (A^2, F) and any $\theta \in \operatorname{Con} A$ is a subalgebra of (A^2, F), we obtain that $f(\theta)$ is also a subalgebra of it, i.e. $f(\theta)$ is a congruence on A.

(ii) Follows by the definition of $f(\theta)$. $\Diamond$

Definition 3.2. (i) We say that the congruence $\theta \in \operatorname{Con} A$ is *invariated by an* $f \in \operatorname{Aut} A$, if the relation $f(\theta) = \theta$ holds. $\theta \in \operatorname{Con} A$ is *invariated by the group* $F \leq \operatorname{Aut} A$, if we have $f(\theta) = \theta$ for all $f \in F$.

(ii) The set of those automorphisms of A under which all $\theta \in \operatorname{Con} A$ are invariant is denoted by $\operatorname{Inv} A$ and we call it *the invariant part* of $\operatorname{Aut} A$.

Lemma 3.3. *Any* $f \in \operatorname{Aut} A$ *induces a lattice automorphism* $\overline{f}$ *of* $\operatorname{Con} A$ *in the following way:* $\overline{f}(\theta) = f(\theta)$, $\theta \in \operatorname{Con} A$. *The mapping* $\Psi(f) = \overline{f}$ *is a group homomorphism of* $\operatorname{Aut} A$ *into* $\operatorname{Aut}(\operatorname{Con} A)$ *and* $\operatorname{Ker} \Psi = \operatorname{Inv} A$.

Proof. First we check $\Psi(f_1 \circ f_2) = \Psi(f_1) \circ \Psi(f_2)$, i.e. that $\overline{f_1 \circ f_2} = \overline{f_1} \circ \overline{f_2}$. We have $(\overline{f_1} \circ \overline{f_2})(\theta) = f_2(f_1(\theta))$ by definition. By Remark 3.1 (ii) we can write $(a, b) \in f_2(f_1(\theta)) \iff (f_1^{-1}(f_2^{-1}(a)), f_1^{-1}(f_2^{-1}(b))) \in \theta \iff ((f_2^{-1} \circ f_1^{-1})(a), (f_2^{-1} \circ f_1^{-1})(b)) \in \theta \iff ((f_1 \circ f_2)^{-1}(a), (f_1 \circ f_2)^{-1}(b)) \in \theta \iff (a, b) \in (f_1 \circ f_2)(\theta)$. Thus $(\overline{f_1} \circ \overline{f_2})(\theta) = \overline{f_1 \circ f_2}(\theta)$, $\theta \in \operatorname{Con} A$, i.e.: $\overline{f_1} \circ \overline{f_2} = \overline{f_1 \circ f_2}$. It follows that: $\overline{f} \circ \overline{f^{-1}} = \overline{f \circ f^{-1}} = \overline{1^A} = 1^{\operatorname{Con} A}$ and symmetrically $\overline{f^{-1}} \circ \overline{f} = 1^{\operatorname{Con} A}$. Thus we conclude that $\overline{f}$ and $\overline{f^{-1}}$ are inverses of each other. Since $\overline{f}$ and $\overline{f^{-1}}$ are both isotone mappings, we conclude that $\overline{f}$ is a lattice automorphism. Since $\Psi: \operatorname{Aut} A \longrightarrow \operatorname{Aut}(\operatorname{Con} A)$ satisfies the rule $\Psi(f_1 \circ f_2) = \Psi(f_1) \circ \Psi(f_2)$, it is a group homomorphism. Take now $f \in \operatorname{Aut} A$ such that $\overline{f} = 1^{\operatorname{Con} A}$. This relation is equivalent to $f(\theta) = \theta$, $\forall \theta \in \operatorname{Con} A$, i.e. to $f \in \operatorname{Inv} A$. Thus we obtain $\operatorname{Ker} \Psi = \operatorname{Inv} A$. $\Diamond$

Corollary 3.4. (i) $\operatorname{Inv} A$ *is a normal subgroup of* $\operatorname{Aut} A$.

(ii) *The factor group* $\mathrm{Aut}\, A/\,\mathrm{Inv}\, A$ *can be embedded in the automorphism group of the congruence lattice of* A.

Proof. (i) It is trivial.

(ii) Applying for Ψ the homomorphism theorem of groups, we get $\mathrm{Aut}\, A/\,\mathrm{Inv}\, A \cong \Psi(\mathrm{Aut}\, A) \leq \mathrm{Aut}(\mathrm{Con}\, A)$. $\Diamond$

We note that $\theta \in \mathrm{Con}\, A$ is invariated by the group $F \leq \mathrm{Aut}\, A$ iff $f(\theta) \leq \theta$, $\forall f \in F$, i.e. iff $(a,b) \in \theta \Longrightarrow (f(a), f(b)) \in \theta$, $\forall f \in F$. Indeed, it is clear that for any $f \in F$ we have $f(\theta) \leq \theta$ iff $(a,b) \in \in \theta \Longrightarrow (f(a), f(b)) \in \theta$. On the other hand if $f(\theta) \leq \theta$, $f \in F$, then $f^{-1}(\theta) \leq \theta$ implies: $\theta = f(f^{-1}(\theta)) \leq f(\theta)$.

Now we shall concentrate on some particular properties of $\mathrm{Inv}\, A$ in the case of unary algebras.

Proposition 3.5. *Let* (A, Γ) *be a unary algebra,* $F, G \leq \mathrm{Aut}\, A$ *and* $\theta \in \mathrm{Con}\, A$. *Then the following statements are true:*

(i) *If* θ *is invariated by* F, *then we have* $\theta \circ \theta_F = \theta_F \circ \theta$.

(ii) *If* $G \lhd F$, *then* θ_G *is invariated by* F.

(iii) *If* $G \leq F$, $\mathrm{Fix}\, F \neq A$ *and* θ_G *is invariated by* F, *then we have* $G \lhd F$.

Proof. (i) Take $(a,b) \in \theta \circ \theta_F$. Then there exist a $c \in A$ and an $f \in F$ such that $(a,c) \in \theta$ and $f(c) = b$. Since θ is invariated by f, we have $(f(a), f(c)) \in \theta$. But now from $(a, f(a)) \in \theta_F$ and $(f(a), b) \in \theta$ follows $(a,b) \in \theta_F \circ \theta$. Thus we obtain: $\theta \circ \theta_F \subseteq \theta_F \circ \theta$. Symmetrically we can prove $\theta_F \circ \theta \subseteq \theta \circ \theta_F$.

(ii) If $(a,b) \in \theta_G$ then we have $g(a) = b$ for some $g \in G$. Since $G \lhd F$, for any $f \in F$ there is a $g' \in G$ such that $g \circ f = f \circ g'$, so we obtain that $f(b) = f(g(a)) = g'(f(a))$, i.e. that $(f(a), f(b)) \in \theta_G$.

(iii) Take $a \in A \setminus \mathrm{Fix}\, F$ and $g \in G$ arbitrary; since θ_G is invariated by any $f \in F$, we can write $(f(a), f(g(a))) \in \theta_G$. Thus for any $f \in F$ there exists a $g' \in G$ such that $g'(f(a)) = f(g(a))$, i.e. $(f \circ g')(a) = (g \circ \circ f)(a)$. We obtain $(f \circ g' \circ f^{-1} \circ g^{-1})(a) = a$. Now $f \circ g' \circ f^{-1} \circ g^{-1} \in F$ and $a \notin \mathrm{Fix}\, F$ implies $f \circ g' \circ f^{-1} \circ g^{-1} = 1^A$, i.e. $f^{-1} \circ g \circ f = g' \in G$. Thus we conclude $G \lhd F$. $\Diamond$

Lemma 3.6. (i) *Any subalgebra* $B \leq A$ *with at least two elements is invariated by the whole* $\mathrm{Inv}\, A$.

(ii) *If* $F \leq \mathrm{Inv}\, A$, $F \neq \{1^A\}$, *then either every singleton of* A *is contained in* $\mathrm{Fix}\, F$, *or* $\mathrm{Fix}\, F = \emptyset$, A *has exactly two singletons and the order of* F *is two.*

Proof. (i) We have $f(\rho_B) = \rho_B$, $\forall f \in \mathrm{Inv}\, A$. Since f is injective, for all $x, y \in B$, $x \neq y$ we obtain $f(x), f(y) \in B$. Thus we conclude $f(B) \subseteq B$, for all $f \in \operatorname{Inv} A$.

(ii) If $\operatorname{Fix} F \neq \emptyset$ and there is a singleton $\{a\} \subseteq A \setminus \operatorname{Fix} F$, then the subalgebra $\operatorname{Fix} F \cup \{a\}$ is invariated by all $f \in F$, according to (i). Since $A \setminus \operatorname{Fix} F$ is also invariated by any $f \in F$ (see Remark 1.4), we get $f(a) = a$, $f \in F$ — which is a contradiction. Thus $\operatorname{Fix} F \neq \emptyset$ implies that all the singletons of A are contained in $\operatorname{Fix} F$. Assume now that $\operatorname{Fix} F = \emptyset$ and A contains at least one singleton. If A has exactly one singleton, then according to Remark 1.3. (i), it is a common fixpoint for all $f \in F$ — contradicting $\operatorname{Fix} F = \emptyset$. If A contains 3 different singletons $\{a\}$, $\{b\}$, $\{c\}$, then the fact that the subalgebras $\{a, b\}$ and $\{b, c\}$ are both invariated by F implies $f(b) = b$, $f \in F$ — contradiction. Thus in the case of our latter assumption A must contain exactly two singletons, — let us denote them by $\{a\}$, $\{b\}$, $a, b \in A$. Since for any $f \in F \setminus \{1^A\}$ we have $f(a) = b$ by assumption, we get $f_1(a) = f_2(a)$, $\forall f_1, f_2 \in F \setminus \{1^A\}$. But in this case $(f_1 \circ f_2^{-1})(a) = a$ and $a \notin \operatorname{Fix} F$ implies $f_1 = f_2$ — i.e. F contains only one element different from 1^A. $\Diamond$

Proposition 3.7. (i) *If* $\operatorname{Fix}(\operatorname{Inv} A) \neq A$ *then* $\operatorname{Inv} A$ *is a Hamiltonian group and* $\operatorname{S}(\operatorname{Inv} A)$ *is isomorphic to an interval of* $\operatorname{Con} A$.

(ii) *For any* $F \leq \operatorname{Inv} A$, θ_F *is permutable with any congruence* $\theta \in \operatorname{Con} A$.

Proof. (i) Since for any subgroup $G \leq \operatorname{Inv} A$, θ_G is invariated by $\operatorname{Inv} A$, by Prop. 3.5 (iii) we obtain that any subgroup G of $\operatorname{Inv} A$ is a normal subgroup, i.e. that $\operatorname{Inv} A$ is a Hamiltonian group. We can assume now that $\operatorname{Inv} A \neq \{1^A\}$. If $A \setminus \operatorname{Fix}(\operatorname{Inv} A)$ contains an element A such that (a) has at least two elements, then the statement follows by Lemma 3.6 (i) and Th. 2.15 (ii). If all the elements of $A \setminus \operatorname{Fix}(\operatorname{Inv} A)$ are singletons, then according to Lemma 3.6 (ii), we have $\operatorname{Fix}(\operatorname{Inv} A) = \emptyset$, A consists of exactly two singletons, and the group $\operatorname{Inv} A$ is of order two. Thus both $\operatorname{S}(\operatorname{Inv} A)$ and $\operatorname{Con} A$ are chains with two elements.

(ii) It follows directly from Prop. 3.5 (i) and from the definition of $\operatorname{Inv} A$. $\Diamond$

Corollary 3.8. *If* (A, Γ) *is a cyclic unary algebra, then* $\operatorname{Inv} A$ *is a Hamiltonian group. If* (A, Γ) *is subdirectly irreducible, then* $\operatorname{Inv} A$ *is either the quaternion group or it is a subdirectly irreducible Abelian group.*

Proof. In both of these cases we have $\operatorname{Fix}(\operatorname{Inv} A) \neq A$, so Prop. 3.7 (i) applies. If (A, Γ) is subdirectly irreducible, then by Th. 2.19 (ii) $\operatorname{S}(\operatorname{Inv} A)$ is isomorphic to a principal ideal of $\operatorname{Con} A$. Since this one contains a least nonzero element, $\operatorname{Inv} A$ is also subdirectly irreducible. Now the statement follows from the fact, that a subdirectly irreducible Hamiltonian group is either the quaternion group or it is a subdirectly irreducible Abelian group. ◊

Let us consider again an arbitrary algebra (A, F).

Lemma 3.9. *If (A, F) is a subdirect product of algebras (A_i, F), $i \in \in I$, $(I \neq \emptyset)$, then* $\operatorname{Inv} A$ *is a subdirect product of some subgroups G_i of* $\operatorname{Inv} A_i$.

Proof. Let π_i denote the natural projection of A onto A_i $(i \in I)$. For any $f \in \operatorname{Inv} A$ and any $i \in I$ we define a mapping $f_i: A_i \longrightarrow A_i$ as follows: $f_i(\pi_i(x)) = \pi_i(f(x))$, $x \in A$. f_i is well defined: If we have $\pi_i(x_1) = \pi_i(x_2)$ for some $x_1, x_2 \in A$, then $(x_1, x_2) \in \ker \pi_i$ and $f \in \operatorname{Inv} A$ implies $(f(x_1), f(x_2)) \in \ker \pi_i$, i.e. $\pi_i(f(x_1)) = \pi_i(f(x_2))$. Let us show that f_i is an automorphism of A_i: It is easy to check that the mapping $f_i^*: A_i \longrightarrow A_i$, defined by $f_i^*(\pi_i(x)) = \pi_i(f^{-1}(x))$ is the inverse of f_i. Thus f_i is bijective. Take now an n-ary operation $p \in F$ and $z_1, z_2, \ldots, z_n \in A_i$. Then there are $x_1, x_2, \ldots, x_n \in A$ such that $z_1 = \pi_i(x_1)$, $z_2 = \pi_i(x_2), \ldots, z_n = \pi_i(x_n)$. We can write: $p(f_i(z_1), \ \ldots \ , f_i(z_n)) = p(\pi_i(f(x_1)) \ \ldots \ \pi_i(f(x_n))) = \pi_i(f(p(x_1 \ , \ldots \ldots, x_n))) = f_i(\pi_i(p(x_1, \ldots, x_n))) = f_i(p(z_1, \ldots, z_n))$.

We claim $f_i \in \operatorname{Inv} A_i$. It is enough to show that for any $\theta \in \in \operatorname{Con} A_i$ and $(a, b) \in A_i^2$, we have $(a, b) \in \theta \iff (f_i(a), f_i(b)) \in \theta$. If $\theta \in \operatorname{Con} A_i$, then it is known that $\pi_i^{-1}(\theta) = \{(u, v) \in A^2 | (\pi_i(u), \pi_i(v)) \in \in \theta\}$ is a congruence of A. Take $(a, b) \in A_i^2$; then there are some $u, v \in A$ such that $\pi_i(u) = a$ and $\pi_i(v) = b$. Now for any $f \in \operatorname{Inv} A$ we have

$$(a, b) \in \theta \iff (u, v) \in \pi_i^{-1}(\theta) \iff (f(u), f(v)) \in \pi_i^{-1}(\theta) \iff$$
$$\iff (\pi_i(f(u)), \pi_i(f(v))) \in \theta \iff (f_i(u), f_i(v)) \in \theta.$$

Let us consider now for an arbitrary, but fixed $i \in I$ the mapping $\mu_i: \operatorname{Inv} A \longrightarrow \operatorname{Inv} A_i$, $\mu_i(f) = f_i (f \in \operatorname{Inv} A)$. We can write: $[\mu_i(f \circ g)](\pi_i(x)) = \pi_i(g(f(x))) = g_i(\pi_i(f(x))) = g_i(f_i(\pi_i(x))) = (f_i \circ \circ g_i)(\pi_i(x)) = [\mu_i(f) \circ \mu_i(g)](\pi_i(x))$. Since $\pi_i: A \longrightarrow A_i$ is onto, we obtain $\mu_i(f \circ g) = \mu_i(f) \circ \mu_i(g)$, $\forall f, g \in \operatorname{Inv} A$ — i.e. μ_i is a group homomorphism. Let us denote the group $\mu_i(\operatorname{Inv} A)$ by G_i $(i \in I)$. We have $\operatorname{Inv} A / \ker \mu_i \cong G_i \leq \operatorname{Inv} A_i$, $\forall i \in I$ and $\ker \mu_i = \{f \in \in \operatorname{Inv} A | f_i = 1^{A_i}\} = \{f \in \operatorname{Inv} A | \pi_i(f(x)) = \pi_i(x), \forall x \in A\} = \{f \in \in \operatorname{Inv} A | (x, f(x)) \in \ker \pi_i, \forall x \in A\}$. Since A is a subdirect product of algebras A_i, $i \in I$, we have $\bigwedge_{i \in I} \ker \pi_i = 0_A$. We get: $f \in \bigwedge_{i \in I} \ker \mu_i \iff \iff (x, f(x)) \in \bigwedge_{i \in I} \ker \pi_i = 0_A$, $\forall x \in A$. As the latter relation is equivalent to $f = 1^A$, we conclude: $\bigwedge_{i \in I} \ker \mu_i = \{1^A\}$. According to this result $\mathrm{Inv}\, A$ is a subdirect product of the groups $G_i \leq \mathrm{Inv}\, A_i$, $i \in I$. ◊

Corollary 3.10. *If $(A, F) \leq \prod_{i \in I} A_i$ is a subdirect product of algebras (A_i, F), $i \in I$, then for any $f \in \mathrm{Inv}\, A$ and for any $x = (x_i)_{i \in I} \in A$ we have $f(x) = (f_i(x_i))_{i \in I}$.*

Proof. Let us introduce the notation $g(x) = (f_i(x_i))_{i \in I} \in \prod_{i \in I} A_i$. We can write $\pi_i(f(x)) = f_i(\pi_i(x)) = f_i(x_i) = \pi_i(g(x))$, $\forall i \in I$. Since A is a subdirect product of algebras A_i, these relations imply $f(x) = g(x)$, i.e.: $f(x) = (f_i(x_i))_{i \in I}$, $x \in A$. ◊

Theorem 3.11. *If (A, Γ) is an arbitrary unary algebra, then* $\mathrm{Inv}\, A$ *is either the quaternion group, or it is a subdirect irreducible Abelian group, or it is a subdirect product of some groups of the above type. These groups can be chosen to be subgroups of the invariant part of the automorphism groups of some subdirectly irreducible factors of A.*

Proof. If (A, Γ) is subdirectly irreducible, then the statement of this theorem is the same as of the Cor. 3.8. If $A \leq \prod_{i \in I} A_i$ is a representation of A as a subdirect product of subdirectly irreducible algebras A_i, then according to Lemma 3.9, there exist $G_i \leq \mathrm{Inv}\, A_i$, $i \in I$, such that $\mathrm{Inv}\, A$ is a subdirect product of these groups G_i. By Cor. 3.8, any G_i is either a subgroup of a quaternion group or of a subdirectly irreducible Abelian group. Thus any G_i, $i \in I$ itself is either a quaternion group or it is a subdirectly irreducible Abelian group. ◊

Corollary 3.12. *If (A, Γ) is a unary algebra such that* $\mathrm{Con}\, A$ *is a rigid lattice, then* $\mathrm{Aut}\, A$ *is either the quaternion group or it is a subdirectly irreducible Abelian group, or it is a subdirect product of some groups of these type.*

Proof. Now by Cor. 3.4 (ii) $\mathrm{Aut}\, A = \mathrm{Inv}\, A$, thus the statement follows by Th. 3.11. ◊

For unary algebras we can formulate the following

Proposition 3.13. (i) *If* $\mathrm{Con}\, A$ *is a finite lattice, then* $\mathrm{Aut}\, A$ *is a finite group.*

(ii) *If (A, Γ) is a semisimple unary algebra, then* $\mathrm{Inv}\, A$ *is an Abelian group and* $\mathrm{S}(\mathrm{Inv}\, A)$ *is isomorphic to an interval of* $\mathrm{Con}\, A$.

Proof. (i) If (A, Γ) is subdirectly irreducible then $\mathrm{S}(\mathrm{Aut}\, A)$ is finite according to Th. 2.19 and to the assumption. Thus $\mathrm{Aut}\, A$ is a finite group. If (A, Γ) is subdirectly reducible then it can be represented as a finite subdirect product of some subdirectly irreducible algebras (A_i, Γ), $1 \leq i \leq n$ $(n \in \mathbb{N})$. Any $\mathrm{Con}\, A_i$, being isomorphic to a principal filter of $\mathrm{Con}\, A$, is a finite lattice. Since $\mathrm{S}(\mathrm{Inv}\, A_i)$ is isomorphic to a principal ideal of $\mathrm{Con}\, A_i$, it is finite too. Thus any $\mathrm{Inv}\, A_i$ is a finite group, which implies according to Th. 3.11 that $\mathrm{Inv}\, A$ itself is finite. Since by Cor. 3.4 (ii) $\mathrm{Aut}\, A / \mathrm{Inv}\, A$ can be embedded in $\mathrm{Aut}(\mathrm{Con}\, A)$ which is obviously is a group of finite order, the factor group $\mathrm{Aut}\, A / \mathrm{Inv}\, A$ is finite. From this the evident result is that the group $\mathrm{Aut}\, A$ has finite order.

(ii) Since any subdirectly irreducible factor of (A, Γ) is congruence simple, according to [3], $\mathrm{Aut}\, A_i$ is either a cyclic group of prime order, or it is trivial. Evidently the same is true for $\mathrm{Inv}\, A_i$, $i \in I$. Thus $\mathrm{Inv}\, A$ — as a subdirect product of Abelian groups — is itself Abelian. If (A, Γ) is subdirectly irreducible then the second part of the statement (ii) is obvious. Assume now that A is a subdirect product of simple unary algebras A_i, $i \in I$ and $x = (x_i)_{i \in I} \in A$ is a fixpoint of an $f \in \mathrm{Inv}\, A$, $f \neq 1^A$. According to Cor. 3.10, we have $(f_i(x_i))_{i \in I} = = (x_i)_{i \in I}$, i.e. $f_i(x_i) = x_i$, $\forall i \in I$ — where $x_i \in A_i$. Since any A_i is congruence simple, $\mathrm{Aut}\, A_i \neq \{1^{A_i}\}$ implies, according to [3], that A_i is either subalgebra-simple or it consists of two singletons. But in both of these cases, according to Lemma 2.17. we have $\mathrm{Fix}(\mathrm{Aut}\, A_i) = \emptyset$. Thus $f_i(x_i) = x_i$, $i \in I$ implies $f_i = 1^{A_i}$ for all $i \in I$. Summarizing, we obtain $f = 1^A$, contradicting our assumption. Thus we have $\mathrm{Fix}(\mathrm{Inv}\, A) = \emptyset$ and now the second part of statement (ii) follows by Prop. 3.7 (i). ◇

Proposition 3.14. *If (A, Γ) is a subalgebra-simple unary algebra with the property that $\theta_{\mathrm{Aut}\, A}$ has at most one complement in any interval of $\mathrm{Con}\, A$ containing it, then the following statements are equivalent:*

(i) $\mathrm{Inv}\, A = \mathrm{Aut}\, A$

(ii) $\mathrm{Aut}\, A$ *is a Hamiltonian group.*

Proof. According to Cor. 3.8, the implication (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i): We have only to prove $f(\theta) = \theta$ for all $f \in \mathrm{Aut}\, A$ and $\theta \in \mathrm{Con}\, A$. It is obvious, that for any $\theta \geq \theta_{\mathrm{Aut}\, A}$ we have $f(\theta) = = \theta$, $f \in \mathrm{Aut}\, A$. Since (A, Γ) is subalgebra-simple, for any $\theta \leq \theta_{\mathrm{Aut}\, A}$, there exists an $G \leq \mathrm{Aut}\, A$ such that $\theta_G = \theta$. Since $\mathrm{Aut}\, A$ is Hamiltonian, according to Prop. 3.5 (ii), we have $f(\theta_G) = \theta_G$ for any $G \leq \leq \mathrm{Aut}\, A$. Thus $f(\theta) = \theta$. Take now a $\theta \in \mathrm{Con}\, A$ arbitrary. We can write:

$$f(\theta)\wedge\theta_{\mathrm{Aut}\,A}=f(\theta)\wedge f(\theta_{\mathrm{Aut}\,A})=f(\theta\wedge\theta_{\mathrm{Aut}\,A})=\theta\wedge\theta_{\mathrm{Aut}\,A},$$

$$f(\theta)\vee\theta_{\mathrm{Aut}\,A}=f(\theta)\vee f(\theta_{\mathrm{Aut}\,A})=f(\theta\vee\theta_{\mathrm{Aut}\,A})=\theta\vee\theta_{\mathrm{Aut}\,A}.$$

Thus θ and $f(\theta)$ are both complements of $\theta_{\mathrm{Aut}\,A}$ in the interval $[\theta\wedge\wedge\theta_{\mathrm{Aut}\,A},\theta\vee\theta_{\mathrm{Aut}\,A}]$. According to our assumptions we obtain $f(\theta)=\theta$, i.e.: $f\in\mathrm{Inv}\,A$. ◊

Corollary 3.15. *If* (A,Γ) *is a subalgebra-simple unary algebra and if* $\theta_{\mathrm{Aut}\,A}$ *is a standard element of* $\mathrm{Con}\,A$, *then* $\mathrm{Inv}\,A=\mathrm{Aut}\,A\iff\mathrm{Aut}\,A$ *is Hamiltonian.*

Proof. It is known, that a standard element of a lattice has at most one complement in any interval containing it (see for ex. [2]) — thus the conditions of Prop. 3.14 hold. ◊

Corollary 3.16. *If* (A,Γ) *is a subalgebra-simple unary algebra with* $\mathrm{Con}\,A$ *distributive, then* $\mathrm{Inv}\,A=\mathrm{Aut}\,A$ *and* $\mathrm{Aut}\,A$ *is a locally cyclic group.*

Proof. Since $\mathrm{Fix}(\mathrm{Aut}\,A)=\emptyset$, the second assertion follows from Cor. 2.16. Since $\mathrm{Aut}\,A$ is Abelian and any element of the distributive lattice $\mathrm{Con}\,A$ is a standard, applying Cor. 3.15 we get $\mathrm{Inv}\,A=\mathrm{Aut}\,A$. ◊

Problems

1) Find a necessary and sufficient condition for $\Upsilon\colon \mathrm{S}(\mathrm{Aut}\,A)\longrightarrow\longrightarrow[0_A,\theta_{\mathrm{Aut}\,A}]$ to be: a) injective, b) lattice isomorphism.

2) Find a necessary and sufficient condition to have: a) $\mathrm{Fix}(\mathrm{Inv}\,A)\neq A$, b) $\mathrm{Inv}\,A$ Hamiltonian group.

3) If $|\,\mathrm{Con}\,A|=n(n\in\mathbb{N})$, find an upper bound for $|\,\mathrm{Aut}\,A|$ or $|\mathrm{S}(\mathrm{Aut}\,A)|$.

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